

# Infinitesimal Calculus

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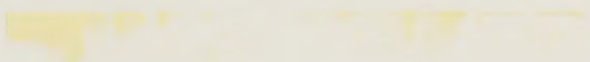
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# Infinitesimal Calculus



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## Editors' Introduction

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This book containing many of the central ideas of analysis covers the material for a modern junior-level advanced calculus course in American colleges and universities. Stress is placed on the calculational aspects of the subject to develop not only calculational ability but also an understanding of the methods of calculation and of the role of calculation and approximation in analysis. The reader is assumed to be familiar with the rudiments of real analysis, including the basic properties of the real and complex numbers, continuous functions, derivatives, and primitives. These topics are outlined in Chapter 0—*Introduction*.

The book is roughly divided into three parts. The first part (Chapters I–V) deals with the general question of approximation. Topics include the local question of approximation by asymptotic developments, the global theory of approximation to roots of equations, and uniform convergence.

The second part (Chapters VI–X) is an introduction to the theory of functions of a complex variable. The usual local theory of the Cauchy integral, power series development, and a study of singularities and residues is followed by applications of the theory of functions of a complex variable to the theory of approximation and a discussion of conformal mapping.

The third part (Chapters XI–XV) is a study of differential equations in the real and complex domains. Primary consideration is given to linear equations. Perturbations of linear differential systems are included, and the section concludes with a study of Bessel's equation.

The book can form the basis of an excellent upper division course in real and complex analysis. It provides an excellent background for further study in analysis, and can also be used as a text for a course in mathematical methods in physics or astronomy. It can serve as an excellent review of undergraduate analysis for graduate students. In addition, material can be extracted from the first ten chapters (particularly Chapters VI–X) to form a text for a one-semester junior- or senior-level course in complex analysis.

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DALE HUSEMOLLER, Haverford College

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# Preface

“Today’s students can no longer calculate”: such is the grievance frequently directed against current teaching of mathematics by physicists and engineers, and it must be admitted that this criticism is often justified. When one has seen a second or third year undergraduate toil over a change of variable or an integration by parts, one can scarcely be other than alarmed, particularly (as is sometimes the case) when the same student seasons his ignorance and clumsiness with a pretentious and useless jargon which he has also failed to understand.

It must be continually repeated that there is no “modern mathematics” as opposed to “classical mathematics” but simply the mathematics of today, which continues that of yesterday without any deep rupture, and which above all is dedicated to solving the great problems left by our predecessors. To do this, mathematics has gradually developed a profusion of new abstract concepts, which, by concentrating on the heart of a given problem and by eliminating trivial details, have made possible a steady advance in areas still considered inaccessible scarcely fifty years ago. Those mathematicians who create abstraction for the sake of abstraction are mostly mediocrities.

A by no means negligible consequence of this tendency to abstraction has been a “tidying up” which these new concepts have helped to create in the teaching of the fundamentals of mathematics (particularly in algebra and geometry). Prior to this, ridiculous traditions had encumbered teaching with trivialities and with useless and even harmful developments. Nevertheless the substance of so-called “classical mathematics” has remained intact, and the basis of modern analysis is still the wonderful tool wrought by the mathematicians of the last three centuries, the Infinitesimal Calculus. To pretend to neglect it in order to plunge immediately into the most recent functional analysis is to build on sand and can produce nothing but sterility and verbiage.

Until this year this stumbling block was hardly avoidable. Trapped on the one hand by a secondary teaching in the hands of a mandarin cut off from living mathematics for 80 years and exclusively devoted to the contemplation of its navel, and on the other hand by the teaching of modern analysis given in the Faculties, which “stick” to research in order to prepare for it efficiently, the unfortunate student had just one year to initiate himself into the classical Infinitesimal Calculus and to learn how to handle its techniques fluently. Experience soon showed that this was insufficient, and the palliative introduced under the title of “Mathematical Techniques of Physics” given by mathematicians more concerned with rigor than with efficiency, achieved in many Faculties

the teaching of a painless version of abstract analysis, stressing principles rather than calculation.

The new syllabuses, by stretching the "first cycle" over two years, should re-establish the equilibrium and give the conscientious student the solid technical basis which will enable him later to assimilate more abstract concepts without falling into psittacism. Essential parts of classical analysis, which can and should be approached without too much abstract preparation, like the theory of analytic functions and of differential equations, have fortunately been included in these syllabuses, particularly in the second year. This book is above all devoted to the development of these fundamental techniques, assuming known the fundamentals of the differential and integral Calculus taught in the first year of the first cycle.

We must therefore "know how to calculate" before claiming access to modern analysis. But what does "to calculate" mean? There are in fact two types of "calculus" which there is a tendency to confuse. On the one hand, there is the "algebraic calculus", which (oversimplifying the issue) can be characterized as the establishing of *equalities*; the prototype is given by the formulae for the solution of equations (the "closed formulae" of the Anglo-Saxons) which wield a strange kind of fascination on the users of mathematics: how many times have I met an engineer or a physicist who wants mathematics to be a kind of automatic machine producing formulae for the solution of problems!

This kind of relation also exists in analysis and can often be of great importance—Cauchy's formula and the development into Fourier series are typical examples of this. But in my opinion the essence of the Infinitesimal Calculus does not lie here. Physicists insist, with good reason, that for them a theorem is without interest if it does not entail at least the possibility of calculating numerically the numbers or functions under consideration. They will have nothing to do with those "existence theorems" of the pure mathematicians which do not fulfil these conditions. But to speak of numerical calculation is to speak of *approximation*, a real number being "known" only when a method to approximate it has been given (with an approximation which the mathematician wants to be arbitrarily small, whereas the user of mathematics is content with much less). If it is remembered that the teaching of mathematics, in the first cycle, is addressed at least as much to the future physicists and chemists as to the mathematicians, it will be understood why this side of analysis is particularly insisted upon in this work. I have not tried to write a treatise on the Numerical Calculus proper, which should be the object of specialized teaching, but no concept has been introduced which is not susceptible to numerical evaluation. At each stage the theoretical means of obtaining such calculations has been indicated, if required.

The pure mathematicians would in fact be wrong to despise this "down to earth" side of the Infinitesimal Calculus. To acquire a "feeling for analysis" indispensable even in the most abstract speculations, one must have learnt to distinguish between what is "large" and what is "small", what is "dominant" and what is "negligible". In other words, Infinitesimal Calculus, as it is presented in this book, is an apprenticeship in the handling of *inequalities* far more than of equalities and can be summed up in three words:

MAJORIZE, MINORIZE, APPROXIMATE.

The adoption of this point of view by no means implies that I have sacrificed rigor to convenience, or reduced the Infinitesimal Calculus to a series of recipes. We have to shape thinking beings, not robots, to induce the student to understand what he is doing, not to teach him mechanical methods. To have a "feeling for analysis" is to have acquired an "intuitive" idea of the operations of the Infinitesimal Calculus and this is obtained only through use and numerous concrete examples. But the test which proves that one has really reached this stage is to know how to give precise definitions of the notions used and to employ these to build correct proofs; for these last are no more, in the end, than a "pulling into shape" of intuition.

On this point, the physicists often jeer at the pure mathematician for always wanting to prove everything and for "splitting hairs" to establish "self-evident" results. They are not always wrong, and a beginner would do well to accept plausible results without encumbering his mind with subtle proofs,<sup>1</sup> so that he can reserve his efforts for the assimilation of new and not "self-evident" ideas. I have therefore had no hesitation in admitting a certain number of basic theorems of analysis<sup>2</sup> nor in pointing out to students that they may, at first reading, dispense with knowing certain long or slightly delicate proofs, by printing the latter in small print.

The physicists venture onto dangerous ground where they have a tendency to accept as "evident" that which is not so at all and to forget that our intuition is but a rudimentary instrument, which at times leads us into gross errors. Contrary to what many of them believe, it is not necessary to look for functions as "monstrous" as continuous functions without derivatives in order to fault them in results which they accept without discussion. The "Runge phenomenon" (Chapter IX, Appendix) shows that the classical method of polynomial interpolation can diverge for analytic functions as "nice" as we could wish; and there are functions *analytic* for  $|z| < 1$ , *continuous* in the whole disc  $|z| \leq 1$ , which however transform the circle  $|z| = 1$  onto a Peano curve filling a square.<sup>3</sup>

Implicit faith therefore has its perils. In any case one cannot meet serious experimentalists without being struck by the extreme care which they take in making sure of the correctness of their measurements and in avoiding fallacious interpretations. To handle mathematics correctly requires an equal care, and I do not think it is good teaching practice to try to inculcate strict habits of work in some spheres, while allowing (or even encouraging) slackness and vagueness in others.

I have not adhered slavishly to the official syllabuses, and I have stressed particularly that which seemed to me most important for the student who completes his first cycle with a view to going on to his License or Maitrise in Physics or Mathematics (pure or applied). Thus I have omitted everything concerning multiple integrals and differential forms. I have said elsewhere what I have thought of the "Stokes mania" of some of my colleagues, and the coverage of the subject in the first year of the first cycle seems quite sufficient to me, without trying to enter into refinements which at this level can only be sterile.<sup>4</sup> On the other hand I have included a number of topics of the Infinitesimal Calculus which do not expressly appear in the syllabus, or which, like the serious study of differential equations, are in my opinion left too late, at the level of the Maitrise. Roughly speaking, it can be said that the analysis expounded in this book is essentially analysis "of one variable", real or complex.<sup>5</sup> All mathematicians know that the passage from one to several variables is a brutal "jump" which gives rise to great

difficulties, and necessitates quite new methods. On the other hand, analysis of one variable is an essential tool for working towards more general questions. I have thought it wholly appropriate to put this "mutation" at the junction of the two cycles.

The present timetables do not therefore permit the teaching of the whole of this book in the second year of the first cycle, and the teacher or student who uses it will make his own choice. Nevertheless one may be forgiven for hoping that one day secondary teaching will place in the lumber-room of history the fossilized mathematics at present taught and that the time thus gained will be usefully employed in teaching in the last three years at high school what is now taught in the first year of the first cycle.<sup>6</sup> The first four chapters of this book, which are only complementary to the syllabus of the present first year (and usually omitted), could then be advantageously incorporated into the first year, and all of the remaining chapters into the second year. A student who had properly assimilated them would, in my opinion, be well prepared either to apply his mathematical knowledge to concrete problems, or to move to a higher level of abstraction and begin the present syllabus of the Maitrise in pure mathematics.<sup>7</sup>

I have used profitably the notes of this course written out very carefully by Mme. C. Lassalle, assistant in the Faculty of Sciences at Nice, who has also kindly assisted me in checking the proofs. I should like here to express my gratitude to her.

#### NOTES

1. In the end this simply means increasing the number of axioms, an inflation against which only the logicians protest.
2. All these results are proved in my book *Foundations of Modern Analysis* (Academic Press, New York and London, 1960) quoted [FA] in this volume.
3. See [FA], Chapter IX, section 12, problem 5.
4. Stokes' formula has now found its natural place in the new syllabuses, in the certificate C3 of the Maitrise in pure mathematics.
5. Naturally there are some "problems in two dimensions" in the theory of functions of a complex variable and this is what increases the difficulties compared to the elementary theory of functions of one real variable. Nevertheless, as long as one does not begin the study of analytic functions from the point of view of harmonic functions, the theory with its use of "paths" and "loops" remains basically "one-dimensional".
6. Belgian experiences show that with a rational secondary teaching, pupils *well-prepared since the first year* of high school can fruitfully begin the Integral Calculus in the junior year with no psychological hurdles to overcome.
7. In this book I have not dealt with those parts of the syllabus which are concerned with algebra, numerical calculus and the elementary ideas of probability theory.

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# Notations

In the following definitions Roman numerals indicate the Chapter and arabic numerals the section or problem in the Chapter.

$\mathbf{X} - \mathbf{Y}$	set of elements of $\mathbf{X}$ not belonging to $\mathbf{Y}$ : 0, 1
$(a, b)$	pair of two objects: 0, 1
$\mathbf{X} \times \mathbf{Y}$	product of two sets: 0, 1
$\text{pr}_1 c, \text{pr}_2 c$	first and second projection: 0, 1
$\mathbf{X}_1 \times \mathbf{X}_2 \times \cdots \times \mathbf{X}_n$	product of $n$ sets: 0, 1
$\text{pr}_j x$	$j^{\text{th}}$ projection of $x$ : 0, 1
$\mathbf{X}^n$	product of $n$ sets equal to $\mathbf{X}$ : 0, 1
$f _{\mathbf{A}}$	restriction of the function $f$ to the set $\mathbf{A}$ : 0, 1
$f(\mathbf{A})$	image of the set $\mathbf{A}$ under $f$ : 0, 1
$f^{-1}(\mathbf{B})$	inverse image of the set $\mathbf{B}$ under $f$ : 0, 1
$f^{-1}$	inverse function of a bijective mapping $f$ : 0, 1
$\mathbf{R}$	set of real numbers: 0, 2
$\mathbf{C}$	set of complex numbers: 0, 2
$\Re z, \Im z$	real and imaginary parts of $z$ : 0, 2
$[a, b], ]a, b[$	intervals in $\mathbf{R}$ : 0, 2
$[a, b[, ]a, b]$	
$[a, +\infty[, ]a, +\infty[$	unbounded (or infinite) intervals: 0, 2
$] -\infty, a], ] -\infty, a[$	
$\sup \mathbf{A}$	least upper bound of a set $\mathbf{A} \subset \mathbf{R}$ : 0, 2
$\inf \mathbf{A}$	greatest lower bound of a set $\mathbf{A} \subset \mathbf{R}$ : 0, 2
$\sup_{x \in \mathbf{A}} f(x), \inf_{x \in \mathbf{A}} f(x)$	least upper bound and greatest lower bound of the real function $f$ in $\mathbf{A}$ : 0, 3
$f(a+), f(a-)$	right and left limits at the point $a \in \mathbf{R}$ : 0, 4
$f'(t)$	derivative of a vector function: 0, 4
$\int_a^b f(t) dt$	integral of a vector function: 0, 4
$\text{sgn } x$	sign of $x \in \mathbf{R}$ : 0, 4
$d(a, \mathbf{F})$	distance of a point $a \in \mathbf{C}$ from a closed set $\mathbf{F} \subset \mathbf{C}$ : 0, 5

$\ \mathbf{z}\ , \ A\ $	norm of a vector, of a matrix: I, 1
$\sum_{n \in I} u_n$	sum of a partial series: I, 2
$\sum_{n=-\infty}^{+\infty} a_n$	sum of an infinite series in both directions: I, 2
$\sum_{n=0}^{\infty} \mathbf{a}_n$	sum of a series of vectors: I, 2
$N(A)$	norm of a matrix: I, problem 16
$f \sim g, f(x) \sim g(x)$	equivalence of two functions: III, 3
$f = O(g), f = o(g)$	Landau's notations for the comparison of two functions: III, 3
$f \leqslant g, f \ll g,  f  \gg g$	Hardy's notations for the comparison of two functions: III, 3
$\mathcal{O}$	comparison scale: III, 3
$\mathcal{C}$	set of coefficients in the generalized asymptotic developments: III, 7
$\int_a^{+\infty} f(t) dt$	improper integral: III, 9
$\int_a^b f(t) dt, \int_a^b f(t) dt$	improper integral in the neighbourhood of $a$ or of $b$ : III, 9
$\Gamma(x)$	Eulerian integral of the second kind, or gamma function: III, 9
$\gamma$	Euler's constant: III, 11
$B(p, q)$	Eulerian integral of the first kind, or beta function: IV, 3
$d(f, g)$	distance between two functions: V, 1
$f * \varphi$	convolution of two functions: V, 4
$B_n(f)$	Bernstein polynomial: V, Appendix
$f'(z), Df(z), \frac{df}{dz}$	derivative of an analytic function: VI, 6
$e^z, \exp(z)$	complex exponential: VI, 8
$\cos z, \sin z$	cosine and sine of a complex number: VI, 8
$\tan z, \cot z$	tangent and cotangent of a complex number: VI, 8
$e^{Az}, \exp(Az)$	exponential of a matrix: VI, 8
$\gamma^0$	opposite of a path $\gamma$ : VII, 1
$\gamma_1 \vee \gamma_2$	juxtaposition of two paths $\gamma_1, \gamma_2$ : VII, 1
$\int_{\gamma} f(z) dz$	integral of a function along a path $\gamma$ : VII, 2
$\int_{z_0}^z f(u) du$	integral from $z_0$ to $z$ of an analytic function $f$ admitting a primitive: VII, 3
$j(a; \gamma)$	index of a point $a$ with respect to a loop $\gamma$ : VII, 6
$\int_{\gamma} f(z) dz$	integral along a path $\gamma$ without endpoints: VII, 10
$\omega(a; f)$	order of a meromorphic function $f$ at the point $a$ : VIII, 3
$\text{Res}_a f$	residue of $f$ at the isolated singularity $a$ : VIII, 4
$\arg w$	argument (amplitude) of $w$ : VIII, 9
$\log w$	principal determination of the logarithm of $w$ : VIII, 9

$z^\lambda$	principal determination of the $\lambda^{\text{th}}$ power of $z$ : VIII, 9
$\prod_{n=1}^{\infty} a_n$	infinite product: VIII, 11
$\Gamma(z)$	gamma function in the complex domain: IX, 4
$B_n$	Bernoulli number: IX, 5
$\varphi_n(z)$	Bernoulli polynomial: IX, 5
$\tilde{\varphi}_n(x)$	Bernoulli polynomial continued by periodicity: IX, 5
$d_1(f, g), d_p(f, g)$	distance between two functions: IX, 9
$\operatorname{sn} z, \operatorname{cn} z, \operatorname{dn} z$	jacobian elliptic functions: X, 7
$R(t, s)$	resolvent matrix: XII, 2
$H_\lambda^1(z), H_\lambda^2(z)$	Hankel functions: XV, 2
$J_\lambda(z)$	Bessel functions: XV, 4
$N_\lambda(z)$	Neumann functions: XV, 4



# Introduction

It is assumed that the reader is familiar with the basic ideas of Algebra and Analysis that any student should know at the end of his first year. A good summary of these is given in *Formules commentées* by J. Klein and G. Reeb hereafter referred to by the initials K-R.† This preliminary chapter adds a few complementary remarks, a few further comments and a little advice.

## 1. Sets and functions

The handling of the symbols  $\in$ ,  $\notin$ ,  $\subset$ ,  $\not\subset$ ,  $\cup$ ,  $\cap$  of Set Theory is supposed known; if  $Y \subset X$ ,  $X - Y$  is the set of elements of  $X$  which do not belong to  $Y$ . To two objects  $a, b$  we associate a new object, the *pair* (or *ordered pair*)  $(a, b)$ ; two pairs  $(a, b)$  and  $(a', b')$  are equal if and only if  $a = a'$  and  $b = b'$ ; care should be taken to confuse neither the pairs  $(a, b)$  and  $(b, a)$  nor the pair  $(a, a)$  and the object  $a$ . Given two sets  $X, Y$  (distinct or not) their *product*  $X \times Y$  is the set of pairs  $(a, b)$  where  $a \in X$  and  $b \in Y$ . The most frequent example is the plane  $\mathbf{R} \times \mathbf{R}$ , which should always be kept in mind when using products. For each element  $c \in X \times Y$  there is by definition a unique element  $a \in X$  and a unique element  $b \in Y$  such that  $c = (a, b)$ ; we put  $a = \text{pr}_1 c$ ,  $b = \text{pr}_2 c$  and say that they are the *first* and *second projection* respectively of  $c$ . More generally we define for a finite number of sets  $X_1, \dots, X_n$  (distinct or not), the *product*  $X_1 \times X_2 \times \dots \times X_n$  as the set of the *finite sequences of  $n$  elements*  $(x_1, x_2, \dots, x_n)$  where  $x_j \in X_j$  for  $1 \leq j \leq n$ ;  $x_j$  is called the  *$j^{\text{th}}$  projection of  $x = (x_1, x_2, \dots, x_n)$*  and is written  $\text{pr}_j x$ . When all the  $X_j$  are equal to the same set  $X$ , we write  $X^n$  instead of  $X_1 \times X_2 \times \dots \times X_n$ . Most sets occurring in this book will be *subsets of the products  $\mathbf{R}^n$* .

We suppose known the idea of *function* or *mapping*  $f: X \rightarrow Y$  of a set  $X$  into a set  $Y$ ; we shall always bear in mind that a function defined on  $X$  has *only one value*  $f(x)$  for each  $x \in X$ . The *graph* of  $f$  is the set of pairs  $(x, f(x)) \in X \times Y$  where  $x$  takes all values in  $X$ . We shall also write  $x \rightarrow f(x)$  for a function  $f$ . One should get used to considering a function  $f$  as a new object and take care not to confuse a function  $f$  with its values  $f(x)$ .

If  $A$  is a subset of  $X$ , the *restriction to  $A$*  of the function  $f: X \rightarrow Y$  is the function

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† Gauthier-Villars, 1964

defined on  $A$  whose values are  $f(x)$  for each  $x \in A$ ; we write  $f|A$  and one must take care *not to confuse it* with  $f$ : the set on which a function is defined is part of the definition of the function. On the other hand we say that  $f$  is an *extension of*  $f|A$  to  $X$ ; a function may have many extensions.

If  $f: X \rightarrow Y$  is a mapping of  $X$  into  $Y$ , the *image*  $f(A)$  by  $f$  of a subset  $A$  of  $X$  is the set of elements  $f(x)$  where  $x$  describes  $A$ . The *inverse image*  $f^{-1}(B)$  of a subset  $B$  of  $Y$  is the set of all  $x \in X$  such that  $f(x) \in B$ .

For example,  $\text{pr}_1: X \times Y \rightarrow X$  is a mapping of  $X \times Y$  into  $X$ ; for every subset  $C \subset X \times Y$ ,  $\text{pr}_1(C)$  is the set of all  $x \in X$  for which there exists *at least one*  $y \in Y$  such that  $(x, y) \in C$ . For each subset  $A$  of  $X$ ,  $\text{pr}_1^{-1}(A)$  is the set  $A \times Y$ .

A function  $f: X \rightarrow Y$  is said to be *surjective* if  $f(X) = Y$ , *injective* if the relation  $f(x_1) = f(x_2)$  implies  $x_1 = x_2$ , *bijective* if it is both injective and surjective. In this last case there exists one and only one mapping  $f^{-1}: Y \rightarrow X$  such that  $f^{-1}(f(x)) = x$  for every  $x \in X$ ; it is called the *inverse* of  $f$ , and  $f$  is the inverse of  $f^{-1}$ .

If  $f: X \rightarrow Y$ ,  $g: Y \rightarrow Z$  are two mappings, the mapping  $x \rightarrow g(f(x))$  of  $X$  into  $Z$  is called the *composition* of  $g$  and  $f$  and written  $g \circ f$ .

## 2. Real numbers and complex numbers

The real number and the complex number are the two essential tools of Analysis; it is important to be able to handle these without hesitation, and particularly to bear in mind always their *geometrical* significance, notably for the complex numbers the interpretation of  $\Re z$ ,  $\Im z$  (*real part* and *imaginary part* of  $z$ ),  $|z|$ ,  $\bar{z}$ ,  $1/z$ ,  $z + z'$ ,  $zz'$ . The set of real numbers (or *real line*) is written  $\mathbf{R}$ ; there is *no* distinction made between points  $(x, y)$  of the plane  $\mathbf{R} \times \mathbf{R}$  and complex numbers  $x + iy$ ; if we write  $\mathbf{C}$  for the set of complex numbers instead of  $\mathbf{R} \times \mathbf{R}$  or  $\mathbf{R}^2$ , it is to remind ourselves that besides the addition of the vectors in  $\mathbf{R} \times \mathbf{R}$ , a multiplication operation is defined which makes  $\mathbf{C}$  a *field*.

(2.1) On the line  $\mathbf{R}$ , we shall carefully distinguish the four kinds of *intervals*: the *closed* interval  $[a, b]$ , being the set of all  $x$  such that  $a \leq x \leq b$ , the *open* interval  $]a, b[$ , being the set of all  $x$  such that  $a < x < b$  (which is non-empty only if  $a < b$ ), the interval *half-open on the right*  $[a, b[$ , being the set of all  $x$  such that  $a \leq x < b$ , and the interval *half-open on the left*  $]a, b]$ , being the set of all  $x$  such that  $a < x \leq b$ .

We write  $[a, +\infty[$  (resp.  $]a, +\infty[$ ) for the set of all  $x \geq a$  (resp.  $x > a$ ), and  $] -\infty, a]$  (resp.  $] -\infty, a[$ ) for the set of all  $x \leq a$  (resp.  $x < a$ ); they are said to be *infinite closed* (resp. *open*) *intervals*.

(2.2) A set  $A \subset \mathbf{R}$  is said to be *bounded above* (resp. *bounded below*) if there exists at least one  $a \in \mathbf{R}$  such that  $x \leq a$  (resp.  $x \geq a$ ) for all  $x \in A$ ; the numbers having this property are called the *upper bounds* (resp. *lower bounds*) of  $A$ . A fundamental property of  $\mathbf{R}$  is that for every non-empty subset  $A$ , which is bounded above, there is a *least upper bound*  $b$  of  $A$  called the *supremum* of  $A$  and written  $\sup A$ . The number  $b$  may or may not belong to  $A$  and is also characterized by the following two properties:

1. For every  $x \in A$ , we have  $x \leq b$ .
2. For every number  $\varepsilon > 0$ , there exists at least one  $x \in A$  such that  $b - \varepsilon < x \leq b$ .

In the same way a non-empty set  $A$ , which is bounded below in  $\mathbf{R}$ , has a *greatest lower bound* called its *infimum* and written  $\inf A$ . We have  $\inf A = -\sup(-A)$  (where  $-A$  is the image of  $A$  under the symmetry  $x \rightarrow -x$ ).

A set  $A \subset \mathbf{R}$ , which is both bounded above and bounded below, is said to be *bounded*.

(2.3) A set  $U \subset \mathbf{R}$  is said to be *open* if it is a union of open intervals; both  $\mathbf{R}$  and the empty set are open. Every union of open sets is an open set; the intersection of two open sets is an open set.

A set  $F \subset \mathbf{R}$  is said to be *closed* if its complement  $\mathbf{R} - F$  is open;  $\mathbf{R}$  and the empty set are closed, as well as every closed interval (finite or not). Every intersection of closed sets is a closed set; the same is true for the union of two closed sets.

If  $f$  is a *continuous* real function in an open (resp. closed) interval  $I \subset \mathbf{R}$ , the set of all  $x \in I$  such that  $f(x) > 0$  (resp.  $f(x) \geq 0$  or  $f(x) = 0$ ) is *open* (resp. *closed*) in  $\mathbf{R}$ .

If  $F$  is *closed* and *bounded above* (resp. *bounded below*) in  $\mathbf{R}$ , the number  $\sup F$  (resp.  $\inf F$ ) *belongs to*  $F$  and is therefore the *largest* (resp. *smallest*) element of  $F$ . In particular if  $f$  is a continuous real function in a bounded closed interval  $I$ , there is a *smallest root* and a *largest root* of the equation  $f(x) = 0$  in  $I$ .

### 3. Continuous functions of a real variable

A real-valued function  $f$  defined on a set  $X$  is said to be *bounded above* (resp. *bounded below*) if the set  $f(X) \subset \mathbf{R}$  is bounded above (resp. bounded below); the number  $\sup f(X)$  (resp.  $\inf f(X)$ ) is then called the *least upper bound* (resp. *greatest lower bound*) of  $f$  in  $X$  and written  $\sup_{x \in X} f(x)$  (resp.  $\inf_{x \in X} f(x)$ ). The function  $f$  is called *bounded* if it is both bounded above and bounded below; equivalently  $f$  is bounded if the function  $|f|: x \rightarrow |f(x)|$  is bounded above.

The continuous real functions in a *finite closed interval*  $I = [a, b]$  enjoy properties constantly used and of a very intuitive nature.

(3.1) *Every continuous real function in  $I$  is bounded in  $I$ .*

(3.2) *If the real function  $f$  is continuous in  $I$ , there exists at least one point  $x_1 \in I$  such that  $f(x_1) = \sup_{x \in I} f(x)$  and at least one point  $x_2 \in I$  such that  $f(x_2) = \inf_{x \in I} f(x)$  ( $f$  is also said to attain its absolute maximum at at least one point and attain its absolute minimum at at least one point).*

(3.3) *If  $f(a)f(b) < 0$ , there is at least one point  $c \in I$  such that  $f(c) = 0$  (a continuous function cannot change sign without vanishing).*

(3.4) *A real function  $f$  continuous in  $I$  is uniformly continuous, i.e. for each  $\varepsilon > 0$  there exists a  $\delta > 0$ , depending only on  $\varepsilon$ , such that relations  $x' \in I, x'' \in I$  and  $|x' - x''| \leq \delta$  imply  $|f(x') - f(x'')| \leq \varepsilon$ .*

It is convenient to admit these theorems without proof; nevertheless it is instructive to see how one can prove them using exclusively the definition of the continuous functions and the existence of the supremum of a set bounded above in  $\mathbf{R}$ , as below. But

even if one is not interested in these proofs, it is useful to notice that with the exception of (3.3) the above theorems lose their validity when one omits the hypothesis that the interval  $I$  is closed, or the hypothesis that it is finite. Of course there are some continuous functions in  $\mathbf{R}$  which are not bounded, for example  $x$  or  $x^2$ . Since

$$|(x+h)^2 - x^2| = |(2x+h)h| \geq (2x-1)|h|$$

for  $x > 0$  and  $|h| < 1$ , the continuous function  $x \rightarrow x^2$  in  $\mathbf{R}$  is not uniformly continuous (which shows that the two notions are distinct, though at first sight very close ones). The function  $1/(1+x)$  is continuous and bounded in  $[0, +\infty[$  but does not attain its infimum 0 in this interval. Lastly, if one of the theorems (3.1), (3.2) or (3.4) is false for a function  $f$  continuous on the open interval  $]1, +\infty[$ , then the corresponding theorem is false for the function  $x \rightarrow f(1/x)$  in the finite (but not closed) interval  $]0, 1[$ .

*Proof of (3.1)* Let  $A \subset I$  be the set of those points  $c$  such that  $f$  is bounded in the interval  $[a, c]$ ; we have to show that  $A = I$ . The set  $A$  being bounded above has a least upper bound  $\beta$  in  $\mathbf{R}$ , such that  $\beta \leq b$ . It will be proved that  $\beta = b$  and that  $b \in A$ , which will prove the theorem. Now  $f$  is continuous at the point  $\beta$ ; there is therefore an interval  $J = [\beta - h, \beta + h]$  with  $h > 0$  such that

$$|f(x) - f(\beta)| \leq 1$$

in  $I \cap J$ . By definition of  $\beta$ , there is a point  $c \in A$  such that  $\beta - h < c \leq \beta$ ;  $f$  being bounded in  $[a, c]$  and in  $I \cap J$ , is bounded in the intersection of  $I$  with  $[a, \beta + h]$ . If we had  $\beta < b$ , there would be a point  $c'$  such that  $\beta < c' < \beta + h$  and  $\beta < c' < b$ , and  $f$  would be bounded in  $[a, c']$ , in other words we would have  $c' \in A$ . Therefore  $\beta$  would not be an upper bound of  $A$ , contrary to the hypothesis. It follows that  $\beta = b$  and as  $I \cap [a, b + h] = I$ ,  $f$  is bounded in  $I$ .

*Proof of (3.2)* Let us set  $\mu = \sup_{x \in I} f(x)$ ; then  $f(x) \leq \mu$  for all  $x \in I$ . If there is no point  $x \in I$  such that  $f(x) = \mu$  the function  $x \rightarrow 1/(\mu - f(x))$  is continuous in  $I$ , therefore bounded by (3.1). But for all integers  $n$ , there exists by hypothesis  $x \in I$  such that  $\mu - (1/n) < f(x) < \mu$ , and therefore  $1/(\mu - f(x)) > n$ , a contradiction.

*Proof of (3.3)* Suppose for example that  $f(a) < 0$ ,  $f(b) > 0$ , and consider the non-empty set  $A$  of all  $x \in I$  such that  $f(x) < 0$ ; it has a least upper bound  $\beta$ . We cannot have  $f(\beta) < 0$ ; indeed in this case  $\beta < b$ ; since  $f$  is continuous in  $I$ , there would exist an interval  $J = [\beta - h, \beta + h]$  with  $h > 0$  such that  $|f(x) - f(\beta)| \leq \frac{1}{2}|f(\beta)|$  for  $x \in I \cap J$ , and therefore  $f(x) \leq \frac{1}{2}f(\beta) < 0$  in  $I \cap J$ . Now this interval must contain some points  $y > \beta$ , and for such a point  $y \in A$ , so  $\beta$  is not an upper bound of  $A$ , which is absurd. Neither can we have  $f(\beta) > 0$ , since then we would again have  $|f(x) - f(\beta)| \leq \frac{1}{2}f(\beta)$  in an interval  $I \cap J$ , and therefore  $f(x) \geq \frac{1}{2}f(\beta) > 0$  in this interval. But clearly  $\beta > a$ , and so there would exist a point  $y \in A$  such that  $\beta - h < y \leq \beta$ , which is again absurd. The only other possibility is  $f(\beta) = 0$ .

*Proof of (3.4)* Choose  $\varepsilon > 0$  and let  $A$  be the set of points  $c \in I$  such that there exists  $\delta > 0$  (depending on  $\varepsilon$  and  $c$ ) such that the relations  $x' \in [a, c]$ ,  $x'' \in [a, c]$  and  $|x' - x''| \leq \delta$  imply  $|f(x') - f(x'')| \leq \varepsilon$ . Let  $\beta$  be the least upper bound of  $A$ . We shall prove that  $\beta = b$  and  $b \in A$ . Since  $f$  is continuous, there is an interval  $J = [\beta - h, \beta + h]$  with  $h > 0$  such that for  $x \in I \cap J$  we have  $|f(x) - f(\beta)| \leq \frac{1}{2}\varepsilon$ . By hypothesis there is a point  $c \in A$  such that  $\beta - \frac{1}{2}h < c \leq \beta$ . Let  $\delta > 0$  be chosen such that the relations  $a \leq x' \leq x'' \leq c$ ,  $x'' - x' \leq \delta$  imply  $|f(x'') - f(x')| \leq \varepsilon$ . Let  $\delta' = \inf(\delta, \frac{1}{2}h)$  and let us show that for  $x'$  and  $x''$  in  $I \cap [a, \beta + h]$ , satisfying  $x' \leq x''$  and  $x'' - x' \leq \delta'$ , we have again  $|f(x'') - f(x')| \leq \varepsilon$ ;

this is evident if  $x'' \leq c$ . In the contrary case  $x'$  and  $x''$  both belong to  $J$  and therefore  $|f(x') - f(\beta)| \leq \frac{1}{2}\epsilon$ ,  $|f(x'') - f(\beta)| \leq \frac{1}{2}\epsilon$ , hence  $|f(x') - f(x'')| \leq \epsilon$ . Thus  $\beta \in A$ , and if  $\beta < b$ , there would be points  $c \in A$  such that  $\beta < c < \beta + h$ , and  $\beta$  would therefore not be an upper bound of  $A$ , which is absurd.

## t. Extensions of the concepts of derivative and primitive

(4.1) In Analysis one should become as familiar with the handling of vector functions of a real variable with values in any space  $\mathbf{R}^n$  as with functions with real values, especially for the calculus of their derivatives and primitives. A function  $t \rightarrow \mathbf{f}(t)$  with values in  $\mathbf{R}^n$  is *differentiable* at a point  $t_0$  if the components  $f_j$  of  $\mathbf{f}$ , for  $1 \leq j \leq n$ , are differentiable at  $t_0$  and the derivative  $\mathbf{f}'(t_0)$  is the vector with components  $f'_j(t_0)$  ( $1 \leq j \leq n$ ). This is applied in particular to the *matrix* function  $t \rightarrow A(t)$ , a square matrix  $A(t)$  of order  $n$  being a vector of  $\mathbf{R}^{n^2}$ . From these definitions and from the formula giving the derivative of a product the following formulae arise:

$$(4.1.1) \quad \frac{d}{dt}(A(t) \cdot \mathbf{f}(t)) = A'(t) \cdot \mathbf{f}(t) + A(t) \cdot \mathbf{f}'(t)$$

$$(4.1.2) \quad \frac{d}{dt}(A(t) \cdot B(t)) = A'(t)B(t) + A(t)B'(t)$$

where  $\mathbf{f}$  is a vector function taking values in  $\mathbf{R}^n$ ,  $A$  and  $B$  are matrix functions taking values in the space of square matrices of order  $n$ . (In (4.1.2) care must be taken to respect the *order* of the factors in the second expression.) Similarly, we shall constantly work with the complex functions of a real variable; if  $f$  and  $g$  are two complex differentiable functions in  $I$ , we have again the formula for the differentiation of a product

$$(4.1.3) \quad \frac{d}{dt}(f(t)g(t)) = f'(t)g(t) + f(t)g'(t).$$

From (4.1.3) it follows that formulae (4.1.1) and (4.1.2) are still valid when  $\mathbf{f}$  takes values in  $\mathbf{C}^n$ ,  $A$  and  $B$  being in the space of matrices of order  $n$  with *complex* elements.

(4.2) We define similarly the integral  $\int_a^b \mathbf{f}(t) dt$  of a vector function  $\mathbf{f} = (f_j)$  continuous in a finite closed interval  $[a, b]$  taking values in  $\mathbf{R}^n$ , as the vector of  $\mathbf{R}^n$  whose components are the numbers  $\int_a^b f_j(t) dt$ . As  $x$  varies in  $[a, b]$ , the vector function  $\mathbf{F}(x) = \int_a^x \mathbf{f}(t) dt$  therefore has the derivative  $\mathbf{f}(x)$  at every point  $x$  such that  $a < x < b$ . At the point  $a$  it is more correct to say that  $\mathbf{F}$  admits a *derivative on the right*

$$\lim_{h \rightarrow 0, h > 0} \frac{\mathbf{F}(a+h) - \mathbf{F}(a)}{h}$$

equal to  $\mathbf{f}(a)$ ; similarly at the point  $b$ ,  $\mathbf{F}$  admits a derivative on the left

$$\lim_{h \rightarrow 0, h > 0} \frac{\mathbf{F}(b-h) - \mathbf{F}(b)}{h}$$

equal to  $\mathbf{f}(b)$ . We also say that  $\mathbf{F}$  is a *primitive* of  $\mathbf{f}$

(4.3) It is useful to define the integral for more general functions than the continuous functions. We shall say that in a finite closed interval  $I = [a, b]$  of  $\mathbf{R}$ , a vector function  $\mathbf{f}$  taking values in a space  $\mathbf{R}^n$  is *piecewise-continuous* in  $I$ , if we can decompose  $I$  into a finite number of intervals

$$[a_0, a_1], [a_1, a_2], \dots, [a_{m-1}, a_m]$$

with  $a = a_0 < a_1 < a_2 < \dots < a_{m-1} < a_m = b$ , such that  $\mathbf{f}$  is *continuous* at every point  $x$  interior to one of these intervals, and also such that the limits on the right  $\mathbf{f}(a_k+) = \lim_{x \rightarrow a_k, x > a_k} \mathbf{f}(x)$  exist for  $0 \leq k \leq m-1$ , and the limits on the left  $\mathbf{f}(a_k-) = \lim_{x \rightarrow a_k, x < a_k} \mathbf{f}(x)$  for  $1 \leq k \leq m$ . No condition whatever is imposed on the values of  $\mathbf{f}$  at the points of discontinuity. For example the function  $\text{sgn } x$  ("sign of  $x$ ") equal to  $-1$  for  $x < 0$ , to  $0$  for  $x = 0$ , to  $1$  for  $x > 0$  is piecewise-continuous. The integral  $\int_a^b \mathbf{f}(t) dt$  of such a function

is then defined as equal to  $\sum_{k=0}^{m-1} \int_{a_k}^{a_{k+1}} \mathbf{f}_k(t) dt$  where in the closed interval  $[a_k, a_{k+1}]$   $\mathbf{f}_k(t)$  is taken equal to  $\mathbf{f}(t)$  if  $a_k < t < a_{k+1}$ , to  $\mathbf{f}(a_k+)$  for  $t = a_k$ , to  $\mathbf{f}(a_{k+1}-)$  for  $t = a_{k+1}$ , and so  $\mathbf{f}_k$  is continuous in this finite closed interval. When we refer to the integral of a function in a *finite closed* interval of  $\mathbf{R}$ , it will always be understood that the function is *piecewise-continuous* in this interval. Note that the integral of such a function  $\mathbf{f}$  *does not depend* on the values of  $\mathbf{f}$  at the points of discontinuity.

(4.4) Let  $\mathbf{f}$  be a piecewise-continuous vector function in  $[a, b]$ . The function

$$(4.4.1) \quad \mathbf{F}(x) = \int_a^x \mathbf{f}(t) dt$$

is then continuous in  $[a, b]$  and has a derivative equal to  $\mathbf{f}(x)$  at every point  $x$  not one of the points of discontinuity of  $\mathbf{f}$ . At a point  $c$  of discontinuity of  $\mathbf{f}$  interior to  $[a, b]$ ,  $\mathbf{F}$  has a derivative on the right equal to  $\mathbf{f}(c+)$  and a derivative on the left equal to  $\mathbf{f}(c-)$ . Finally, at the point  $a$ ,  $\mathbf{F}$  has a derivative on the right equal to  $\mathbf{f}(a+)$  and at the point  $b$ ,  $\mathbf{F}$  has a derivative on the left equal to  $\mathbf{f}(b-)$ . We also say that  $\mathbf{F}$  is a *primitive* of  $\mathbf{f}$ , as also for all functions of the form  $\mathbf{F}(x) + \mathbf{A}$ , where  $\mathbf{A}$  is a constant vector.

(4.5) Under the same hypotheses one defines

$$\int_b^a \mathbf{f}(t) dt = - \int_a^b \mathbf{f}(t) dt = \mathbf{F}(a) - \mathbf{F}(b).$$

The usual rules of the Integral Calculus apply

$$(4.5.1) \quad \int_a^b \mathbf{f}(t) dt + \int_b^c \mathbf{f}(t) dt + \int_c^a \mathbf{f}(t) dt = 0$$

whatever the values of  $a, b, c$ ,

$$(4.5.2) \quad \int_a^b (\mathbf{f}(t) + \mathbf{g}(t)) dt = \int_a^b \mathbf{f}(t) dt + \int_a^b \mathbf{g}(t) dt$$

$$(4.5.3) \quad \int_a^b \lambda \mathbf{f}(t) dt = \lambda \int_a^b \mathbf{f}(t) dt$$

for every scalar  $\lambda \in \mathbf{R}$ , a relation which generalizes to

$$(4.5.4) \quad \int_a^b (A \cdot \mathbf{f}(t)) \, dt = A \cdot \int_a^b \mathbf{f}(t) \, dt$$

for every matrix  $A$  of order  $n$  with real elements (*constant* in  $[a, b]$ ). Moreover, if  $\mathbf{f}$  takes its values in  $\mathbf{C}^n$ , we again have (4.5.3) for a constant *complex* scalar  $\lambda$  and (4.5.4) for a constant matrix having *complex* elements. Let  $\varphi$  be a numerical piecewise-continuous function in  $[a, b]$  satisfying  $\varphi(x) > 0$  except at a finite number of points. Then the function  $\Phi(x) = c + \int_a^x \varphi(t) \, dt$  is *strictly increasing* in  $[a, b]$  (K-R, p. 62). For each piecewise continuous vector function  $\mathbf{f}$  in the interval  $[\Phi(a), \Phi(b)]$  we then have the *formula for changing the variable*

$$(4.5.5) \quad \int_{\Phi(a)}^{\Phi(b)} \mathbf{f}(u) \, du = \int_a^b \mathbf{f}(\Phi(t)) \varphi(t) \, dt$$

as seen when decomposing the interval  $[a, b]$  into a finite number of intervals, in each of which  $\mathbf{f}(\Phi(t))$  and  $\varphi(t)$  are continuous and applying to each of these intervals the classical formula (K-R, p. 82).

Lastly, if  $u$  and  $v$  are primitives of real piecewise-continuous functions  $u'$  and  $v'$  in  $[a, b]$ , we have the usual formula for *integrating by parts*

$$(4.5.6) \quad \int_a^b u(t)v'(t) \, dt = u(b)v(b) - u(a)v(a) - \int_a^b u'(t)v(t) \, dt$$

as can again be seen by decomposition of the interval  $[a, b]$  into subintervals, where  $u'$  and  $v'$  are continuous. This formula is also valid (by (4.1.3)) when the functions  $u$  and  $v$  have *complex* values. There are similar formulae for the functions having vector values or matrix values, corresponding to (4.1.1) and (4.1.2),

$$(4.5.7) \quad \int_a^b A(t) \cdot \mathbf{f}'(t) \, dt = A(b) \cdot \mathbf{f}(b) - A(a) \cdot \mathbf{f}(a) - \int_a^b A'(t) \cdot \mathbf{f}(t) \, dt,$$

$$(4.5.8) \quad \int_a^b A(t)B'(t) \, dt = A(b)B(b) - A(a)B(a) - \int_a^b A'(t)B(t) \, dt.$$

The importance in Analysis of the preceding formulae cannot be exaggerated, as we shall notice repeatedly; it is important to become used to handling them without hesitation and in an almost mechanical way.

(4.6) Finally we shall often consider functions defined as *integrals depending on a parameter*

$$(4.6.1) \quad \mathbf{I}(t) = \int_a^b \mathbf{F}(x, t) \, dx$$

where  $\mathbf{F}$  is a function of the two real variables  $x, t$ ;  $x$  varying in a finite closed interval  $[a, b]$  of  $\mathbf{R}$ ,  $t$  in any open interval  $J$  of  $\mathbf{R}$ .  $\mathbf{F}$  may be a vector function and  $x \rightarrow \mathbf{F}(x, t)$  is assumed piecewise-continuous for each  $t \in J$ . The following two theorems are stated without proof:

(4.6.2) If the function of two real variables  $(x, t) \rightarrow \mathbf{F}(x, t)$  is continuous in the product  $[a, b] \times J \subset \mathbf{R}^2$ , then  $\mathbf{I}(t)$  is continuous in  $J$ .

(4.6.3) If the partial derivative  $(\partial \mathbf{F} / \partial t)(x, t)$  exists for each point of  $[a, b] \times J$  and is continuous in this product, then  $\mathbf{I}(t)$  admits a continuous derivative in  $J$ , given by *Leibniz's Formula* for "differentiation under the integral sign"

$$(4.6.4) \quad \mathbf{I}'(t) = \int_a^b \frac{\partial \mathbf{F}}{\partial t}(x, t) dx.$$

These theorems can be generalized to the case of several parameters: if

$$\mathbf{I}(t, s) = \int_a^b \mathbf{F}(x, t, s) dx$$

where the function of three variables  $(x, t, s) \rightarrow \mathbf{F}(x, t, s)$  is continuous, then  $(t, s) \rightarrow \mathbf{I}(t, s)$  is continuous in *both* variables.

## 5. Topology of the plane

We shall often consider functions defined on subsets of the vector space  $\mathbf{R}^n$  or taking their values in such a space. It will be necessary to use a certain number of properties of these spaces, called topological properties, assumed without proof, and which we shall first state in the case of the plane  $\mathbf{R}^2$ . It will be convenient (notably for the applications from Chap. VI onwards) to consider the points of  $\mathbf{R}^2$  as complex numbers (i.e. to speak of  $\mathbf{C}$  rather than  $\mathbf{R}^2$ ).

(5.1) Let us recall that for  $a \in \mathbf{C}$  and for all  $r > 0$ , the set of points  $z = x + iy \in \mathbf{C}$  such that  $|z - a| = r$  is a *circle* of center  $a$  and radius  $r$ . The set of all  $z$  such that  $|z - a| < r$  is called the *open disc* of center  $a$  and radius  $r$ ; the set of all  $z$  such that  $|z - a| \leq r$  is called the *closed disc* of centre  $a$  and radius  $r$ . A set  $A \subset \mathbf{C}$  is called *bounded* if it is contained in a closed disc; this is equivalent to saying that the set of all points  $|z|$  where  $z$  describes  $A$  is bounded above in  $\mathbf{R}$ .

(5.2) A set  $U \subset \mathbf{C}$  is called *open* if for each  $a \in U$  there exists  $r > 0$  such that the open disc  $|z - a| < r$  is *contained in*  $U$ . The whole plane  $\mathbf{C}$  is open, as well as the empty set (since this latter set contains no points, a definition which concerns a property of the points of the set is automatically satisfied). An open disc is an open set, as well as the *half-plane*  $\Re z > a$  or  $\Im z > a$ , or  $\Re z < a$  or  $\Im z < a$ ; an open annulus  $r_1 < |z - a| < r_2$  ( $0 \leq r_1 < r_2$ ) is an open set. More generally, if  $f$  is continuous in  $\mathbf{C}$  with values in  $\mathbf{R}$ , the set of all  $z \in \mathbf{C}$  such that  $f(z) > 0$  is open. Every union of open sets is an open set. Every intersection of a finite number of open sets is an open set.

(5.3) A set  $F \subset \mathbf{C}$  is called *closed* if its complement  $\mathbf{C} - F$  is open. The whole plane  $\mathbf{C}$  is closed as well as the empty set. A closed disc is a closed set; if  $f$  is a continuous function in  $\mathbf{C}$  with real values, the set of all  $z \in \mathbf{C}$  such that  $f(z) \geq 0$  (or the set of all  $z \in \mathbf{C}$  such that  $f(z) = 0$ ) is closed. Every intersection of closed sets is a closed set; every finite union of closed sets is a closed set.

(5.4) Let us recall that given a sequence  $(z_n)$  of points of  $\mathbf{C}$ , we say that the sequence has the *limit*  $a \in \mathbf{C}$  (or *converges to*  $a$ ) if the sequence of the distances  $|z_n - a|$  tends to 0. A closed set  $F \subset \mathbf{C}$  can also be defined by the following condition: for each sequence of points  $z_n$  belonging to  $F$  which has a limit  $a$  in  $\mathbf{C}$  we have  $a \in F$ . If  $F$  is a closed set non-empty and distinct from  $\mathbf{C}$ , for each point  $a \in \mathbf{C} - F$  we write  $d(a, F) = \inf_{z \in F} |z - a|$ , which is  $> 0$ , and call  $d(a, F)$  the *distance from*  $a$  to  $F$ . It is the radius of the largest open disc of center  $a$  contained in the open set  $\mathbf{C} - F$ . There is always at least one point  $z_0 \in F$  such that  $|z_0 - a| = d(a, F)$  (the “nearest” point to  $a$ ). There may be an infinity of such points.

(5.5) If  $U$  is an open set in  $\mathbf{C}$ , non-empty and distinct from  $\mathbf{C}$ , there is at least one *boundary point* of  $U$ , namely a point  $a \notin U$  which is the limit of a sequence of points in  $U$ . The set of these points is called the *boundary of*  $U$  and is closed in  $\mathbf{C}$ . For example if  $U$  is the complement of a finite set  $F$ , then  $F$  is the boundary of  $U$ . The boundary of the open disc  $|z - a| < r$  is the circle  $|z - a| = r$ ; it is also the boundary of the open set of all  $z \in \mathbf{C}$  such that  $|z - a| > r$ , called the *exterior* of the disc  $|z - a| < r$ .

Given a boundary point  $a$  of an open set  $U$ , and a vector function  $f$  defined in  $U$  (but *not* at the point  $a$ )  $f(z)$  is said to *tend to the limit*  $c$  when  $z \in U$  tends to  $a$ , if the sequence  $f(z_n)$  tends to  $c$  for each sequence  $(z_n)$  of points in  $U$  tending to  $a$ . This notion of limit is more subtle than for functions of real variables. When for instance  $U = \mathbf{C} - \{0\}$ ,  $a = 0$ , it is not sufficient that the restriction of  $f$  to  $L \cap U$ , for each half-line  $L$  with end-point at 0, has a limit at 0, for it to be true that  $f$  has a limit at 0. For example if  $f(z) = z/|z|$ , for each half-line  $L: t \rightarrow t e^{i\theta}$  ( $t \geq 0$ ) we have  $f(t e^{i\theta}) = e^{i\theta}$ , so there is a limit for the restriction of  $f$  to  $L \cap U$ , but this limit *depends on*  $L$ , so there is no limit for  $f$  at the point 0.

Note that to say a function  $f$  defined in an open set  $U \subset \mathbf{C}$  is continuous at a point  $a \in U$ , means that the limit of  $f(z)$  as  $z$  tends to  $a$ , while remaining in  $U - \{a\}$ , exists and is equal to  $f(a)$ .

If  $F$  is a closed set in  $\mathbf{C}$ , the *boundary points* of the open set  $\mathbf{C} - F$  are by definition the boundary points of  $F$ ; they belong to  $F$ . The points of  $F$  which are not boundary points are called *interior points* of  $F$ : such a point  $a$  is characterized by the property that there is an open disc  $|z - a| < r$  ( $r > 0$ ) contained in  $F$ .

(5.6) The bounded closed sets in  $\mathbf{C}$  (also called *compact sets*) play an important role because of their special properties. If such a set  $K$  is contained in an open set  $U$ , then the greatest lower bound in  $K$  of the distances  $d(z, \mathbf{C} - U)$  is *strictly positive*; i.e. there exists a number  $\alpha > 0$  such that for each point  $z \in K$  the disc of centre  $z$  and radius  $\alpha$  is contained in  $U$ . This is not true for unbounded closed sets, as seen for instance from the closed set  $F$  defined by the relations  $x > 0$ ,  $xy = 1$  and the open set  $y > 0$  which contains it.

A real function  $f$  continuous on a compact set  $K \subset \mathbf{C}$  has properties generalizing (3.1), (3.2) and (3.4):  $f$  is bounded in  $K$  and attains there its absolute minimum and its absolute maximum; moreover  $f$  is *uniformly continuous* in  $K$ , i.e. for each  $\varepsilon > 0$  there exists a number  $\delta > 0$  depending only on  $\varepsilon$ , and such that the relations  $z' \in K$ ,  $z'' \in K$  and  $|z' - z''| \leq \delta$  imply  $|f(z') - f(z'')| \leq \varepsilon$ .

Another property concerns the *complex* continuous functions on the compact set  $K$ : for such a function  $f$ , the set  $f(K)$  is again a compact set in  $\mathbf{C}$ .

(5.7) An *affine linear function* on  $\mathbf{R}$  with values in a vector space  $E$  is a function defined on  $\mathbf{R}$  and of the form  $\sigma: t \mapsto at + b$ , where  $a$  and  $b$  are vectors of  $E$ . The image by such a function of a bounded closed interval  $[\alpha, \beta]$  of  $\mathbf{R}$  is called a *closed segment* in  $E$ ,  $\sigma(\alpha)$  and  $\sigma(\beta)$  the *endpoints* of the segment.

A *piecewise affine linear function* with values in  $E$  is a function  $l$  defined in a bounded closed interval  $[\alpha, \beta]$  of  $\mathbf{R}$  and having the following property: there exist a finite number of points

$$\alpha_0 = \alpha < \alpha_1 < \alpha_2 < \cdots < \alpha_{n-1} < \alpha_n = \beta$$

such that in each of the intervals  $[\alpha_k, \alpha_{k+1}]$  ( $0 \leq k \leq n-1$ ),  $l$  coincides with an affine linear function. The image of  $[\alpha, \beta]$  by  $l$  is called a *polygon* in  $E$ .

(5.8) An open set  $U \subset \mathbf{C}$  is said to be *connected* if for each pair of points  $a$  and  $b$  of  $U$ , there exists a piecewise affine linear function  $l$  defined in an interval  $[\alpha, \beta]$  of  $\mathbf{R}$  and with values in  $U$ , such that  $l(\alpha) = a$ ,  $l(\beta) = b$ ; we also say that any two points in  $U$  can be *joined by a polygon contained in  $U$* . One can show that an equivalent definition is the following:  $U$  is connected if it is not possible to write  $U$  as a union of two *open, non-empty, disjoint sets*  $U_1$  and  $U_2$ . For example, the open set of all  $z \in \mathbf{C}$  such that  $\Re z \neq 0$  is not connected, for it is the union of the non-empty, disjoint, open sets  $\Re z > 0$ ,  $\Re z < 0$ . On the other hand the plane  $\mathbf{C}$  itself, an open disc, the exterior of a disc and an open annulus are connected sets. A union of open connected sets all having a common point is connected; but the intersection of two open connected sets is not necessarily connected.

(5.9) A vector function  $f$  defined in an open set  $U \subset \mathbf{C}$  is said to be *locally constant* if, for each  $z_0 \in U$ , there exists a number  $\delta > 0$  such that the disc  $|z - z_0| < \delta$  is contained in  $U$  and such that  $f$  is constant in this disc. In an open, *connected* set  $U \subset \mathbf{C}$  a locally constant function  $f$  is *constant*. Let  $l: [\alpha, \beta] \rightarrow U$  be a piecewise linear function; it is sufficient with the notations of (5.7) to show that

$$f(l(\alpha_{k+1})) = f(l(\alpha_k)).$$

But in the interval  $[\alpha_k, \alpha_{k+1}]$  the function  $t \mapsto f(l(t))$  is continuous and has zero derivative, so is constant, which proves our assertion.

(5.10) It is easy to extend to the spaces  $\mathbf{R}^n$  or  $\mathbf{C}^n$  the definitions and results given above for  $\mathbf{C}$ ; these extensions will be used very little, and are left to the reader.

# Majorant, minorant

## Elementary operations

(1.1) To *majorize* a real unknown number  $x$  is to find a known number  $b$  such that we can show that  $x \leq b$ ; such a number  $b$  is called a *majorant* (or *upper bound*) of  $x$ . Similarly, to *minorize*  $x$  is to find a known number  $c$  such that  $c \leq x$  and  $c$  is called a *minorant* (or *lower bound*) of  $x$ . A majorant of  $x$  is a minorant of  $-x$ , and conversely. Very often we shall majorize and minorize  $x$  simultaneously, by enclosing  $x$  in a known interval. This is equivalent to majorizing the *absolute value*  $|x - a|$  where  $a$  is the midpoint of the interval:

$$|x - a| \leq r.$$

(1.2) For the *complex numbers* we majorize or minorize their *real parts*, their *imaginary parts* and most often their *absolute values*; in other words, the complex numbers enter into the calculus of majorization or minorization only by the intermediary of *real functions* of these numbers; remember that an inequality between complex numbers is *meaningless*.

(1.3) The rules of the calculus of majorization and minorization are simple translations of the rules for elementary inequalities among real numbers. From the inequalities

$$(1.3.1) \quad \begin{cases} c \leq x \leq b \\ c' \leq x' \leq b' \end{cases}$$

between real numbers, it follows that

$$(1.3.2) \quad c + c' \leq x + x' \leq b + b'$$

$$(1.3.3) \quad c - b' \leq x - x' \leq b - c'.$$

In other words:

*To majorize or minorize a sum of two real numbers, majorize (or minorize) each of the terms. To majorize a difference  $x - x'$ , majorize  $x$  and minorize  $x'$ ; to minorize  $x - x'$ , minorize  $x$  and majorize  $x'$ .*

Similarly, when it is a matter of strictly positive numbers, from the inequalities

$$(1.3.4) \quad \begin{cases} 0 < c \leq x \leq b \\ 0 < c' \leq x' \leq b' \end{cases}$$

we have

$$(1.3.5) \quad 0 < cc' \leq xx' \leq bb'$$

$$(1.3.6) \quad 0 < c/b' \leq x/x' \leq b/c'.$$

In other words:

To majorize (or minorize) a product of two strictly positive numbers, majorize (minorize) each of the factors. To majorize a quotient  $x/x'$  of two strictly positive numbers, majorize the numerator and minorize the denominator; to minorize  $x/x'$  minorize the numerator and majorize the denominator.

(1.4) For the majorization and minorization of the *absolute values* of complex numbers it must be borne in mind that sum and difference play the same role for complex numbers although not for positive numbers. The previous rules for the sum must therefore be modified for the complex numbers as follows: From the inequalities

$$(1.4.1) \quad 0 < c \leq |z| \leq b, \quad 0 < c' \leq |z'| \leq b'$$

we have

$$(1.4.2) \quad \max(0, c - b', c' - b) \leq |z \pm z'| \leq b + b';$$

the inequality on the left being of interest only if one of the two numbers  $c - b'$ ,  $c' - b$  is *strictly positive* (in geometrical terms, each of the inequalities (1.4.1) means that  $z$  (resp.  $z'$ ) is situated in a circular annulus of centre 0, and we can minorize the distance between  $z$  and  $z'$  by a number  $> 0$  only if these annuli fail to intersect) (Fig. 1).

For the products and quotients of complex numbers, on the other hand, the rules of

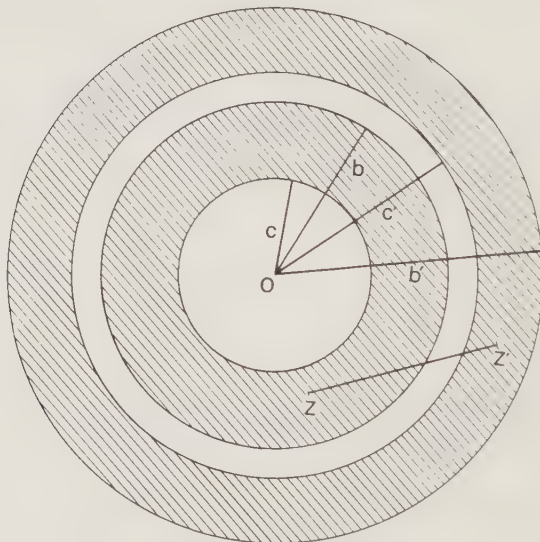


FIGURE 1

(1.3) require no modification, on account of the formula  $|zz'| = |z| \cdot |z'|$ ; in other words from the inequalities (1.4.1), we have

$$(1.4.3) \quad 0 < cc' \leq |zz'| \leq bb'$$

$$(1.4.4) \quad 0 < c/b' \leq |z/z'| \leq b/c'.$$

(1.5) We shall constantly have to apply these rules in succession; for example, for four complex numbers  $x, y, z, t$ ,

$$\left| \frac{x+y}{z+t} \right| \leq \frac{|x|+|y|}{|z|-|t|}$$

provided that  $|z| > |t|$ . For absolute values minorization is always more difficult than majorization: for instance, to minorize usefully the absolute value of a sum  $|x+y+z|$  of three complex numbers we must already have some idea of the “orders of magnitude” of  $|x|, |y|, |z|$ .

As an example of the repeated application of majorization, we quote the *rule of addition of errors*, which is very often used. From the inequalities

$$|z_1 - a_1| \leq \varepsilon_1, \quad |z_2 - a_2| \leq \varepsilon_2, \quad \dots, \quad |z_n - a_n| \leq \varepsilon_n$$

where the  $a_k$  and  $z_k$  are complex, it follows that

$$|(z_1 + z_2 + \dots + z_n) - (a_1 + a_2 + \dots + a_n)| \leq \varepsilon_1 + \varepsilon_2 + \dots + \varepsilon_n.$$

(1.6) For a “*complex vector*”, or system of  $n$  complex numbers

$$\mathbf{z} = (z_1, z_2, \dots, z_n) \in \mathbf{C}^n$$

we set

$$(1.6.1) \quad \|\mathbf{z}\| = \sup (|z_1|, |z_2|, \dots, |z_n|)$$

a non-negative number called the *norm* of  $\mathbf{z}$ . It follows from this definition that for two complex vectors  $\mathbf{z}', \mathbf{z}''$

$$(1.6.2) \quad \|\mathbf{z}' \pm \mathbf{z}''\| \leq \|\mathbf{z}'\| + \|\mathbf{z}''\|$$

and for every complex number  $t$

$$(1.6.3) \quad \|t\mathbf{z}\| = |t| \cdot \|\mathbf{z}\|.$$

For a *matrix* of order  $n$ ,  $A = (a_{jk})$ , with complex elements, considered as a vector of  $\mathbf{C}^{n^2}$ , we have by definition,  $\|A\| = \sup_{j,k} (|a_{jk}|)$ . It follows that for every vector  $\mathbf{z} \in \mathbf{C}^n$  and every square matrix  $B$  of order  $n$ , we have the majorizations

$$(1.6.4) \quad \|A \cdot \mathbf{z}\| \leq n \|A\| \cdot \|\mathbf{z}\|,$$

$$(1.6.5) \quad \|AB\| \leq n \|A\| \cdot \|B\|.$$

The majorizations of norms of complex vectors immediately reduce to majorizations of absolute values of complex numbers.

The previous definition, in particular, applies to real vectors

$$\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbf{R}^n \subset \mathbf{C}^n.$$

(1.7) Consider real *functions* defined in any set  $E$ ; we say that a function  $f$  *majorizes* a function  $g$  (or that  $g$  *minorizes*  $f$ ) if  $g(x) \leq f(x)$  for any  $x \in E$ . Finding majorizations or minorizations for functions then reduces to the corresponding problems for the values of the functions considered at *each point* of  $E$ . We shall often have to decompose  $E$  into several subsets and to majorize (or minorize) a given function by *different* functions in the different parts of  $E$  considered.

(1.8) The majorizations and minorizations of functions allow us to “solve” inequalities of the form

$$(1.8.1) \quad f(x) \leq A$$

( $f$  a real function,  $A$  constant) in the following sense: we do not try to determine the *whole* set  $F$  of all  $x \in E$  satisfying the inequality (1.8.1), but only to prove that it contains a non-empty subset  $G$  easy to determine. To do this look for a *majorization*  $f(x) \leq g(x)$  of  $f$  such that the set  $G$  of all  $x \in E$  such that  $g(x) \leq A$  can be determined exactly. It is then clear that  $F \supset G$ .

## 2. Series and limits

(2.1) At the very elementary level of this book we shall mostly have to consider series (of complex numbers or real numbers) which are *absolutely convergent*. The study of convergent series which are not absolutely convergent is often a difficult problem for which there are few general methods of attack. The study of the absolute convergence of a series  $z_1 + z_2 + \cdots + z_n + \cdots$  is by definition the study of the convergence of the series

$$|z_1| + |z_2| + \cdots + |z_n| + \cdots$$

with *non-negative* terms, and for these we shall always eventually apply the unique *Comparison Principle*:

(2.2) If  $u_n, v_n$  are the general terms of two series with non-negative terms, and if  $0 \leq u_n \leq v_n$  for all  $n$  (or only for  $n \geq n_0$ ), then if the series

$$v_1 + v_2 + \cdots + v_n + \cdots$$

is convergent, so is the series  $u_1 + u_2 + \cdots + u_n + \cdots$  and

$$(2.2.1) \quad u_1 + u_2 + \cdots + u_n + \cdots \leq v_1 + v_2 + \cdots + v_n + \cdots$$

If the series of general term  $u_n$  is divergent, so also is the series of general term  $v_n$ .

Take care *never* to apply this principle when the terms of the considered series are not  $\geq 0$  from a certain term onwards.

(2.3) The study of the convergence of a series with non-negative terms (or the absolute convergence of a series with complex terms) amounts, then, to a problem of *majorization* for its general term. Chap. III shows how to treat this problem in most cases that are met in practice. For the sums of absolutely convergent series use the rules of majorization and minorization

$$(2.3.1) \quad \left| \sum_{n=1}^N u_n \right| - \sum_{n=N+1}^{\infty} |u_n| \leq \left| \sum_{n=1}^{\infty} u_n \right| \leq \sum_{n=1}^{\infty} |u_n| \quad \text{for all } N.$$

(2.4) The interest of absolutely convergent series is that for such a series with general term  $u_n$ , any series obtained by *arbitrarily changing the order of the terms is still absolutely convergent*, and has the *same sum* as the given series. To be precise, this means that for each *bijective* transformation  $\sigma$  of the set of integers  $\geq 1$  onto itself, if  $v_n = u_{\sigma(n)}$ , then the series with general term  $v_n$  is absolutely convergent, and  $s' = \sum_{n=1}^{\infty} v_n$  is *equal to*  $s = \sum_{n=1}^{\infty} u_n$ .

If  $m$  is the largest of the integers  $\sigma(1), \sigma(2), \dots, \sigma(n)$ , we have by definition

$$|v_1| + |v_2| + \dots + |v_n| \leq |u_1| + |u_2| + \dots + |u_m| \leq \sum_{n=1}^{\infty} |u_n|$$

and from this the absolute convergence of the series with general term  $v_n$  follows immediately.

Let us set  $s_n = \sum_{k=1}^n u_k$ ,  $s'_n = \sum_{k=1}^n v_k$ . For each  $\varepsilon > 0$ , there exists an integer  $n_0$  such that for  $n \geq n_0$  we have  $|u_{n+1}| + \dots + |u_{n+p}| \leq \varepsilon$  for every  $p \geq 1$ . Let  $m_0$  be the largest of the integers  $\sigma^{-1}(1), \dots, \sigma^{-1}(n_0)$ ; then if  $n \geq m_0$ , we have  $\sigma(n) \geq n_0$ , and deduce from the preceding and the definition of  $v_n$  that for  $n \geq m_0$ , we have  $|v_{n+1}| + \dots + |v_{n+p}| \leq \varepsilon$  for every  $p \geq 1$ . On the other hand the difference  $s'_{m_0} - s_{n_0}$  is equal to the sum of all  $v_k$  such that  $1 \leq k \leq m_0$  and that  $k$  is not of the form  $\sigma^{-1}(h)$  with  $h \leq n_0$ . This sum is therefore the sum of a certain number of terms  $u_n$  for each of which  $h \geq n_0$ ; we have then  $|s'_{m_0} - s_{n_0}| \leq \varepsilon$ . As seen above, for  $n \geq n_0$  and  $n \geq m_0$  we have  $|s_n - s_{n_0}| \leq \varepsilon$  and  $|s'_n - s'_{m_0}| \leq \varepsilon$ , therefore  $|s'_n - s_n| \leq 3\varepsilon$ , and in the limit  $|s' - s| \leq 3\varepsilon$ , hence the conclusion, since  $\varepsilon$  is arbitrary.

Note carefully that the previous proposition is *no longer valid* if the series is convergent but *not absolutely convergent* (problem 1).

(2.5) Let  $(n_k)$  be an infinite strictly increasing sequence of integers  $\geq 1$ ; for each series with general term  $u_n$  call the series with general term  $v_k = u_{n_k}$  a *partial series* of the considered series corresponding to the *partial sequence* (or *subsequence*)  $(n_k)$ . In general a partial series of a convergent series is *not* necessarily itself convergent; e.g. consider the series  $\sum_{n=1}^{\infty} (-1)^n/n$ , where the partial series of the terms of even order is not convergent.

On the other hand, for an *absolutely convergent* series with general term  $u_n$ , *every partial series is absolutely convergent*, and

$$(2.5.1) \quad \left| \sum_{k=1}^{\infty} u_{n_k} \right| \leq \sum_{k=1}^{\infty} |u_{n_k}| \leq \sum_{n=1}^{\infty} |u_n|.$$

Indeed, it is clear that for each integer  $p$

$$\sum_{k=1}^p |u_{n_k}| \leq |u_1| + |u_2| + \dots + |u_{n_p}| \leq \sum_{n=1}^{\infty} |u_n|$$

which proves the absolute convergence of the partial series, as well as the relation (2.5.1) when passing to the limit.

It is clear that the sum of the partial series with general term  $u_{n_k}$  is also the sum of the series with general term  $w_n$  where  $w_n = u_n$  if  $n$  is of the form  $n_k$ ,  $w_n = 0$  if  $n$  is not equal to one of the  $n_k$ . If  $I$  designates the set of integers of the form  $n_k$ , the sum  $\sum_{n=1}^{\infty} w_n$  is also written  $\sum_{n \in I} u_n$  (which is not ambiguous, since the order of the terms does not matter in virtue of (2.4)). With this notation, it is clear that

$$(2.5.2) \quad \left| \sum_{n \in I} u_n \right| \leq \sum_{n \in I} |u_n|$$

and if  $I$  and  $J$  are two infinite subsets *with no common element* of the integers  $\geq 1$ ,

$$(2.5.3) \quad \sum_{n \in I \cup J} u_n = \sum_{n \in I} u_n + \sum_{n \in J} u_n.$$

(2.6) We sometimes have to consider “series infinite in two directions”, where the indices of the terms  $a_n$  vary in the set  $\mathbf{Z}$  of integers (positive or negative). By definition, a series  $\sum_{n=-\infty}^{\infty} a_n$  is said to be *convergent* (resp. *absolutely convergent*) if *each* of the two series  $\sum_{n=0}^{\infty} a_n$ ,  $\sum_{n=1}^{\infty} a_{-n}$  is convergent (resp. absolutely convergent), and the sum  $\sum_{n=-\infty}^{\infty} a_n$  is by definition equal to  $\sum_{n=0}^{\infty} a_n + \sum_{n=1}^{\infty} a_{-n}$ .

(2.7) A series whose terms  $\mathbf{a}_n$  are vectors of  $\mathbf{C}^p$  is said to be *convergent* (resp. *absolutely convergent*) if each of the series with complex terms  $\text{pr}_j(\mathbf{a}_n)$  ( $1 \leq j \leq p$ ) is convergent (resp. absolutely convergent). The sum  $\sum_{n=0}^{\infty} \mathbf{a}_n$  is then by definition the vector whose components are the sums  $\sum_{n=0}^{\infty} \text{pr}_j(\mathbf{a}_n)$ ; it is also the *limit* in  $\mathbf{C}^p$  of the sequence of the partial sums  $\mathbf{s}_n = \sum_{k=0}^n \mathbf{a}_k$ . By virtue of definition (1.6.1) and of (2.2), in order that the series of general term  $\mathbf{a}_n$  be absolutely convergent, it is necessary and sufficient that the series of the *norms*  $\|\mathbf{a}_n\|$  be convergent, and

$$(2.7.1) \quad \left\| \sum_{n=0}^{\infty} \mathbf{a}_n \right\| \leq \sum_{n=0}^{\infty} \|\mathbf{a}_n\|.$$

The reader may extend the previous results to series of vectors.

(2.8) When it is desired to prove the existence of the limit of a *sequence*  $(\mathbf{a}_n)_{n \geq 1}$  of vectors, it is often convenient to *transform the sequence into a series*. This means that the  $\mathbf{a}_n$  are considered as the *partial sums* of the series with general term  $\mathbf{u}_n = \mathbf{a}_n - \mathbf{a}_{n-1}$  (agreeing to take  $\mathbf{a}_0 = 0$ ) so that the *sequence*  $\mathbf{a}_n$  has a *limit* equal to  $\mathbf{s}$  if, and only if, the *series* with general term  $\mathbf{u}_n$  is convergent with the *sum*  $\mathbf{s}$ . This process will be particularly helpful when we can prove the *absolute* convergence of the series with general term  $\mathbf{u}_n$  by application of (2.2) (therefore by a calculus of *majorization*).

### Theorem of the mean

(3.1) If  $a < b$  and if  $f$  and  $g$  are real, piecewise-continuous functions in  $[a, b]$  and  $f(x) \leq g(x)$  at all the points of continuity of  $g$  and  $f$  (so with the exception of at most a *finite* number of points), then

$$(3.1.1) \quad \int_a^b f(t) dt \leq \int_a^b g(t) dt$$

equality occurring only if  $f(x) = g(x)$  at all the points of continuity of  $f$  and  $g$ . By subdividing the interval  $[a, b]$  the known case where  $f$  and  $g$  are both continuous is obtained.

In particular, if it is known that:

1.  $f$  is piecewise-continuous in  $[a, b]$ ;
  2.  $f(x) \geq 0$  except at the points of discontinuity;
  3.  $\int_a^b f(t) dt = 0$ ,
- then, except at the points of discontinuity,  $f(x) = 0$  in  $[a, b]$ .

(3.2) From the inequality (3.1.1) comes the most important method in Analysis for obtaining majorants and minorants, the *theorem of the mean*. If  $a < b$  and  $m \leq f(x) \leq M$  in  $[a, b]$  except at the points of discontinuity of  $f$ , for every function  $g \geq 0$  in  $[a, b]$

$$(3.2.1) \quad m \int_a^b g(t) dt \leq \int_a^b f(t)g(t) dt \leq M \int_a^b g(t) dt$$

and in particular for  $g = 1$

$$(3.2.2) \quad m(b - a) \leq \int_a^b f(t) dt \leq M(b - a).$$

Take care *not to apply* (3.2.1) when  $g$  is not positive! The inequality (3.1.1) gives in particular for every real piecewise-continuous function  $f$

$$(3.2.3) \quad \int_a^b f(t) dt \leq \int_a^b |f(t)| dt$$

hence, if  $|f(t)| \leq M$  at the points of continuity of  $f$  and if  $g \geq 0$  in  $[a, b]$ ,

$$(3.2.4) \quad \left| \int_a^b f(t)g(t) dt \right| \leq M \int_a^b g(t) dt.$$

The majorizations (3.2.3) and (3.2.4) are the most frequently used.

(3.3) The theorem of the mean applies to the integrals of complex functions by virtue of the following proposition:

(3.3.1) For every piecewise-continuous complex function  $f$  in  $[a, b]$

$$(3.3.2) \quad \left| \int_a^b f(t) dt \right| \leq \int_a^b |f(t)| dt$$

(in other words (3.2.3) is also valid for complex functions).

Let us in fact write the complex number  $\int_a^b f(t) dt$  in polar coordinates  $re^{i\alpha}$  ( $r \geq 0$ ), and put  $g(t) = e^{-i\alpha} f(t)$ ; then  $\int_a^b g(t) dt = r$ , or decomposing  $g$  into its real and imaginary parts,

$$g = g_1 + ig_2, \quad r = \int_a^b g(t) dt = \int_a^b g_1(t) dt,$$

since the first integral is real. As  $g_1 \leq |g|$ , from (3.1.1)  $\int_a^b g_1(t) dt \leq \int_a^b |g(t)| dt$ ; but since  $|g| = |f|$ , this proves the inequality (3.3.2).

Note that  $\int_a^b g_1(t) dt = \int_a^b g(t) dt$  only if

$$g_1(t) = |g(t)|$$

except at the points of discontinuity of  $g$  (or of  $f$ ) by (3.1); in other words, there is equality in (3.3.2) only if  $f(t) = e^{i\alpha} |f(t)|$  (except at the points of discontinuity) for some constant  $\alpha$ .

(3.4) When a complex function  $F$  defined and continuous in  $[a, b]$  admits a derivative in  $[a, b]$ , except at a finite number of points, and when this derivative is piecewise-continuous

$$F(x) - F(a) = \int_a^x F'(t) dt$$

(where an arbitrary value is given to the integrand at its points of discontinuity) and so the theorem of the mean gives by (3.3.2) the inequality

$$(3.4.1) \quad |F(b) - F(a)| \leq (b - a) \sup_{a \leq t \leq b} |F'(t)|.$$

(3.5) When  $\mathbf{f}$  is a function defined in  $[a, b]$  and whose values are complex vectors  $\mathbf{f}(t) = (f_1(t), \dots, f_n(t))$ , it follows immediately from the definitions and from (3.3.2) that

$$(3.5.1) \quad \left\| \int_a^b \mathbf{f}(t) dt \right\| \leq \int_a^b \|\mathbf{f}(t)\| dt.$$

As in (3.4), it is immediately deduced that if  $\mathbf{F}$  is a function with values in  $\mathbf{C}^n$ , admitting a derivative in  $[a, b]$  except at a finite number of points and such that this derivative  $\mathbf{F}'$  is piecewise-continuous

$$(3.5.2) \quad \|\mathbf{F}(b) - \mathbf{F}(a)\| \leq (b - a) \sup_{a \leq t \leq b} \|\mathbf{F}'(t)\|.$$

(3.6) The theorem of the mean for functions of several real variables reduces at once to the case of only one variable. Let us confine ourselves to the case of a real function  $f$  defined on an open set  $U$  of  $\mathbf{R}^n$ , and consider two points  $a$  and  $b$  of  $U$  such that the segment which

joins them (viz. the set of points  $(1 - t)a + tb$  for  $0 \leq t \leq 1$ ) is contained in  $U$ . Suppose that  $f$  admits continuous partial derivatives in  $U$ ; then the function

$$g(t) = f((1 - t)a + tb)$$

is differentiable in  $[0, 1]$  and

$$g'(t) = \sum_{j=1}^n (b_j - a_j) \frac{\partial f}{\partial x_j} ((1 - t)a + tb).$$

Thus, from (3.4.1)

$$|f(b) - f(a)| = |g(1) - g(0)| \leq \sup_{0 \leq t \leq 1} |g'(t)|$$

that is

$$(3.6.1) \quad |f(b) - f(a)| \leq \sup_{0 \leq t \leq 1} \left| \sum_{j=1}^n (b_j - a_j) \frac{\partial f}{\partial x_j} ((1 - t)a + tb) \right| \leq nM \|b - a\|,$$

where  $M$  is a majorant of all the functions  $|\partial f / \partial x_j|$  on the segment  $S$  with endpoints  $a$  and  $b$ .

By applying in the same way the formula of the mean to the function

$$g(t) - tg'(t_0), \quad \text{where } 0 \leq t_0 \leq 1,$$

writing  $z_0 = (1 - t_0)a + t_0b$ ,

$$(3.6.2) \quad \left| f(b) - f(a) - \sum_{j=1}^n (b_j - a_j) \frac{\partial f}{\partial x_j} (z_0) \right| \leq \sum_{j=1}^n (b_j - a_j) \sup_{z \in S} \left| \frac{\partial f}{\partial x_j} (z) - \frac{\partial f}{\partial x_j} (z_0) \right|$$

which gives a more precise evaluation of the difference  $f(b) - f(a)$  when it is known how to majorize the differences  $|(\partial f / \partial x_j)(z) - (\partial f / \partial x_j)(z_0)|$ . When it is supposed that  $f$  admits continuous partial derivatives up to order  $p$  in  $U$ , even better estimates of  $f(b) - f(a)$  are obtained by applying to  $g$  Taylor's formula with majorization of its remainder (K-R, p. 63).

These results extend immediately to functions with values in  $\mathbf{C}^r$ .

(3.7) The interest of the theorem of the mean is that from a majorization of the *derivative* of a function a majorization of the function itself can be deduced. There is nothing similar in the opposite direction, in other words, a function can be arbitrarily small and its derivative arbitrarily large, as is shown by the function  $(1/n) \sin n^2x$ . This implies that in Analysis the operation of *integration is much easier to handle than that of differentiation*; we shall see this later when dealing with many very different questions.

#### 4. Cauchy-Schwarz inequality

First note the following elementary lemma:

(4.1) *For any choice of numbers  $a > 0$ ,  $b > 0$ , for every  $t > 0$ ,*

$$(4.1.1) \quad 2ab \leq t^2 a^2 + t^{-2} b^2$$

*where equality occurs only if  $t^2 = b/a$ .*

This is evident, if the inequality is rewritten  $(ta - t^{-1}b)^2 \geq 0$ .

(4.2) (Cauchy-Schwarz inequality for finite sums) *For any choice of complex numbers  $a_j$ ,  $b_j$  ( $1 \leq j \leq n$ )*

$$(4.2.1) \quad \left| \sum_{j=1}^n a_j b_j \right| \leq \left( \sum_{j=1}^n |a_j|^2 \right)^{1/2} \left( \sum_{j=1}^n |b_j|^2 \right)^{1/2}.$$

Since the first member is majorized by  $\sum_j |a_j| \cdot |b_j|$  we can limit ourselves to the case where the numbers  $a_j$ ,  $b_j$  are  $\geq 0$ . Moreover, since  $a_j b_j = 0$  if one of the numbers  $a_j$ ,  $b_j$  is zero, we may suppose  $a_j > 0$  and  $b_j > 0$  for every  $j$ . Then from (4.1.1) for every  $j$ ,

$$2a_j b_j \leq t^2 a_j^2 + t^{-2} b_j^2$$

and so adding the terms,

$$(4.2.2) \quad 2 \sum_{j=1}^n a_j b_j \leq t^2 \left( \sum_{j=1}^n a_j^2 \right) + t^{-2} \left( \sum_{j=1}^n b_j^2 \right)$$

for every  $t > 0$ . But if  $A = \sum_{j=1}^n a_j^2$ ,  $B = \sum_{j=1}^n b_j^2$ , as  $t$  varies between 0 and  $+\infty$ , the minimum of the second member of (4.2.2) is  $2(AB)^{1/2}$  in virtue of (4.1); hence the proposition.

(4.3) (Cauchy-Schwarz inequality for series). *Let  $(a_n)$ ,  $(b_n)$  be two infinite sequences of numbers  $\geq 0$ . If the two series with general terms  $a_n^2$  and  $b_n^2$  respectively are convergent, then so is the series with general term  $a_n b_n$ , and*

$$(4.3.1) \quad \sum_{n=1}^{\infty} a_n b_n \leq \left( \sum_{n=1}^{\infty} a_n^2 \right)^{1/2} \left( \sum_{n=1}^{\infty} b_n^2 \right)^{1/2}.$$

In fact, the partial sums  $\sum_{n=1}^N a_n b_n$  are majorized by a number independent of  $N$ , for by (4.2.1),

$$\sum_{n=1}^N a_n b_n \leq \left( \sum_{n=1}^N a_n^2 \right)^{1/2} \left( \sum_{n=1}^N b_n^2 \right)^{1/2} \leq \left( \sum_{n=1}^{\infty} a_n^2 \right)^{1/2} \left( \sum_{n=1}^{\infty} b_n^2 \right)^{1/2}.$$

The inequality (4.3.1) is then deduced from the previous one by letting  $N$  tend to infinity.

(4.4) (Cauchy-Schwarz inequality for integrals). *If  $f$  and  $g$  are two piecewise-continuous complex functions in  $[a, b]$*

$$(4.4.1) \quad \left| \int_a^b f(t) g(t) dt \right| \leq \left( \int_a^b |f(t)|^2 dt \right)^{1/2} \left( \int_a^b |g(t)|^2 dt \right)^{1/2}.$$

If one of the factors of the second member is zero, for instance the first one, then  $f(t)$  is zero except at a finite number of points (3.1.1), so the first member is zero. It may therefore be

supposed that each of the two integrals of the second member is positive. For each  $s > 0$  and each  $t \in [a, b]$ , by (4.1.1)

$$(4.4.2) \quad 2|f(t)g(t)| \leq s^2|f(t)|^2 + s^{-2}|g(t)|^2$$

and integrating, using (3.3.1),

$$2 \int_a^b |f(t)g(t)| dt \leq s^2 \int_a^b |f(t)|^2 dt + s^{-2} \int_a^b |g(t)|^2 dt$$

and, taking into account (3.3.2), the reasoning is completed as in (4.2).

(4.5) Note that in (4.4.1) there is equality of the two members only if (*except at the points of discontinuity of  $f$  or  $g$* )

$$(4.5.1) \quad \alpha f(t) = \beta \overline{g(t)}$$

for two constants  $\alpha, \beta$ , not both zero. Indeed the relation (4.4.2) can be written  $(s|f(t)| - s^{-1}|g(t)|)^2 \geq 0$ , and the integral of the first member can be zero only if  $s|f(t)| - s^{-1}|g(t)| = 0$  except at the points of discontinuity, because of (3.1). Moreover, to have  $|\int_a^b f(t)g(t) dt| = \int_a^b |f(t)g(t)| dt$ , it is necessary and sufficient that  $f(t)g(t) = |f(t)g(t)| e^{i\theta}$  for some constant  $\theta$ , except at the points of discontinuity. This relation and the preceding remark imply (4.5.1).

There are similar results for the cases of equality in (4.2.1) and (4.3.1).

(4.6) (Minkowski's inequality). *With the same hypotheses as in (4.4)*

$$(4.6.1) \quad \left( \int_a^b |f(t) + g(t)|^2 dt \right)^{1/2} \leq \left( \int_a^b |f(t)|^2 dt \right)^{1/2} + \left( \int_a^b |g(t)|^2 dt \right)^{1/2}.$$

It is sufficient to remark that

$$\int_a^b |f(t) + g(t)|^2 dt \leq \int_a^b |f(t)|^2 dt + \int_a^b |g(t)|^2 dt + 2 \int_a^b |f(t)g(t)| dt$$

and to majorize the last integral by the Cauchy-Schwarz inequality.

Minkowski's inequality for finite sums or series is deduced from (4.2) and (4.3) by the same reasoning.

## PROBLEMS

1. Consider the alternating series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{(-1)^{n-1}}{n} + \cdots$$

which is convergent, but not absolutely convergent. The series

$$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \cdots + \frac{1}{2^n + 1} + \frac{1}{2^n + 3} + \cdots + \frac{1}{2^{n+1} - 1} - \frac{1}{2^n} + \cdots$$

is obtained by permuting the terms of the alternating series, but is divergent (minorize the sum of each group of consecutive positive terms).

2. Let  $f$  be a complex function defined in an interval  $[a, b] \subset \mathbf{R}$ , and the primitive of a piecewise-continuous function  $f'$ . Suppose that  $f(a) = f(b) = 0$  and that  $|f'(x)| \leq M$  in  $[a, b]$ . Show that

$$\left| \int_a^b f(t) dt \right| \leq M \frac{(b-a)^2}{4}.$$

For which functions is there equality of the two members?

3. Let  $f$  be a complex function defined in an interval  $[0, a] \subset \mathbf{R}$ , and the primitive of a piecewise-continuous function  $f'$ , with  $f(0) = 0$ . Show that

$$\int_0^a |f'(t)f(t)| dt \leq \frac{a}{2} \int_0^a |f'(t)|^2 dt.$$

(Consider the function  $u$ , primitive of  $|f'|$ , in order to reduce to the case where  $f'(t) \geq 0$  in  $[0, a]$ . In this latter case use the Cauchy-Schwarz inequality to majorize  $f^2(a)$ .) When is there equality?

4. Let  $f$  be a continuous real function, strictly increasing in an interval  $[0, a] \subset \mathbf{R}$  and such that  $f(0) = 0$ ; let  $g$  be the inverse function, defined and strictly increasing in  $[0, f(a)]$ . Show that for  $0 \leq x \leq a$  and  $0 \leq y \leq f(a)$ , we have ("Young's inequality")

$$xy \leq \int_0^x f(t) dt + \int_0^y g(u) du$$

equality occurring only if  $y = f(x)$ . (Study the variations of the function  $x \rightarrow xy - \int_0^x f(t) dt$ .) Deduce the following inequalities:

$$xy \leq ax^p + by^q$$

for  $x \geq 0, y \geq 0, p > 1, q = p/(p-1), a > 0, b > 0$  and  $(pa)^q(qb)^p \geq 1$ ;

$$xy \leq x \log x + e^{y-1}$$

for  $x > 0, y$  real. Case of equality?

5. Let  $r_1, r_2, \dots, r_n$  be any real numbers. Put

$$f(x, y) = (x + r_1 y)(x + r_2 y) \dots (x + r_n y).$$

Show that for integers  $h$  and  $k$  satisfying  $h + k < n$ , the polynomial  $\frac{\partial^{h+k} f(x, y)}{\partial x^h \partial y^k}$  is a product of  $h + k$  factors of the form  $x + t_j y$ , where the  $t_j$  are real (apply Rolle's theorem). Deduce from this that if

$$f(x, y) = x^n + \binom{n}{1} p_1 x^{n-1} y + \binom{n}{2} p_2 x^{n-2} y^2 + \dots + p_n y^n$$

$p_{k-1} p_{k+1} \leq p_k^2$  for  $1 \leq k \leq n-1$  (with  $p_0 = 1$ ). Hence show that if all  $r_j$  are  $> 0$ , we have  $p_k^{1/k} \geq p_{k+1}^{1/(k+1)}$ . In particular

$$\frac{1}{n} (r_1 + r_2 + \dots + r_n) \geq (r_1 r_2 \dots r_n)^{1/n}$$

(geometric mean inequality).

6. A real function  $f$  defined in an interval  $I \subset \mathbf{R}$  (bounded or not) is said to be *convex* in  $I$  if, for any two points  $x < x'$  of  $I$  and every  $z$  such that  $x < z < x'$ , the point  $(z, f(z))$  lies under the line segment joining the points  $(x, f(x))$  and  $(x', f(x'))$ . In other words if  $0 < \lambda < 1$

$$f(\lambda x + (1 - \lambda)x') \leq \lambda f(x) + (1 - \lambda)f(x').$$

We say that  $f$  is *concave* in  $I$  if  $-f$  is convex.

(a) Show that for every finite set of points  $(x_1, x_2, \dots, x_p)$  in  $I$  and every set  $(\lambda_1, \lambda_2, \dots, \lambda_p)$  of real numbers such that  $\lambda_j \geq 0$ ,  $\sum_{j=1}^p \lambda_j = 1$

$$f(\lambda_1 x_1 + \lambda_2 x_2 + \dots + \lambda_p x_p) \leq \lambda_1 f(x_1) + \lambda_2 f(x_2) + \dots + \lambda_p f(x_p).$$

(b) A real function  $f$  is convex in  $I$  if, and only if, for each  $a \in I$  the *gradient*

$$p_f(a, x) = \frac{f(x) - f(a)}{x - a}$$

is a function increasing in  $I - \{a\}$ . Deduce from this that for each point  $a$  interior to  $I$  the right derivative  $f'_R(a)$  and the left derivative  $f'_L(a)$  exist and  $f'_L(a) \leq f'_R(a)$ . If  $a < b$  are two points interior to  $I$ , then  $f'_R(a) \leq p_f(a, b) \leq f'_L(b)$ .

(c) Show that a function  $f$  with an increasing derivative in an open interval  $I \subset \mathbf{R}$  is convex. (Use contradiction and the mean value theorem.) In particular if  $f''(x)$  exists and is  $\geq 0$  in  $I$ ,  $f$  is convex.

(d) The following three functions are convex for  $x > 0$ :

$$-\log x, \quad x^p \quad \text{for } p \geq 1, \quad (1 + x^p)^{1/p} \quad \text{for } p \geq 1.$$

By using (a) deduce from these again the geometric mean inequality (problem 5) as well as the following two inequalities:

$$\sum_j x_j y_j \leq (\sum_j x_j^p)^{1/p} (\sum_j y_j^q)^{1/q}$$

where the  $x_j, y_j$  are  $\geq 0$ ,  $p > 1$  and  $q = (p - 1)/p$  (Hölder's inequality);

$$(1) \quad (\sum_j x_j^{1/p})^p + (\sum_j y_j^{1/p})^p \leq (\sum_j (x_j + y_j)^{1/p})^p$$

where the  $x_j, y_j$  are  $\geq 0$ ,  $p \geq 1$  (Minkowski's inequality). By considering in the same way the function  $(1 - x^{1/p})^p$ , which is convex for  $p > 1$  and  $0 < x < 1$ , prove the inequality

$$(2) \quad (\sum_j (x_j + y_j)^p)^{1/p} \leq (\sum_j x_j^p)^{1/p} + (\sum_j y_j^p)^{1/p}$$

where the  $x_j, y_j$  are  $\geq 0$  and  $p > 1$  (Minkowski's inequality).

7. Deduce the second inequality of Minkowski ((2) of problem 6) from Hölder's inequality, noting that  $(\sum_j (x_j + y_j)^p)^{1/p}$  is the supremum of the set of numbers  $\sum_j (x_j + y_j) z_j$ , where the

sequence  $(z_j)$  ranges over the set of sequences satisfying  $\sum_j^n z_j^q = 1$ . In a similar manner prove the inequality (for  $x_j \geq 0, y_j \geq 0$ )

$$(x_1 x_2 \dots x_n)^{1/n} + (y_1 y_2 \dots y_n)^{1/n} \leq ((x_1 + y_1)(x_2 + y_2) \dots (x_n + y_n))^{1/n}$$

noting that  $(x_1 x_2 \dots x_n)^{1/n}$  is the infimum of the set of numbers  $(1/n) \sum_{j=1}^n x_j z_j$ , where the sequence  $(z_j)$  ranges over the set of  $n$  numbers  $> 0$  satisfying  $z_1 z_2 \dots z_n = 1$ .

8. Prove the inequality

$$\left( \frac{\sum_j (x_j + y_j)^p}{\sum_j (x_j + y_j)^r} \right)^{1/(p-r)} \leq \left( \frac{\sum_j x_j^p}{\sum_j x_j^r} \right)^{1/(p-r)} + \left( \frac{\sum_j y_j^p}{\sum_j y_j^r} \right)^{1/(p-r)}$$

where  $x_j$  and  $y_j$  are positive,  $0 < r \leq 1 \leq p$ . (Consider  $\left(\sum_j x_j^p\right)^{1/p}$  as the supremum of all  $\sum_j x_j z_j$ , for  $z_j \geq 0$  and  $\sum_j z_j^q = 1$ , as in problem 7. Then put

$$a = \left( \frac{\left(\sum_j x_j z_j\right)^p}{\sum_j x_j^r} \right)^{1/(p-r)}, \quad b = \left( \frac{\left(\sum_j y_j z_j\right)^p}{\sum_j y_j^r} \right)^{1/(p-r)}$$

and use Hölder's inequality and the first inequality of Minkowski (1).)

9. Let  $f$  be a function differentiable and convex in an open interval  $I$ ; show that for each  $x \in I$ ,  $f(x)$  is the supremum of the set of numbers  $f(y) + (x - y)f'(y)$ .

10. Let  $x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n$  be  $2n$  real numbers such that

$$\begin{aligned} x_1 &\geq x_2 \geq \dots \geq x_n, & y_1 &\geq y_2 \geq \dots \geq y_n \\ x_1 &\geq y_1 \\ x_1 + x_2 &\geq y_1 + y_2 \\ \cdot & \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ x_1 + x_2 + \dots + x_{n-1} &\geq y_1 + y_2 + \dots + y_{n-1} \\ x_1 + x_2 + \dots + x_n &= y_1 + y_2 + \dots + y_n. \end{aligned}$$

Show that for each differentiable convex function  $f(x)$ ,

$$f(x_1) + f(x_2) + \dots + f(x_n) \geq f(y_1) + f(y_2) + \dots + f(y_n).$$

(Use problem 9 to obtain the result that if

$$z_1 \leq z_2 \leq \dots \leq z_n$$

then

$$x_1 f'(z_1) + x_2 f'(z_2) + \dots + x_n f'(z_n) \geq y_1 f'(z_1) + y_2 f'(z_2) + \dots + y_n f'(z_n);$$

and use the fact that  $f'$  is increasing.)

11. For  $p > 1$ , let  $D$  be the subset of  $\mathbf{R}^n$  defined by the relations  $x_j \geq 0$  ( $1 \leq j \leq n$ ),  $x_1^p > x_2^p + x_3^p + \dots + x_n^p$ . Show that if  $f(x) = (x_1^p - x_2^p - \dots - x_n^p)^{1/p}$ , then for  $x, y$  in  $D$

$$f(x + y) \geq f(x) + f(y).$$

(Show with the aid of Hölder's inequality, that  $f(x)$  is the infimum of  $\sum_{j=1}^n x_j z_j$ , where  $z_1 \geq 1$ ,  $z_j \geq 0$  for  $2 \leq j \leq n$ , and  $z_2^q + \dots + z_n^q \leq z_1^q - 1$ , where  $q = p/(p-1)$ .)

12. Let  $a_j, b_j$  be real numbers  $> 0$  ( $1 \leq j \leq n$ ) such that

$$0 < m \leq \frac{b_j}{a_j} \leq M \quad \text{for } 1 \leq j \leq n.$$

(a) Show that  $b_j^2 + mMa_j^2 \leq (M + m)a_j b_j$ , and hence

$$\sum_j b_j^2 + mM \sum_j a_j^2 \leq (M + m) \sum_j a_j b_j.$$

(b) Deduce from (a) that

$$\frac{(a_1^2 + a_2^2 + \cdots + a_n^2)(b_1^2 + b_2^2 + \cdots + b_n^2)}{(a_1 b_1 + a_2 b_2 + \cdots + a_n b_n)^2} \leq \frac{(M + m)^2}{4Mm}$$

(use the geometric mean inequality with  $n = 2$ ).

13. Let  $f, g$  be two piecewise-continuous functions in  $[a, b]$ , such that  $f$  is decreasing and  $0 \leq g(t) \leq 1$  in  $[a, b]$ . If  $\lambda = \int_a^b g(t) dt$ , then

$$\int_{b-\lambda}^b f(t) dt \leq \int_a^b f(t) g(t) dt \leq \int_a^{a+\lambda} f(t) dt$$

equality occurring only if  $f$  is constant in  $[a, b]$  or if  $g = 0$  except at its points of discontinuity, or if  $g = 1$  except at its points of discontinuity. (Vary  $x$  in the integral  $\int_a^x f(t) g(t) dt$  and compare to the integral  $\int_a^{a+G(x)} f(t) dt$ , where  $G(x) = \int_a^x g(t) dt$ .)

14. Let  $w_j, a_j$  be real numbers such that

$$1 \geq w_1 \geq w_2 \geq \cdots \geq w_n \geq 0, \quad a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$$

and let  $h(x)$  be a convex function differentiable in the interval  $[0, a_1]$ . Show that

$$\begin{aligned} h(w_1 a_1 - w_2 a_2 + \cdots + (-1)^{n-1} w_n a_n) \\ \leq (1 - w_1 + w_2 + \cdots + (-1)^{n-1} w_n) h(0) + w_1 h(a_1) - w_2 h(a_2) \\ + \cdots + (-1)^{n-1} w_n h(a_n). \end{aligned}$$

(Apply problem 13 by taking  $f(t) = -h'(t)$  and  $g(t)$  piecewise-constant (step function).)

15. We consider in the space  $\mathbf{C}^n$  the positive non-degenerate Hermitian form  $(\mathbf{x} | \mathbf{y}) = \sum_{j=1}^n x_j \bar{y}_j$  (Hermitian scalar product); the notions of orthogonality and of orthonormal base always relate to this form in the paragraphs below.

(a) Let  $A$  be a square matrix of order  $n$  with complex elements. Show that there exists a unitary matrix  $U$  such that  $UAU^{-1}$  has only zeros underneath its diagonal (which is to say that taking a proper orthonormal base of  $\mathbf{C}^n$ , the endomorphism  $g$  of  $\mathbf{C}^n$  corresponding to  $A$  has a matrix of this form; proceed by induction on  $n$ , by considering an eigenvector of  $g$ ).

(b) Show that if  $H$  is the positive square root of the positive Hermitian matrix  $A^*A$ , there exists a unitary matrix  $U$  such that  $A = UH$  (if  $g$  and  $h$  are the endomorphisms of  $\mathbf{C}^n$  corresponding to  $A$  and  $H$ , show that  $g^{-1}(0) = h^{-1}(0)$  and that if  $V$  is the subspace orthogonal to  $g^{-1}(0)$ , then  $(g(\mathbf{x}) | g(\mathbf{x})) = (h(\mathbf{x}) | h(\mathbf{x}))$  for every  $\mathbf{x} \in V$ ).

(c) Let  $H_1, H_2$  be two Hermitian matrices of order  $n$ ; show that  $\text{Tr}(H_1 H_2) \geq 0$  (using the formula  $\text{Tr}(UAU^{-1}) = \text{Tr}(A)$ , reduce to the case where  $H_1$  is a diagonal matrix). Deduce from this that if  $\varphi(H_1)$  is the largest absolute value of the elements of  $H_1$ , then

$$\text{Tr}(H_1 H_2) \leq \varphi(H_1) \text{Tr}(H_2) \leq \text{Tr}(H_1) \text{Tr}(H_2).$$

16. For every square matrix  $A$  of order  $n$  with complex elements set  $N(A) = (\text{Tr}(A^*A))^{1/2}$ , we have  $N(A) \leq n\|A\|$ ,  $N(A^*) = N(A)$ , and for any unitary matrix  $U$ ,  $N(U) = \sqrt{n}$ . Show that for any two square matrices whatever

$$\begin{aligned} |\text{Tr}(AB)| &\leq N(A)N(B) \\ |\text{Tr}(A)| &\leq \sqrt{n}N(A) \end{aligned}$$

(for the first inequality, consider the product  $(A^* + tB^*)(A + tB)$ , which is positive Hermitian for any  $t \in \mathbf{R}$ ; for the second inequality, use problem 15(a)). Deduce that

$$N(A + B) \leq N(A) + N(B).$$

Show, using problem 15(c), that

$$N(AB) \leq N(A)N(B).$$

Let  $\det(\lambda I - A) = \prod_{j=1}^n (\lambda - \lambda_j)$  be the characteristic polynomial of the matrix  $A$ . Show that  $\sum_j |\lambda_j|^2 \leq (N(A))^2$  (use problem 15(a)).

17. Let  $H$  be a positive invertible Hermitian matrix; show that for any two vectors  $x, y$  of  $\mathbf{C}^n$

$$(x | y)^2 \leq (x | H \cdot x)(y | H^{-1} \cdot y)$$

(if  $H_1$  is the positive square root of  $H$ , observe that  $(x | y) = (H_1 \cdot x | H_1^{-1} \cdot y)$ ). Deduce from this that for every vector  $y \in \mathbf{C}^n$ ,

$$\psi_y(H) = (y | H^{-1} \cdot y)^{-1} = \inf \frac{(x | H \cdot x)}{(x | y)^2}$$

where  $x$  ranges over the set of vectors such that  $(x | y) \neq 0$ . Deduce that

$$\psi_y(H_1 + H_2) \geq \psi_y(H_1) + \psi_y(H_2)$$

for two positive Hermitian invertible matrices. In particular, if we take for  $y$  a vector  $e$ , of the canonical basis of  $\mathbf{C}^n$

$$\frac{\det(H_1 + H_2)}{\det(H_1^{(j)} + H_2^{(j)})} \geq \frac{\det(H_1)}{\det(H_1^{(j)})} + \frac{\det(H_2)}{\det(H_2^{(j)})}$$

where  $H^{(j)}$  designates the square matrix of order  $n - 1$  obtained by suppressing in  $H$  the row and column of index  $j$ .

18. Let  $a_1, a_2, a_3$  be three real numbers  $> 0$ , and  $c_1, c_2, c_3$  three real numbers contained in the smallest interval containing  $a_1, a_2, a_3$ .

(a) Show that if  $c_1 + c_2 + c_3 \geq a_1 + a_2 + a_3$ , then also  $c_1 c_2 c_3 \geq a_1 a_2 a_3$  and

$$c_1 c_2 + c_2 c_3 + c_3 c_1 \geq a_1 a_2 + a_2 a_3 + a_3 a_1.$$

(b) Show that if  $a_1 a_2 + a_2 a_3 + a_3 a_1 \geq c_1 c_2 + c_2 c_3 + c_3 c_1$ , then also

$$a_1 + a_2 + a_3 \geq c_1 + c_2 + c_3.$$

(c) Show that if  $a_1 a_2 a_3 \geq c_1 c_2 c_3$ , then also

$$a_1 + a_2 + a_3 \geq c_1 + c_2 + c_3.$$

19. Let  $f$  be a function  $\geq 0$  and increasing in the interval  $[0, 1]$ . Show that there exists a function  $g \geq 0$ , convex in  $[0, 1]$  and satisfying  $g(x) \leq f(x)$  and  $\int_0^1 g(t) dt \geq \frac{1}{2} \int_0^1 f(t) dt$ .

20. Let  $u$  be a function twice continuously differentiable in an interval  $[a, b] \subset \mathbf{R}$ , such that  $u(a) = u(b) = 0$  and  $u(t) > 0$  for  $a < t < b$ . Show that the integral  $\int_a^b |u''(t)/u(t)| dt$  is either divergent, or converges and satisfies

$$\int_a^b \left| \frac{u''(t)}{u(t)} \right| dt > \frac{4}{b-a}.$$

(If  $M$  is the maximum of  $u(t)$ , note that

$$\int_a^b \left| \frac{u''(t)}{u(t)} \right| dt > \frac{1}{M} \sup_{a < t_1 < t_2 < b} |u'(t_2) - u'(t_1)|$$

and if  $c$  is a point in  $[a, b]$  such that  $u(t) = M$ , apply to  $u$  the mean value theorem in each of the intervals  $[a, c]$  and  $[c, b]$ ).

# Approximation of the roots of an equation

## I. Outline of problem

(1.1) Let us suppose we are given a *real* function  $f$  of a real variable  $x$ ,  $x$  varying in an interval  $I$  of  $\mathbf{R}$  ( $I$  can be the whole of  $\mathbf{R}$ ); we shall suppose in this chapter that  $f$  possesses a second derivative continuous in  $I$  (in most cases  $f$  will even be infinitely differentiable in  $I$ ). The problem is to obtain, with an *arbitrary* approximation (cf. Introduction), the roots of the equation

$$(1.1.1) \quad f(x) = 0.$$

Of course, it would be foolish to hope to obtain, in general, precise formulae for the solution of (1.1.1) of the type of the classical formulae solving equations of the first or second degree; one can show that this is already impossible when  $f$  is a polynomial of degree  $\geq 5$ . But even for the more reasonable problem stated here, there is no general method (at least theoretically) leading to the desired result, except when  $f$  is a polynomial. Even in this case, for polynomials of degree  $\geq 5$ , the theoretical methods of approximation (see Appendix) can lead to inextricable numerical calculations, even for computers. We shall confine ourselves to indicating a few cases where methods can be found (both theoretical and practical) for solving the given problem, referring for more details to the specialized works on Numerical Analysis (see Bibliography).

(1.2) Even if the interval  $I$  is bounded, it may happen that the equation (1.1.1) has infinitely many roots, for example, if  $f$  is identically zero on a subinterval; the example  $f(x) = x^6 \sin 1/x$  shows that there may be infinitely many roots even if  $f$  is not identically zero in any interval.

A first step consists in *decomposing* the interval  $I$  into a finite or infinite number of subintervals, in each of which it is known that  $f$  either *has no roots* or *at most one root*. This will certainly be the case if in such an interval the function is strictly monotonic (K-R, p. 222): we have the first case if  $f$  does not change sign in the interval, and the second case if  $f$  takes values of opposite sign at the endpoints of the interval.

*Example* (1.3) The function

$$(1.3.1) \quad f(x) = \frac{b_1}{x - a_1} + \frac{b_2}{x - a_2} + \cdots + \frac{b_n}{x - a_n} + c$$

where  $a_1 < a_2 < \dots < a_n$  and where  $b_j$  are all  $\neq 0$  and of the *same sign*, is defined in each of the open intervals  $]-\infty, a_1[$ ,  $]a_1, a_2[$ ,  $\dots$ ,  $]a_{n-1}, a_n[$ ,  $]a_n, +\infty[$ ; in each of these intervals it is *monotonic*, since its derivative

$$f'(x) = -\frac{b_1}{(x-a_1)^2} - \frac{b_2}{(x-a_2)^2} - \dots - \frac{b_n}{(x-a_n)^2}$$

has the opposite sign to that of the  $b_j$ . Moreover, when  $x$  tends to  $a_j$  on the right (resp. on the left) all terms in (1.3.1) except  $b_j/(x-a_j)$  tend to a finite limit, and so  $f(x)$  tends in absolute value to  $+\infty$ , with the sign of  $b_j$  (resp. the opposite sign). We conclude from this that the equation  $f(x) = 0$  has just one simple root in each of the  $n-1$  intervals

$$]a_j, a_{j+1}[ \quad (1 \leq j \leq n-1).$$

In the interval  $]-\infty, a_1[$  (resp.  $]a_n, +\infty[$ ), it has a simple root if  $c$  has the sign of the  $b_j$ 's (resp. the opposite sign), otherwise it has no root.

(1.4) To obtain a decomposition of  $I$  into intervals of the type considered in (1.2) (or, as is sometimes said, to “*separate the roots*”) one should, theoretically, study the sense of the variation of  $f$ , which is to say the *sign* of the derivative  $f'$ . This would lead to finding the roots of the equation  $f'(x) = 0$ , which, except in very simple cases like (1.3), is a problem of the same order of difficulty as the solution of (1.1.1).

In no. 2 we shall suppose that  $I = ]a, b[$ , that the derivative  $f'(x)$  *does not vanish* in  $I$  (therefore has constant sign) and that

$$f(a)f(b) < 0.$$

## 2. Linear approximation

(2.1) The idea here for obtaining an approximation to the root  $\xi_0$  of the equation  $f(x) = 0$  in  $I$ , is to replace  $f$  by a *first degree polynomial*  $L(x)$ , which coincides with  $f$  at the points  $a$  and  $b$  (the simplest particular case of “*interpolation polynomials*” which will be met later (IX, Appendix)). The hypothesis implies that  $L$  vanishes at a point  $\xi$  of  $I$ , which is taken as our approximation to  $\xi_0$  (Fig. 2). We shall wish to *majorize* the error  $|\xi - \xi_0|$ , and base our reasoning on the following lemma, a consequence of Rolle's theorem:

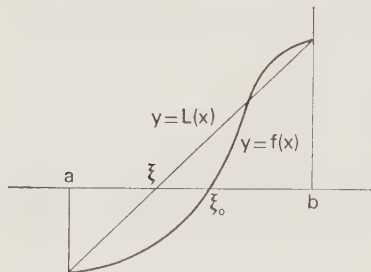


FIGURE 2

(2.2) Let  $J$  be an interval of  $\mathbf{R}$ ,  $f$  a function twice continuously differentiable in  $J$ ,  $x_0$  and  $x_1$  two distinct

points of  $J$ ,  $L$  the polynomial of the first degree

$$(2.2.1) \quad L(x) = \frac{(x-x_0)f(x_1) - (x-x_1)f(x_0)}{x_1-x_0}$$

which coincides with  $f$  at the points  $x_0$  and  $x_1$ : then for each  $x \in J$  there exists a point  $\xi$  (depending on  $x$ ) in the smallest interval containing  $x$ ,  $x_0$  and  $x_1$  such that

$$(2.2.2) \quad f(x) - L(x) = \frac{1}{2}f''(\xi)(x-x_0)(x-x_1).$$

We can, of course, confine ourselves to the case where  $x \neq x_0$  or  $x_1$ ; consider the function of  $z$  in  $J$

$$u(z) = f(z) - L(z) - c(z - x_0)(z - x_1)$$

where  $c$  is determined so that  $u(x) = 0$ . In fact, then  $u(x) = u(x_0) = u(x_1) = 0$ , and therefore by Rolle's theorem there are two *distinct* points  $y_1$  and  $y_2$  in the smallest interval containing  $x$ ,  $x_0$  and  $x_1$  such that  $u'(y_1) = u'(y_2) = 0$ . Applying Rolle's theorem to  $u'$ , we obtain a point  $\zeta$  in the interval with endpoints  $y_1$  and  $y_2$  such that  $u''(\zeta) = 0$ . But  $u''(z) = f''(z) - 2c$ , so  $c = \frac{1}{2}f''(\zeta)$  and (2.2.2) follows on solving  $u(x) = 0$  for  $c$ . The following corollary follows:

(2.3) *If  $f'$  does not vanish in  $J$ , and if  $f(\xi_0) = 0$ ,  $L(\xi) = 0$  for  $\xi$  and  $\xi_0$  in  $J$ , then*

$$(2.3.1) \quad \xi - \xi_0 = \frac{1}{2} \frac{f''(\zeta)}{f'(\zeta')} (\xi - x_0)(\xi - x_1)$$

where  $\zeta$  and  $\zeta'$  belong to  $J$ . Hence if  $|f'(x)| \geq m > 0$  and  $|f''(x)| \leq M$  in  $J$ , then

$$(2.3.2) \quad |\xi - \xi_0| \leq \frac{M}{2m} |\xi - x_0| \cdot |\xi - x_1|.$$

Indeed, from (2.2.2)

$$f(\xi) = \frac{1}{2}f''(\zeta)(\xi - x_0)(\xi - x_1).$$

On the other hand the mean value theorem gives

$$(2.3.3) \quad f(\xi) = (\xi - \xi_0)f'(\zeta')$$

for some  $\zeta'$  in the interval with endpoints  $\xi_0$  and  $\xi$ , hence (2.3.1) and also (2.3.2).

(2.4) If the error  $|\xi - \xi_0|$  estimated by (2.3.2) is not small enough, we can *repeat* the method: calculate  $f(\xi)$  and depending on the sign of  $f(\xi)$  the root  $\xi_0$  is in the interval  $[a, \xi]$  or  $[\xi, b]$  to which the same procedure is applied, obtaining a second approximation  $\xi'$ . Theoretically, this method can be applied indefinitely and one can easily prove that the sequence of numbers thus obtained converges to  $\xi_0$  (see problem 9).

### 3. Solution of $x = g(x)$ by iteration

(3.1) In applying the previous method it is supposed that we already know of the existence of a root in the interval  $I$  under consideration. Under certain circumstances, methods of approximation to a root of (1.1.1) can be found which simultaneously *prove the existence* of the root. The starting point is the mean value theorem (2.3.3): if  $f(\xi_0) = 0$ , and if we can find (by any means) a number  $\xi$  such that  $f(\xi)$  is “small” and  $f'(x)$  is not “too small” in the neighbourhood of  $\xi$ , then the error  $|\xi - \xi_0|$  will be *small*. We shall express this vague idea in precise terms in the two following methods, which are the first examples of an idea which is very fruitful in Analysis—that of *iteration* of a method of approximation (also called “*the method of successive approximations*”). Many other examples will be seen in the remainder of this book.

(3.2) The equation (1.1.1) can always be written in the form

$$(3.2.1) \quad x = g(x)$$

by putting  $g(x) = x - f(x)$ . Then

(3.3) Let  $x_0 \in I$  and suppose that there is an interval  $[x_0 - c, x_0 + c] \subset I$  and a number  $q$ , such that  $0 < q < 1$ , with the following properties:

1.  $|g'(x)| \leq q$  for  $x_0 - c \leq x \leq x_0 + c$ .
2.  $|g(x_0) - x_0| \leq c(1 - q)$ .

Then there exists one, and only one, root  $\xi_0$  of equation (3.2.1) such that

$$x_0 - c \leq \xi_0 \leq x_0 + c.$$

Moreover, a sequence  $(x_n)$  can be defined inductively by

$$(3.3.1) \quad x_{n+1} = g(x_n) \quad \text{for } n \geq 0$$

and then

$$(3.3.2) \quad |\xi_0 - x_n| \leq cq^n,$$

and the sequence  $(x_n)$  tends to  $\xi_0$ .

If  $x_0, \dots, x_n$  are defined by the relations (3.3.1) and belong to  $[x_0 - c, x_0 + c]$ , then  $x_{n+1}$  can be defined by (3.3.1) and

$$x_{n+1} - x_n = g(x_n) - g(x_{n-1}).$$

So from the mean-value theorem (I, 3.4.1) and hypothesis (1)

$$(3.3.3) \quad |x_{n+1} - x_n| \leq q|x_n - x_{n-1}|$$

hence by induction

$$(3.3.4) \quad |x_{n+1} - x_n| \leq q^n |x_1 - x_0| = q^n |g(x_0) - x_0| \leq cq^n (1 - q)$$

using property 2. Since

$$x_{n+1} - x_0 = (x_{n+1} - x_n) + (x_n - x_{n-1}) + \dots + (x_1 - x_0)$$

we get

$$|x_{n+1} - x_0| \leq c(1 - q)(1 + q + \dots + q^n) \leq c.$$

In other words  $x_{n+1}$  again belongs to  $[x_0 - c, x_0 + c]$ , and by induction this holds for every  $x_n$  defined by (3.3.1). The inequality (3.3.4) shows that the series with general term  $x_{n+1} - x_n$  is absolutely convergent, and therefore the sequence  $(x_n)$  has a limit  $\xi_0$ . Since  $g$  is continuous,  $\xi_0 = g(\xi_0)$  by passage to the limit in (3.3.1), and clearly  $\xi_0$  belongs to  $[x_0 - c, x_0 + c]$ . On the other hand from (3.3.4), for  $m \geq n$

$$\begin{aligned} |x_m - x_n| &\leq |x_{n+1} - x_n| + |x_{n+2} - x_{n+1}| + \dots + |x_m - x_{m-1}| \\ &\leq c(1 - q)(q^n + q^{n+1} + \dots + q^{m-1}) \leq cq^n. \end{aligned}$$

Letting  $m$  tend to  $+\infty$  in this relation yields (3.3.2). It remains to show that  $\xi_0$  is the only root of (3.2.1) in this interval. If  $\xi_1$  is a root of (3.2.1) in the interval, then

$$\xi_1 - \xi_0 = g(\xi_1) - g(\xi_0)$$

and by applying the mean value theorem and (1)

$$|\xi_1 - \xi_0| \leq q|\xi_1 - \xi_0|$$

or

$$0 \leq (1 - q)|\xi_1 - \xi_0| \leq 0,$$

which is possible only if  $\xi_1 = \xi_0$  since  $1 - q \neq 0$ .

Q.E.D.

(3.4) With the help of a graph of the function  $g$  the successive constructions of the  $x_n$  (Fig. 3) can easily be represented.

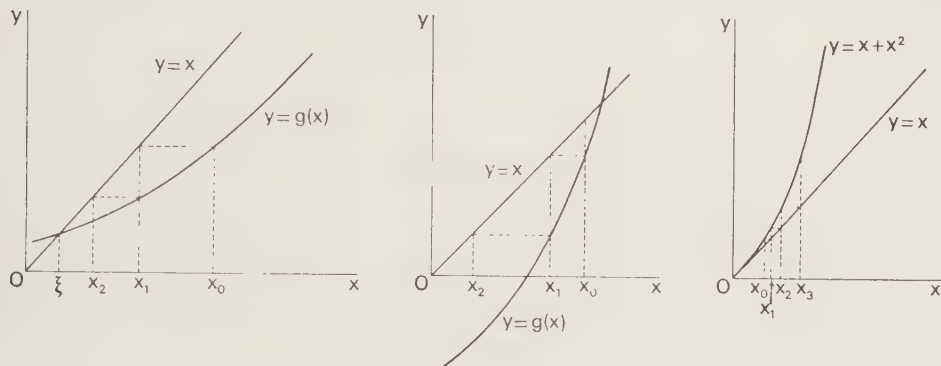


FIGURE 3

Moreover, this shows that the method would not give a sequence converging to a root of (3.2.1) if the condition that  $g'$  have absolute value  $< 1$  is suppressed; it is enough, for example, to take  $g(x) = x + x^2$  and  $x_0 > 0$  to obtain a sequence  $(x_n)$  which tends to  $+\infty$ .

## 4. Newton's method

(4.1) The idea of Newton's method is very similar to that of the earlier method (2.1); instead of taking a polynomial  $L(x)$  representing a *secant* of the graph of the function  $f$ ,  $L(x)$  is taken to be the equation of a line *parallel to a tangent* to this graph (Fig. 4), which is to say

$$(4.1.1) \quad L(x) = f'(z)(x - x_0) + f(x_0).$$

This method is also *iterative* and leads to the *existence* of a unique root in an interval  $[x_0 - c, x_0 + c]$  provided that  $f(x_0)$  is *small enough*, that  $f'$  is *not too small* in the interval and moreover varies "*rather slowly*". More precisely:

(4.2) Let  $x_0 \in I$ : suppose there are two numbers  $c \geq 0$ ,  $\lambda > 0$  with the following properties:

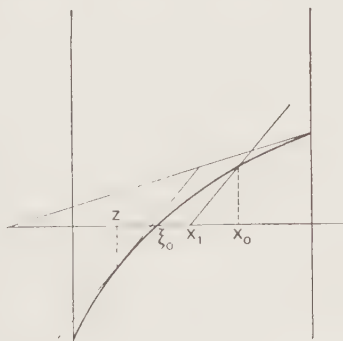


FIGURE 4

$$1. |f(x_0)| \leq c/2\lambda.$$

2. For any points  $x, y$  in  $[x_0 - c, x_0 + c] \subset I$ ,

$$(4.2.1) \quad |f'(x)| \geq 1/\lambda$$

$$(4.2.2) \quad |f'(x) - f'(y)| \leq 1/2\lambda.$$

In these circumstances there exists one, and only one, root  $\xi_0$  of the equation  $f(x) = 0$  in the interval  $[x_0 - c, x_0 + c]$ . Moreover, if  $(z_n)_{n \geq 0}$  is any sequence of points of  $[x_0 - c, x_0 + c]$ , a sequence  $(x_n)$  of points of this interval can be defined such that

$$(4.2.3) \quad x_{n+1} = x_n - \frac{f(x_n)}{f'(z_n)}$$

and this sequence tends to  $\xi_0$ .

We have (I, 3.6.2)

$$(4.2.4) \quad |f(x) - f(y) - f'(z)(x - y)| \leq (x - y) \sup_{x \leq t \leq y} |f'(t) - f'(z)|$$

for any  $x, y, z$  in  $I$  such that  $y < x$ .

It will be shown, by induction on  $n$ , that if  $x_1, x_2, \dots, x_n$  are defined by the relations (4.2.3) (where  $n$  is replaced by  $k$  for  $0 \leq k \leq n - 1$ ) and belong to  $[x_0 - c, x_0 + c]$ , then  $x_{n+1}$  defined by (4.2.3) is also in this interval; moreover, it will be proved at the same time that the sequence  $(x_n)$  satisfies the relations

$$(4.2.5) \quad |x_n - x_{n-1}| \leq c/2^n$$

$$(4.2.6) \quad |f(x_{n-1})| \leq c/2^n \lambda$$

for  $n \geq 1$ . To do this, note that for  $n = 1$  the relation (4.2.6) is true by property 1; on the other hand

$$(4.2.7) \quad |x_1 - x_0| = \left| \frac{f(x_0)}{f'(z_0)} \right| \leq \frac{c}{2}$$

by property 1 and (4.2.1). Let us argue by induction by supposing (4.2.5) and (4.2.6) verified. By (4.2.3),

$$f(x_n) = f(x_n) - f(x_{n-1}) - (x_n - x_{n-1})f'(z_{n-1})$$

therefore by (4.2.4) and using (4.2.2) and (4.2.5)

$$|f(x_n)| \leq \frac{|x_n - x_{n-1}|}{2\lambda} \leq \frac{c}{2^{n+1}\lambda}$$

which proves (4.2.6) where  $n$  is replaced by  $n + 1$ . On the other hand

$$|x_{n+1} - x_n| = \left| \frac{f(x_n)}{f'(z_n)} \right| \leq \frac{c}{2^{n+1}}$$

by (4.2.1) and the preceding result, which proves (4.2.5) where  $n$  is replaced by  $n + 1$ . Hence

$$|x_{n+1} - x_0| \leq c(2^{-1} + 2^{-2} + \dots + 2^{-n-1}) \leq c,$$

so  $x_{n+1}$  indeed belongs to the interval  $[x_0 - c, x_0 + c]$ , and the induction argument is complete. It follows immediately from (4.2.5) that the series with general term  $x_n - x_{n-1}$  is absolutely convergent, and so the sequence  $(x_n)$  tends to a limit  $\xi_0 \in [x_0 - c, x_0 + c]$ ; from the continuity of  $f$  and from (4.2.6),  $f(\xi_0) = 0$ .

It remains to prove the uniqueness of the root of  $f(x) = 0$  in  $[x_0 - c, x_0 + c]$ . If  $\xi_1$  is such a root, by (4.2.4) and (4.2.2)

$$|f'(z_n)(\xi_1 - x_{n+1})| = |f(\xi_1) - f(x_n) - f'(z_n)(\xi_1 - x_n)| \leq (\xi_1 - x_n)/2\lambda$$

hence using (4.2.1)

$$|\xi_1 - x_{n+1}| \leq |\xi_1 - x_n|/2$$

Letting  $n$  tend to  $+\infty$

$$|\xi_1 - \xi_0| \leq |\xi_1 - \xi_0|/2$$

which implies  $\xi_1 = \xi_0$ .

Q.E.D.

(4.3) From the relation (4.2.5), for  $n \leq m$ , we get

$$|x_m - x_n| \leq c(2^{-n-1} + 2^{-n-2} + \dots + 2^{-m}) \leq c/2^n$$

hence, letting  $m$  tend to  $+\infty$ ,

$$(4.3.1) \quad |\xi_0 - x_n| \leq c/2^n.$$

By a suitable choice of the  $z_n$  we can make the sequence  $(x_n)$  tend to  $\xi_0$  more quickly. Suppose that  $|f''(x)| \leq \mu$  in  $I$  and let us take  $z_n = x_n$  for each  $n$ . Taylor's formula then yields

$$-f'(x_n)(\xi_0 - x_{n+1}) = f(\xi_0) - f(x_n) - f'(x_n)(\xi_0 - x_n) = \frac{1}{2}(\xi_0 - x_n)^2 f''(\theta)$$

where  $\theta \in I$ , hence

$$(4.3.2) \quad |\xi_0 - x_{n+1}| \leq \frac{\lambda\mu}{2} |\xi_0 - x_n|^2.$$

Put  $q = \frac{1}{2}\lambda\mu$ ; since  $|\xi_0 - x_0| \leq c$ , it is easily deduced from (4.3.2) by induction on  $n$  that

$$(4.3.3) \quad |\xi_0 - x_n| \leq \frac{1}{q} (cq)^{2^n}$$

which converges very quickly to 0 if  $cq = \frac{1}{2}c\lambda\mu < 1$ . If, for example,  $q > 1$ ,  $cq < 1/10$ , we obtain  $2^n$  exact places of decimals at the  $n^{\text{th}}$  operation. For instance, if  $f(x) = x^2 - N$ , where  $N > 1$ , taking  $x_0 - c > 1$ , we have  $|f'(x)| \geq 2$ ,  $f''(x) = 2$  for  $x \geq x_0 - c$ . Starting from an approximate value  $x_0$  of  $\sqrt{N}$  accurate to  $1/10$ , the sequence  $(x_n)$  defined by

$$x_{n+1} = \frac{1}{2} \left( x_n + \frac{N}{x_n} \right)$$

gives for  $x_n$  a value within  $2 \cdot 10^{-2^n}$  of  $\sqrt{N}$ .

## APPENDIX

## Separation of the roots of a polynomial

In this Appendix we consider polynomials of a *complex* variable with *complex* coefficients; we know by the Fundamental Theorem of Algebra (cf. VI, 9.4) that such a polynomial of degree  $n$  can be written  $c \prod_{k=1}^n (z - z_k)$  where  $c \neq 0$ . The number of indices  $k$  for which  $z_k$  has the same value  $\zeta$  is called the *order of multiplicity* of the root  $\zeta$  of the polynomial. When we speak, without being more precise, of the *number of roots* of the polynomial contained in a set  $E \subset \mathbf{C}$ , we understand that the roots are counted "with their order of multiplicity"—that is, we mean the number of the indices  $k$  for which  $z_k \in E$ .

## 1. Resultant of two polynomials

Let

$$(1.1) \quad \begin{aligned} f(z) &= a_0 z^n + a_1 z^{n-1} + \cdots + a_n \\ g(z) &= b_0 z^m + b_1 z^{m-1} + \cdots + b_m \end{aligned}$$

be two polynomials of degrees  $n \geq 1$ ,  $m \geq 1$ , with  $a_0 \neq 0$ ,  $b_0 \neq 0$ ; let us show that  $f$  and  $g$  have common roots if, and only if, there exist two non-null polynomials  $h(z)$ ,  $k(z)$  with  $\deg(h) \leq m-1$ ,  $\deg(k) \leq n-1$ , satisfying the following identity:

$$(1.2) \quad h(z)f(z) = k(z)g(z).$$

To say that  $f$  and  $g$  have at least one common root, implies that there exists a polynomial  $\varphi(z)$  non-null of degree  $\geq 1$  such that

$$f(z) = \varphi(z)f_1(z), \quad g(z) = \varphi(z)g_1(z)$$

and it is then sufficient to take  $h(z) = g_1(z)$ ,  $k(z) = f_1(z)$  to satisfy (1.2). Conversely, if the identity (1.2) is satisfied and if  $f$  and  $g$  are relatively prime, then  $f$  divides  $k$ , which is absurd, since  $k \neq 0$  and  $\deg(k) < \deg(f)$ .

If we write

$$\begin{aligned} h(z) &= u_0 z^{m-1} + u_1 z^{m-2} + \cdots + u_{m-1} \\ k(z) &= v_0 z^{n-1} + v_1 z^{n-2} + \cdots + v_{n-1} \end{aligned}$$

then on identifying the two members of (1.2), a system of  $m+n$  linear equations is obtained for the  $u_j$  and  $v_j$

$$(1.3) \quad \begin{array}{rcl} a_0 u_0 & = & b_0 v_0 \\ a_1 u_0 + a_0 u_1 & = & b_1 v_0 + b_0 v_1 \\ \cdot & & \cdot \\ \cdot & & \cdot \\ a_n u_{m-2} + a_{n-1} u_{m-1} & = & b_m v_{n-2} + b_{m-1} v_{n-1} \\ a_n u_{m-1} & = & b_m v_{n-1} \end{array}$$

and the condition for  $f$  and  $g$  to have at least one common root is, therefore, that the determinant of this system *vanish*. The determinant is written up to sign

$$(1.4) \quad R(f, g) = \begin{vmatrix} a_0 & a_1 & \dots & a_n & 0 & 0 & \dots & 0 \\ 0 & a_0 & \dots & a_{n-1} & a_n & 0 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \dots & a_0 & a_1 & a_2 & \dots & a_n \\ b_0 & b_1 & \dots & b_{m-1} & b_m & 0 & \dots & 0 \\ 0 & b_0 & \dots & b_{m-2} & b_{m-1} & b_m & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \dots & b_0 & b_1 & \dots & \dots & b_m \end{vmatrix}$$

and is called the *Sylvester Resultant* of  $f$  and  $g$ .

## 2. Number of roots of a polynomial in a half-plane

We shall show that the number of roots of a polynomial of degree  $\geq 1$  situated in the half-plane  $\mathcal{J}z > 0$ , can generally be determined by evaluating the *number of positive squares* in a Hermitian form canonically associated with the polynomial, following a method due to Hermite.

Consider the polynomial

$$(2.1) \quad f(z) = a_0 + a_1 z + \dots + a_n z^n$$

and associate with  $f$  the polynomial

$$(2.2) \quad f^*(z) = \overline{f(\bar{z})} = \bar{a}_0 + \bar{a}_1 z + \dots + \bar{a}_n z^n.$$

If  $f(z) = a_n \prod_{k=1}^n (z - z_k)$  then  $f^*(z) = \bar{a}_n \prod_{k=1}^n (z - \bar{z}_k)$ , which can be expressed briefly by saying that the roots of  $f^*$  are the *conjugates of those of  $f$* . Now form the polynomial symmetric in  $z$  and  $z'$ :

$$(2.3) \quad K(f; z, z') = \frac{f(z)f^*(z') - f(z')f^*(z)}{z - z'} = \sum_{h=0}^{n-1} \sum_{k=0}^{n-1} A_{hk} z^h z'^k$$

It is clear that

$$\begin{aligned} \overline{K(f; z, z')} &= \frac{f^*(\bar{z})f(\bar{z}') - f^*(\bar{z}')f(\bar{z})}{\bar{z} - \bar{z}'} \\ &= -K(f; \bar{z}, \bar{z}') \end{aligned}$$

hence

$$(2.4) \quad \bar{A}_{hkc} = -A_{hkc} = -A_{kch}$$

for any indices  $h$  and  $k$  between 0 and  $n-1$ . Therefore

$$(2.5) \quad H(f; u_0, u_1, \dots, u_{n-1}) = \sum_{h=0}^{n-1} \sum_{k=0}^{n-1} \frac{1}{i} A_{hkc} u_h \bar{u}_k$$

is a *Hermitian form in  $n$  variables*.

Let us now calculate the Hermitian form

$$H(f_1 f_2; u_0, u_1, \dots, u_{n-1})$$

for the product of two polynomials

$$(2.6) \quad \begin{cases} f_1(z) = b_0 + b_1 z + \dots + b_r z^r \\ f_2(z) = c_0 + c_1 z + \dots + c_s z^s \end{cases}$$

with  $r + s = n$ . Clearly

$$K(f_1 f_2; z, z') = f_2(z) f_2^*(z') K(f_1; z, z') + f_1^*(z) f_1(z') K(f_2; z, z').$$

If

$$K(f_1; z, z') = \sum_{h=0}^{r-1} \sum_{k=0}^{r-1} B_{hk} z^h z'^k$$

$$K(f_2; z, z') = \sum_{h=0}^{s-1} \sum_{k=0}^{s-1} C_{hk} z^h z'^k$$

then

$$\begin{aligned} K(f_1 f_2; z, z') &= \sum_{h=0}^{r-1} \sum_{k=0}^{r-1} B_{hk} (c_0 z^h + c_1 z^{h+1} + \dots + c_s z^{h+s}) (\bar{c}_0 z'^k + \dots + \bar{c}_s z'^{k+s}) \\ &\quad + \sum_{k=0}^{s-1} \sum_{h=0}^{s-1} C_{hk} (\bar{b}_0 z^h + \dots + \bar{b}_r z^{h+r}) (b_0 z'^k + \dots + b_r z'^{k+r}) \end{aligned}$$

Hence the identity

$$(2.7) \quad \begin{aligned} H(f_1 f_2; u_0, u_1, \dots, u_{n-1}) &= H(f_1; v_0, v_1, \dots, v_{r-1}) \\ &\quad + H(f_2; w_0, w_1, \dots, w_{s-1}), \end{aligned}$$

where

$$(2.8) \quad v_h = c_0 u_h + c_1 u_{h+1} + \dots + c_s u_{h+s} \quad (h = 0, 1, \dots, r-1),$$

$$(2.9) \quad w_k = \bar{b}_0 u_k + \bar{b}_1 u_{k-1} + \dots + \bar{b}_r u_{k+r} \quad (k = 0, 1, \dots, s-1).$$

It follows from these formulae and (1.4) that the determinant of the matrix of the coefficients of all  $n = r + s$  linear forms  $v_k, w_k$  in the  $u_j$  is the *resultant*  $R(f_1^*, f_2)$ . These forms will therefore be linearly independent if  $f_1^*$  and  $f_2$  have no common root. This will certainly be the case if  $f = f_1 f_2$  has *no real root* and *no pair of conjugate roots*.

Note, on the other hand, that for each  $\zeta \in \mathbf{C}$ , putting  $\varphi_\zeta(z) = z - \zeta$

$$K(\varphi_\zeta; z, z') = \zeta - \bar{\zeta}$$

hence

$$H(\varphi_\zeta; u_0) = (\zeta - \bar{\zeta}) u_0 \bar{u}_0.$$

This being so, the preceding formulae and (2.7) imply, by induction, that

$$(2.10) \quad H(f; u_0, u_1, \dots, u_n) = |a_n|^2 \sum_{k=1}^n \frac{1}{i} (z_k - \bar{z}_k) F_k(u_0, \dots, u_n) \overline{F_k(u_0, \dots, u_n)}$$

where the  $F_k$  are  $n$  linear forms in the  $u_j$ . The above remark shows that if  $f$  has no real root and no pair of imaginary conjugate roots, the  $F_k$  are *linearly independent*.

Suppose, on the contrary, that  $f$  has at least one real root or a pair of imaginary conjugate roots; this is the same as saying that  $f$  and  $f^*$  are not relatively prime. Let  $g$  be the highest

common factor of  $f$  and  $f^*$  and put  $f = gf_1$ ; then  $g^* = g$  and hence  $K(g; z, z') = 0$ , so from (2.7)

$$(2.11) \quad H(f; u_0, u_1, \dots, u_n) = H(f_1; w_0, \dots, w_{n-r-1})$$

if  $\deg(g) = r$ . The  $n - r$  forms  $w_0, \dots, w_{n-r-1}$  are linearly independent.

These remarks and Sylvester's "law of inertia" prove the following theorem.

(2.12) *Let  $g(z)$  be the highest common factor of the polynomials  $f(z)$  and  $f^*(z)$ ; let  $r$  be the degree of  $g$  and put  $f = gf_1$ . Then the rank of the Hermitian form  $H(f)$  is equal to  $n - r$ , and its signature  $(p, q)$  (with  $p + q = n - r$ ) is such that  $p$  is the number of roots of  $f_1(z) = 0$  contained in the half-plane  $\Re z > 0$ ,  $q$  the number of roots of this equation contained in the half-plane  $\Re z < 0$ .*

In particular

(2.13) *The equation  $f(z) = 0$  has all its roots in the half-plane  $\Re z > 0$ , if, and only if, the Hermitian form  $H(f)$  is positive definite.*

The change of variable  $z' = iz$  gives in the same way the number of roots of the polynomial  $f$  which lie in the half-plane  $\Re z > 0$ .

## • Number of real roots in an interval of a polynomial with real coefficients

Suppose now that  $f$  is a polynomial with *real* coefficients; we shall again show, following Hermite, how one can determine the number of *real roots*  $x$  of  $f$  which satisfy  $x > t$  for every real number  $t$ , by evaluating the *number of positive squares* in a *quadratic form* associated with  $f$ .

Consider, then, the polynomial with real coefficients

$$(3.1) \quad f(x) = a_0 + a_1x + \dots + a_nx^n$$

and associate with  $f$  the polynomial

$$(3.2) \quad f^0(x) = (x - t)f'(x).$$

Consider the symmetric polynomial in  $x$  and  $x'$

$$(3.3) \quad L(f; x, x') = \frac{f(x)f^0(x') - f(x')f^0(x)}{x - x'} = \sum_{h=0}^{n-1} \sum_{k=0}^{n-1} A_{hk}x^h x'^k$$

so that  $A_{kh} = A_{hk}$ . It follows that

$$(3.4) \quad Q(f; u_0, u_1, \dots, u_{n-1}) = \sum_{h=0}^{n-1} \sum_{k=0}^{n-1} A_{hk}u_h u_k$$

is a quadratic form in  $n$  real variables.

Consider now the form

$$Q(f_1 f_2; u_0, u_1, \dots, u_{n-1})$$

for the product of two polynomials with real coefficients

$$(3.5) \quad \begin{aligned} f_1(x) &= b_0 + b_1x + \dots + b_r x^r \\ f_2(x) &= c_0 + c_1x + \dots + c_s x^s \end{aligned}$$

with  $r + s = n$ . Clearly

$$L(f_1 f_2; x, x') = f_2(x)f_2(x')L(f_1; x, x') + f_1(x)f_1(x')L(f_2; x, x')$$

whence, as in Section 2

$$(3.6) \quad Q(f_1 f_2; u_0, u_1, \dots, u_{n-1}) = Q(f_1; v_0, \dots, v_{r-1}) + Q(f_2; w_0, \dots, w_{s-1})$$

where

$$(3.7) \quad v_h = c_0 u_h + c_1 u_{h+1} + \dots + c_s u_{h+s} \quad (h = 0, 1, \dots, r-1)$$

$$(3.8) \quad w_k = b_0 u_k + b_1 u_{k+1} + \dots + b_r u_{k+r} \quad (k = 0, 1, \dots, s-1).$$

It is further seen that the determinant of the matrix of the coefficients of these  $n$  forms is the resultant of  $f_1$  and  $f_2$ , therefore these forms are linearly independent if  $f_1$  and  $f_2$  have no common root, which will be the case if  $f = f_1 f_2$  has no multiple root.

We confine ourselves to the case when  $f$  has no multiple root. For each  $\xi \in \mathbf{R}$ , if  $\varphi_\xi(x) = x - \xi$

$$L(\varphi_\xi; x, x') = \xi - t \quad \text{hence} \quad Q(\varphi_\xi; u_0) = (\xi - t)u_0^2.$$

If  $\alpha \in \mathbf{R}$ ,  $\beta \in \mathbf{R}$ ,  $\beta \neq 0$ , and if  $\psi_{\alpha, \beta}(x) = (x - \alpha)^2 + \beta^2$ , then

$$Q(\psi_{\alpha, \beta}; u_0, u_1) = 2(\beta^2(\alpha + t) - t^2(\alpha - t))u_0^2 + 4(\beta^2 + t(\alpha - t))u_0 u_1 - 2(\alpha - t)u_1^2$$

whose discriminant

$$16\beta^2((\alpha + t)^2 + \beta^2)$$

is always  $> 0$ . We can decompose  $f$  into a product of factors of the first degree and of factors of the second degree with non-real conjugate roots, and so by applying induction to (3.6) and the remark on the linear independence of the forms (3.7) and (3.8), the following theorem is proved:

(3.9) *If  $f$  is a polynomial with real coefficients having no multiple root, and if  $t$  is not a root of  $f$ , the quadratic form  $Q(f)$  has rank  $n$  and signature  $(p + r, q + r)$  where  $p$  is the number of real roots  $> t$ ,  $q$  the number of real roots  $< t$ , and  $2r$  the number of non-real roots of  $f$ .*

Knowing how to majorize the absolute value of the roots of  $f$  (problem 1), the application of this result for different values of  $t$  will in principle permit the separation of the real roots of  $f$  by numerical methods.

## PROBLEMS

1. Let  $f(z) = z^n + a_1 z^{n-1} + \dots + a_n$  be a polynomial of degree  $n$  with complex coefficients; let  $f(z) = \prod_{j=1}^n (z - z_j)$  be its decomposition into factors of the first degree, and put  $r_0 = \sup_j |z_j|$ .

(a) Show that if  $r > 0$  satisfies

$$r^n \geq |a_1|r^{n-1} + |a_2|r^{n-2} + \dots + |a_{n-1}|r + |a_n|$$

then  $r_0 \leq r$ ; deduce from this that

$$r_0 \leq \sup \left( 1, \sum_{k=1}^n |a_k| \right).$$

(b) Let  $(\lambda_j)$ ,  $1 \leq j \leq n$ , be  $n$  positive numbers such that  $\sum_{j=1}^n \frac{1}{\lambda_j} = 1$ ; show that

$$r_0 \leq \sup_{1 \leq k \leq n} (\lambda_k |a_k|)^{1/k}$$

(use (a)).

(c) Deduce from (a) that if all the coefficients  $a_k$  are  $\neq 0$ , then

$$r_0 \leq \sup \left( 2|a_1|, 2 \frac{|a_2|}{|a_1|}, \dots, 2 \frac{|a_{n-1}|}{|a_{n-2}|}, \frac{|a_n|}{|a_{n-1}|} \right)$$

(d) Deduce from (a) that

$$r_0 \leq |a_1 - 1| + |a_2 - a_1| + \dots + |a_{n-1} - a_n| + |a_n|$$

(consider the polynomial  $(z - 1)f(z)$ ). Conclude from this that if the  $a_j$  are real and positive, then

$$r_0 \leq \sup \left( a_1, \frac{a_2}{a_1}, \dots, \frac{a_{n-1}}{a_{n-2}}, \frac{a_n}{a_{n-1}} \right).$$

2. Let  $a_j$  ( $0 \leq j \leq n$ ) be real numbers such that  $0 \leq a_j \leq M$ ; suppose further that  $a_n \geq 1$ . Show that for all the roots of the polynomial

$$f(z) = a_0 + a_1z + \dots + a_nz^n,$$

either  $\Re z \leq 0$ , or  $\Re z > 0$  and  $|z| \leq \frac{1}{2}(1 + \sqrt{1 + 4M})$ . (When supposing  $\Re z > 0$ , minorize  $|f(z)/z^n|$  by noting that  $|a_n + \frac{a_{n-1}}{z}| \geq 1$ .)

3. Let  $f(z)$  be a polynomial of degree  $n > 1$  all of whose roots are contained in  $\Re z > 0$ . Show that the roots of  $f'(z)$  are also contained in this half-plane. (Use contradiction by considering the logarithmic derivative  $f'/f$ .)

4. Let  $f(z)$  be a polynomial with real coefficients. Show that the roots of  $f'(z)$  are either real, or lie in the union of the discs whose diameters are the segments joining two conjugate roots of  $f(z)$ . (Use contradiction as in problem 3.)

5. Let  $f(z) = a_0z^n + a_1z^{n-1} + \dots + a_n$  be a polynomial with complex coefficients and put  $a_j = \alpha_j + i\beta_j$ , where  $\alpha_j$  and  $\beta_j$  are real. Let

$$\begin{aligned} P(z) &= \alpha_0z^n + \alpha_1z^{n-1} + \dots + \alpha_n \\ Q(z) &= \beta_0z^n + \beta_1z^{n-1} + \dots + \beta_n. \end{aligned}$$

Show that if all the zeros of  $f(z)$  are contained in the half-plane  $\Re z > 0$ , then all the zeros of  $P(z)$  and  $Q(z)$  are real. (Note that at a zero  $z$  of  $P(z)$ , or of  $Q(z)$ ,  $P(z) + iQ(z) = \pm (P(z) - iQ(z))$ , so also  $f(z) = \pm \overline{f(\bar{z})}$ , and use contradiction.)

6. Let  $f = f_0, f_1, \dots, f_r$  be  $r + 1$  continuous real functions in an interval  $[a, b] \subset \mathbf{R}$ . We say  $(f_j)_{0 \leq j \leq r}$  is a *Sturm sequence* if it satisfies the following conditions:

1. Each  $f_j$  vanishes at most a finite number of times in  $[a, b]$ .
2. If  $f_0(x_0) = 0$ ,  $f_0$  is monotone in a neighbourhood of  $x_0$ ,  $f_1(x_0)$  is  $> 0$  if  $f_0$  is increasing, and  $< 0$  if  $f_0$  is decreasing.
3.  $f_r$  is a constant  $\neq 0$ .
4. There exist continuous functions  $g_1, \dots, g_{r-1}$  in  $[a, b]$  such that

$$\begin{aligned} f_0 &= g_1 f_1 - f_2 \\ f_1 &= g_2 f_2 - f_3 \\ &\vdots \\ f_{r-2} &= g_{r-1} f_{r-1} - f_r. \end{aligned}$$

For a sequence  $(a_j)_{1 \leq j \leq m}$  of real numbers not all zero, let  $(a_{i_k})_{1 \leq k \leq p}$  be the sequence extracted from  $(a_j)_{1 \leq j \leq m}$  formed by the non-zero terms. We call the *variation number* of the

sequence  $(a_j)$  the number of indices  $k \leq p-1$  such that  $a_{i_k}$  and  $a_{i_{k+1}}$  are of opposite sign.

For each  $x \in [a, b]$ , write  $w(x)$  for the variation number of the sequence

$$f_0(x), f_1(x), \dots, f_r(x).$$

Show that, if  $f_0(a) \neq 0$  and  $f_0(b) \neq 0$ , the number of roots of the equation  $f_0(x) = 0$  in  $[a, b]$  is equal to  $w(a) - w(b)$ . (Examine for increasing  $x$  the variation of  $w(x)$  at a root of the equation  $f_j(x) = 0$ .)

Apply to the case where  $f$  is a polynomial with real coefficients, and  $f_1$  its derivative (determine the polynomials  $f_2, \dots, f_r$  by Euclidean division). Consider the case where  $f$  has multiple roots (*Sturm's theorem*).

7. Let  $f$  be a polynomial of degree  $n > 0$ ; for each  $x \in \mathbf{R}$  let  $v(x)$  be the variation number of the sequence  $(f^{(k)}(x))_{0 \leq k \leq n}$  (with  $f^{(0)} = f$ ). If  $\nu$  is the number of roots of  $f(x) = 0$  in the interval  $[a, b]$ , where  $f(a) \neq 0$  and  $f(b) \neq 0$ , show that  $\nu \leq v(a) - v(b)$  and that the difference  $v(a) - v(b) - \nu$  is even. (Partition the interval  $[a, b]$  by means of the roots of all  $f^{(k)}(x) = 0$ , and proceed as in problem 6.) Deduce that if  $f(x) = a_0 + a_1x + \dots + a_nx^n$  with  $a_0 \neq 0$ , then the number of roots  $> 0$  of  $f$  is at most the variation number of the sequence  $(a_k)_{0 \leq k \leq n}$  and that the difference of these two numbers is even (*Descartes' rule*).

8. Let  $a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n$  be real numbers such that  $a_j \neq 0$  and  $b_1 < b_2 < \dots < b_n$ . Show that the number of real roots of the polynomial

$$f(x) = a_1(x - b_1)^m + a_2(x - b_2)^m + \dots + a_n(x - b_n)^m \quad (m \text{ integer } \geq 1)$$

is at most equal to the variation number of the sequence

$$a_1, a_2, \dots, a_n, (-1)^m a_1$$

and that the difference of these two numbers is even. (Use induction on  $n$ , considering the derivative of  $f(x)/(x - b_n)^m$ .)

9. Let  $J = [a, b]$  be an interval of  $\mathbf{R}$  in which the twice continuously differentiable function  $f$  satisfies the relations  $|f'(x)| \geq m$ ,  $|f''(x)| \leq M$ ,  $f(a)$  and  $f(b)$  have opposite sign. Show that if  $(M/4m)(b - a) = q < 1$ , one can, by  $n$  successive applications of the linear method of approximation, find an interval with endpoints  $a_n, b_n$  containing the only root of  $f(x) = 0$  in  $[a, b]$ , with

$$|b_n - a_n| \leq \frac{4m}{M} q^{2^n}.$$

10. With the notations of the Appendix, equations (1.1), show that if

$$f(z) = a_0(z - \xi_1) \dots (z - \xi_n), \quad g(z) = b_0(z - \eta_1) \dots (z - \eta_m)$$

then

$$R(f, g) = a_0^m b_0^n \prod_{\substack{1 \leq j \leq n \\ 1 \leq k \leq m}} (\xi_j - \eta_k).$$

(Observe that we can consider, for  $a_0$  and  $b_0$  fixed,  $f$  and  $g$  as functions of the variables  $\xi_j$  and  $\eta_k$  respectively, and that  $R(f, g)$  is divisible by all the polynomials  $\xi_j - \eta_k$ .)

11. For a polynomial with complex coefficients

$$f(z) = a_0 z^n + a_1 z^{n-1} + \dots + a_n$$

write

$$H(f) = \sup (|a_0|, |a_1|, \dots, |a_n|)$$

(the *height* of  $f$ ).

(a) For each complex number  $A$  satisfying  $|A| \geq 1$ , let  $f_A(z) = f(z - A)$ ; show that

$$H(f_A) \leq (n+1)(|A|+1)^n H(f).$$

(b) Let  $\xi$  be a zero of  $f$ , and let  $\zeta$  be a point of  $\mathbf{C}$  such that  $|\zeta| \geq 1$  and  $|\xi - \zeta| \leq 1$ . Show that

$$|f(\zeta)| \leq 2^n n |\zeta|^n H(f) |\xi - \zeta|$$

(majorize  $|f(\zeta) - f(\xi)| = |f(\zeta)|$  with the help of the mean-value theorem, and note that  $|z| \leq 2|\xi|$  on the segment joining  $\xi$  and  $\zeta$ ).

12. Let  $\xi_1, \dots, \xi_n$  be any  $n$  complex numbers. Show that there exists a complex number  $A$  such that  $1 \leq |A| \leq \sqrt{n+1}$ , and such that

$$\inf_{1 \leq j \leq n} |\xi_j - A| \geq 1$$

(consider the  $n+1$  discs with centres  $0, \xi_1, \dots, \xi_n$  and radius 1, and observe that the sum of their areas is  $(n+1)\pi$ . Deduce from this that there exists at least one point  $A$  belonging to none of these discs, and belonging to the disc  $|z| \leq \sqrt{n+1}$ ). If all the  $\xi_j$  are real, one can take  $A = i$ .

13. With the notations of problem 10, suppose that  $R(f, g) \neq 0$ , so that the zeros  $\xi_j$  of  $f$  are distinct from the zeros  $\eta_k$  of  $g$ . Put

$$\Delta(f, g) = \inf_{\substack{1 \leq j \leq n \\ 1 \leq k \leq m}} |\xi_j - \eta_k|.$$

Show that, either

$$\Delta(f, g) \geq 1$$

or

$$\Delta(f, g) \geq (C(f, g) H(f)^m H(g)^n)^{-1} |R(f, g)|$$

where

$$C(f, g) = (m+1)^{2n} (n+1)^{2m} 4^{mn} c(f, g)^{mn}$$

$c(f, g)$  being equal to 1 if at least one of the polynomials  $f, g$  has all its roots real, and otherwise  $c(f, g)$  is equal to the smallest of the two numbers  $m+1, n+1$ . (It is enough to consider the case where  $\Delta(f, g) \leq 1$  and where  $\Delta(f, g) = |\xi_1 - \eta_1|$ . Suppose first that  $|\eta_k| \geq 1$  for every  $k$ . Write  $R(f, g) = b_0^n \prod_{k=1}^m f(\eta_k)$ ; show first that

$$|R(f, g)| \leq |b_0|^n |f(\eta_1)| \prod_{k=2}^m ((n+1)|\eta_k|^n H(f))$$

then majorize  $|f(\eta_1)|$  with the help of problem 11(b), in terms of  $\Delta(f, g)$ . Pass to the general case by using problems 11(a) and 12.)

14. Let  $g(\mathbf{x})$  be a function defined in a cube  $K: \|\mathbf{x} - \mathbf{x}_0\| \leq c$  in  $\mathbf{R}^n$ , continuously differentiable and with values in  $\mathbf{R}^n$ , such that

$$g(\mathbf{x}) = (g_j(\mathbf{x}))_{1 \leq j \leq n},$$

where the  $g_j$  are scalar functions continuously differentiable in  $K$ . Suppose there exists  $q$ , such that  $0 < q < 1$ , with the following properties:

1.  $\left| \frac{\partial g_j}{\partial x_k} \right| \leq \frac{q}{n}$  in  $K$  for  $1 \leq j \leq n$ ,  $1 \leq k \leq n$ .
2.  $\|g(\mathbf{x}_0) - \mathbf{x}_0\| \leq c(1 - q)$ .

Then there exists one, and only one, root  $\mathbf{z}$  of the vector equation  $\mathbf{g}(\mathbf{x}) = \mathbf{x}$  in  $K$  (generalize the method of (3.3)).

15. (a) Let  $P(x)$  be a polynomial of even degree  $n$  with complex coefficients. Show that  $P(x)P(-x) = P_1(x^2)$ , where  $P_1(x)$  is a polynomial of degree  $n$ ; if  $P(x) = a_0 \prod_{k=1}^n (x - x_k)$ , then  $P_1(x) = a_0^2 \prod_{k=1}^n (x - x_k^2)$ . Define inductively  $P_k(x)$  by the condition  $P_{k-1}(x)P_{k-1}(-x) = P_k(x^2)$ .

(b) Suppose that  $|x_1| > |x_2| > \dots > |x_n|$ . If

$$P_k(x) = a_0^{(k)}x^n + a_1^{(k)}x^{n-1} + \dots + a_n^{(k)}$$

show that  $|x_j| = \lim_{k \rightarrow \infty} |a_j^{(k)} / a_{j-1}^{(k)}|^{2^{-k}}$ . (*Graeffe's method* for calculating the absolute values of the roots of a polynomial; consider the elementary symmetric functions of the roots of  $P_k(x)$ .)

# Asymptotic developments

## Introduction

We shall use (by abuse of language) the expression “neighbourhood on the right” of a point  $x_0 \in \mathbf{R}$  to designate a semi-open interval of the form  $]x_0, x_0 + h]$  ( $h > 0$ ). Similarly a “neighbourhood on the left” of  $x_0$  will mean an interval  $[x_0 - h, x_0[$  ( $h > 0$ ), a “neighbourhood of  $+\infty$ ” will be an interval of the form  $[A, +\infty[$ , and a “neighbourhood of  $-\infty$ ” an interval of the form  $]-\infty, -A]$  ( $A > 0$ ). In numerous problems in Analysis we have to study the “behaviour” of a complex function  $f$  “in the neighbourhood of  $x_0$ ”; we must make our precise meaning clear. In the first place we are interested in the properties which occur in a *non-specified* neighbourhood (on the right or left) of  $x_0$ ; from this point of view, two functions which *coincide in a neighbourhood* of  $x_0$  are thus identical, even if they differ outside this neighbourhood. An example of such a property is the *existence of a limit* (finite or infinite) of  $f$  on the right or on the left at the point  $x_0$ .

In the above,  $x_0$  can be replaced by  $+\infty$  or by  $-\infty$ . In fact we shall systematically study functions *in the neighbourhood of*  $+\infty$ . This will not be a restriction of generality, since the study of a function  $f(x)$  in the neighbourhood of  $x_0$  on the right, for example, is the same as the study of the function  $g(x) = f(x_0 + (1/x))$  in the neighbourhood of  $+\infty$ , and the study of  $f(x)$  in the neighbourhood of  $-\infty$  is the same as the study of  $f(-x)$  in the neighbourhood of  $+\infty$ .

When it is known that a function  $f(x)$  has a limit as  $x$  tends to  $+\infty$  (which is not necessarily the case, e.g.  $f(x) = \sin x$ ) this does not in general constitute sufficient information to treat of problems in which  $f(x)$  occurs. For example, the four functions

$$(1.1) \quad x, \quad x^2, \quad \sqrt{x}, \quad x + \frac{1}{2} \sin \pi x$$

all tend to  $+\infty$  as  $x$  tends to  $+\infty$ . But if we consider the expression  $f(x+1) - f(x)$  for each of these functions, we obtain respectively

$$1, \quad 2x + 1, \quad \sqrt{x+1} - \sqrt{x} = \frac{1}{\sqrt{x+1} + \sqrt{x}}, \quad 1 - \sin \pi x.$$

The first three tend respectively to 1,  $+\infty$ , 0, and the fourth has no limit.

In order to go further we must *compare* functions to functions whose behaviour in the neighbourhood of  $+\infty$  is considered *known*.

## 2. Functions of comparison

We regard as *known* in the neighbourhood of  $+\infty$ , functions of one of the following types

$$1, \quad x^\alpha \ (\alpha \neq 0), \quad (\log x)^\beta \ (\beta \neq 0), \quad e^{cx^\gamma} \ (c \neq 0, \gamma > 0)$$

(where  $\alpha, \beta, \gamma, c$  are real constants) as well as their products, that is to say functions of the form

$$(2.1) \quad g(x) = x^\alpha (\log x)^\beta e^{P(x)}$$

(where  $P(x) = c_1 x^{\gamma_1} + c_2 x^{\gamma_2} + \dots + c_k x^{\gamma_k}$ , with  $\gamma_1 > \gamma_2 > \dots > \gamma_k > 0$ ,  $\alpha, \beta$ , and all  $c_j$  being real constants of any sign whatever.) The set  $\mathcal{E}$  of these functions has the following properties:

- (2.2) (i) *Each function of  $\mathcal{E}$  is  $> 0$  in a neighbourhood of  $+\infty$ .*  
 (ii) *Each function of  $\mathcal{E}$ , other than the constant 1, tends either to 0 or to  $+\infty$  as  $x$  tends to  $+\infty$ .*  
 (iii) *Every product of functions in  $\mathcal{E}$  belongs to  $\mathcal{E}$ ; if  $f \in \mathcal{E}$ , then  $f^\lambda \in \mathcal{E}$  for every real exponent  $\lambda$  (and in particular the quotient of two functions in  $\mathcal{E}$  is in  $\mathcal{E}$ ).*

The assertions (i) and (iii) are trivial. To prove (ii) for a function  $f \neq 1$ , it can be assumed that in  $P$  the coefficient  $c_1$  is  $\geq 0$ , otherwise consider  $1/f$ . If  $P = 0$  (all the  $c_j$  being zero) assertion (ii) is classical ("a power is stronger than a logarithm", K-R, p. 12). We may therefore suppose that  $c_1 > 0$ , and can write

$$P(x) = c_1 x^{\gamma_1} (1 + b_2 x^{\gamma_2 - \gamma_1} + \dots + b_k x^{\gamma_k - \gamma_1})$$

where the factor in parentheses tends to 1. Hence  $P(x) \geq c_1 x^{\gamma_1}/2$  in the neighbourhood of  $+\infty$ , and we can confine ourselves to the case where  $P(x) = cx^\gamma$  with  $c > 0, \gamma > 0$ . Replacing  $f$  by  $f^{1/c}$ , then  $x^\gamma$  by  $y$ , we finally have to prove that  $y^\alpha (\log y)^\beta e^y$  tends to  $+\infty$  with  $y$ , for any real constants  $\alpha$  and  $\beta$ ; but this is evident, since, as soon as  $y$  is sufficiently large,  $e^{y/4} \geq y^{-\alpha}$  and  $e^{y/4} \geq y \geq (\log y)^{-\beta}$ , so  $y^\alpha (\log y)^\beta e^y \geq e^{y/2}$ .

(2.3) From now on we shall use almost exclusively the properties of  $\mathcal{E}$  enumerated in (2.2) and everything stated will therefore apply to any set of functions which (in the neighbourhood of  $+\infty$ ) have these properties. It is easy to form such functions not of the form (2.1). For example, products of functions of the type (2.1) and of "repeated logarithms":  $(\log \log x)^\lambda$ ,  $(\log \log \log x)^\mu \dots$ , or of "repeated exponentials":  $\exp(a \exp(bx^\gamma))$ ,  $\exp(a \exp(b \exp(cx^\gamma)))$ ,  $\dots$ , or of functions of the form  $\exp(x^\alpha (\log x)^\beta)$  (which includes  $x^x = e^{x \log x}$ ), etc.  $\dots$

(2.4) When studying the behaviour of a function in the neighbourhood (on the right for instance) of a point  $x_0 \in \mathbf{R}$ , we shall also consider known the functions  $f(1/(x - x_0))$  where  $f \in \mathcal{E}$ , in other words the functions

$$(2.4.1) \quad (x - x_0)^\alpha |\log(x - x_0)|^\beta e^{P(1/(x - x_0))}$$

( $\alpha, \beta$ , and the  $c_j$  any real numbers). Observe that the exponents of  $(x - x_0)$  in  $P(1/(x - x_0))$  are strictly negative; the function  $e^{x^{-\gamma} - x_0}$  does not belong to (2.4.1) (see (6.4)).

We shall often have to study the behaviour of *sequences*  $(u_n)$ , recalling that a sequence is just a *function*  $n \rightarrow u_n$  defined on the set  $\mathbf{N}$  of integers  $\geq 0$  (or in an interval  $[n_0, +\infty[$  of this set). The restrictions of the functions of  $\mathcal{E}$  to the interval  $[2, +\infty[$  of  $\mathbf{N}$  are considered “known”.

## Relations of comparison

The functions of  $\mathcal{E}$  being considered known *in the neighbourhood of*  $+\infty$ , we consider also known in the neighbourhood of  $+\infty$  every complex function of the form  $f = c.g$ , where  $c$  is a complex constant  $\neq 0$  and  $g \in \mathcal{E}$ . This leads to the following idea for studying the behaviour of a function  $f$  in the neighbourhood of  $+\infty$ : find  $g \in \mathcal{E}$  such that the quotient  $f/g$  has a *non-zero* finite limit  $c$ . When this is the case, the function  $c.g$  is said to be the *principal part of*  $f$ , relative to  $\mathcal{E}$ , and this is written

$$(3.1) \quad f \sim c.g \quad \text{or} \quad f(x) \sim cg(x).$$

More generally, if  $g$  is a function (not necessarily belonging to  $\mathcal{E}$ ) such that  $g(x) \neq 0$  in a neighbourhood of  $+\infty$ , it will be said that  $f$  is *equivalent to*  $g$  in the neighbourhood of  $+\infty$  and we write  $f \sim g$  if the quotient  $f(x)/g(x)$  tends to 1 as  $x$  tends to  $+\infty$ . It is clear that if  $f \sim g$  and  $g \sim h$  then  $f \sim h$ . A function of  $\mathcal{E}$  is obviously its own principal part. For example, of the functions (1.1), the first three are in  $\mathcal{E}$ , and for the fourth  $x + \frac{1}{2} \sin \pi x \sim x$ .

It is possible that for every function  $g \in \mathcal{E}$ , the ratio  $f/g$  does not tend to any limit, or tends to 0, or that  $|f|/g$  tends to  $+\infty$ . To express these last two possibilities, and to shorten the calculation of principal parts (when they exist), the following notations are introduced.

(3.2) Given a function  $g \in \mathcal{E}$  (or more generally a function such that  $g(x) > 0$  in the neighbourhood of  $+\infty$ ), then if the function  $|f|/g$  is *majorized by a constant* in the neighbourhood of  $+\infty$  this is expressed by

$$(3.2.1) \quad f = O(g) \quad (\text{Landau's notation})$$

or

$$(3.2.2) \quad f \leqslant g \quad (\text{or } g \geqslant f) \quad (\text{Hardy's notations}).$$

For example,  $x \sin x = O(x)$ .

If the function  $f/g$  tends to 0 as  $x$  tends to  $+\infty$  this is written

$$(3.2.3) \quad f = o(g) \quad (\text{Landau's notation})$$

or

$$(3.2.4) \quad f \ll g \quad (\text{Hardy's notation})$$

and  $f$  is said to be *negligible compared to*  $g$ . It is clear that  $f \ll g$  implies  $f \leqslant g$ .

Lastly, if  $|f|/g$  tends to  $+\infty$  (which implies that  $f$  does not vanish in a neighbourhood of  $+\infty$ ) this is written

$$(3.2.5) \quad g = o(|f|) \quad (\text{Landau's notation})$$

or

$$(3.2.6) \quad |f| \gg g \quad (\text{Hardy's notation})$$

and we say that  $f$  dominates  $g$ .

In particular, to write  $f = O(1)$  or  $f \leq 1$  is to say that  $f$  is bounded in the neighbourhood of  $+\infty$ , and to write  $f = o(1)$  or  $f \ll 1$  is to say that  $f$  tends to 0 as  $x$  tends to  $+\infty$ . The relation (3.1) is equivalent to

$$(3.2.7) \quad f = c.g + o(g) \quad \text{or} \quad f - c.g \ll g.$$

*Remarks (3.3)* We shall use the preceding notations, with the same significance, in the neighbourhood (on the right or on the left) of a point  $x_0 \in \mathbf{R}$ , or in the neighbourhood of  $-\infty$ . The functions of  $\mathcal{E}$  must of course be replaced by those which can be deduced by the appropriate changes of variables indicated in the introduction of this chapter; for example, in the neighbourhood on the right of  $x_0$ , these will be the functions  $(x - x_0)^\alpha$ ,  $|\log(x - x_0)|^\beta$ ,  $\exp(c/(x - x_0)^\gamma)$  (with  $\gamma > 0$ ) and their products.

(3.4) These notations will also be used for sequences  $(u_n)$  of complex numbers, that is the functions defined on the set  $\mathbf{N}$  of integers (or in an interval  $[n_0, +\infty[$  of this set); the functions of  $\mathcal{E}$  must then be replaced by their restrictions to an interval  $[n_0, +\infty[$  of  $\mathbf{N}$ .

(3.5) One should note the difference between the problems of comparison studied in this chapter and the problems of majorization which form the basis of numerical computations. The relation  $f(x) = O(x^2)$  does not enable us to majorize the value  $f(x)$  for a given  $x$ , since it only means that for a certain constant  $A > 0$  and a certain interval  $[x_0, +\infty[$ , both unknown,  $|f(x)| \leq Ax^2$  for  $x \geq x_0$ . However, it will be noticed that in practice most relations of comparison are obtained by methods which, applied more carefully, can yield genuine majorizations.

#### 4. Computation with relations of comparison

(4.1) The following rules for handling the symbols introduced in no. 3 are evident consequences of the definitions:

(4.1.1) (*Transitivity*) If  $f \leq g$  and  $g \leq h$ , then  $f \leq h$ . If  $f \ll g$  and  $g \leq h$ , or if  $f \leq g$  and  $g \ll h$ , then  $f \ll h$ .

(4.1.2) If  $f_1 \leq g$  and  $f_2 \leq g$ , then  $f_1 \pm f_2 \leq g$  and  $c.f_1 \leq g$  for every complex constant  $c$ . If  $f_1 \ll g$  and  $f_2 \ll g$ , then  $f_1 \pm f_2 \ll g$  and  $c.f_1 \ll g$  for every complex constant  $c$ .

With Landau's notation this is written

$$\begin{aligned} O(g) + O(g) &= O(g), & c.O(g) &= O(g) \\ o(g) + o(g) &= o(g), & c.o(g) &= o(g). \end{aligned}$$

This form is an abuse of notation, since the symbols  $O(g)$  and  $o(g)$  do not represent well-defined functions. These relations will therefore be considered as a convenient shorthand and care will be taken not to calculate mechanically with these symbols, as if they were numbers or functions: from the relation  $O(g) + O(g) = O(g)$  it would be absurd to deduce  $O(g) = 0$ !

(4.1.3) If  $f_1 \leq g_1$  and  $f_2 \leq g_2$ , then  $f_1 f_2 \leq g_1 g_2$ . If  $f_1 \ll g_1$  and  $f_2 \leq g_2$ , then  $f_1 f_2 \ll g_1 g_2$ . If  $f \leq g$  (resp.  $f \ll g$ ), then  $|f|^\lambda \leq g^\lambda$  (resp.  $|f|^\lambda \ll g^\lambda$ ) for every  $\lambda > 0$ .

With Landau's notation this is written

$$\begin{aligned} O(g_1)O(g_2) &= O(g_1 g_2), & o(g_1)O(g_2) &= o(g_1 g_2), \\ |O(g)|^\lambda &= O(g^\lambda), & |o(g)|^\lambda &= o(g^\lambda). \end{aligned}$$

Observe that there is no general rule for *quotients*, since if  $f \leq g$ , the function  $f$  can vanish in every neighbourhood of  $+\infty$ , as is shown for example in the case  $\sin x = o(x)$ . However, if we suppose that  $f(x) > 0$  in a neighbourhood of  $+\infty$ , then the relation  $f \leq g$  is equivalent to  $1/g \leq 1/f$  and the relation  $f \ll g$  to  $1/g \ll 1/f$ .

(4.1.4) Let  $g$  be a function of  $\mathcal{E}$ . If  $f_1 \sim c_1 g, f_2 \sim c_2 g$  ( $c_1, c_2$  complex constants  $\neq 0$ ), then  $f_1 + f_2 \sim (c_1 + c_2)g$  if  $c_1 + c_2 \neq 0$ ; if  $c_1 + c_2 = 0$ , then  $f_1 + f_2 \ll g$ .

(4.1.5) Let  $g_1, g_2$  be two functions of  $\mathcal{E}$ . If  $f_1 \sim c_1 g_1, f_2 \sim c_2 g_2$  ( $c_1, c_2$  complex constants  $\neq 0$ ), then  $f_1 f_2 \sim c_1 c_2 (g_1 g_2)$  and  $f_1/f_2 \sim (c_1/c_2)(g_1/g_2)$ . If  $g \in \mathcal{E}$  and  $f \sim c \cdot g$  with  $c > 0$ , then  $f^\lambda \sim c^\lambda g^\lambda$  for every  $\lambda > 0$ .

(4.2) When it is known how to compare two functions  $f, g$  in the neighbourhood of  $+\infty$ , one often has to compare the *exponentials*  $e^f, e^g$ , or the *logarithms*  $\log |f|, \log |g|$  of these functions. This calls for some caution. In the first place the relation  $f \leq g$  (and even the relation  $f \sim g$ ) does not necessarily imply  $e^f \leq e^g$ : for example  $x^2 + x \sim x^2$ , but  $e^{x^2+x}/e^{x^2}$  tends to  $+\infty$ .

(4.2.1) If  $f \ll g$  and if  $g(x)$  tends, with  $x$ , to  $+\infty$ , then for each  $\delta > 0$ ,  $e^{-\delta g} \ll e^{\mathcal{R}f} \ll e^{\delta g}$ .

Since  $|\mathcal{R}f| \leq |f| \ll g$ , we can confine ourselves to the case when  $f$  is real. We have to prove that  $e^{\pm f - \delta g}$  tends to 0, and so that  $\pm f - \delta g$  tends to  $-\infty$ ; but, since  $\pm f - \delta g = -\delta g(1 \pm (1/\delta)(f/g))$ , this results immediately from the fact that  $g$  tends to  $+\infty$  and  $f/g$  tends to 0.

This result does not extend to the case when  $g$  remains bounded; for example,  $1/x \ll 1$  but  $e^{1/x}/e$  tends to  $1/e$ .

It should be noted that the relation  $f \ll g$  does not necessarily imply that  $\log |f| \ll \log g$ ; for example,  $x \ll x^2$ , but the ratio  $\log x/\log x^2$  is equal to  $\frac{1}{2}$ . On the other hand:

(4.2.2) Suppose that  $g(x)$  tends to  $+\infty$ . Then, if  $f \leq g$  and if there is a constant  $a > 0$  such that  $|f| \geq a$  in the neighbourhood of  $+\infty$ ,  $\log |f| \leq \log g$ . If  $f \sim c \cdot g$  with  $c \neq 0$ ,  $\log |f| \sim \log g$ .

To prove the first assertion, note that  $\log |f| \geq \log a$  and  $|f| \leq Ag$  for some constant  $A > 0$  in the neighbourhood of  $+\infty$ . From the second relation we deduce that  $\log |f| \leq \log g + \log A \leq 2 \log g$  in some neighbourhood of  $+\infty$ , since  $\log g$  tends to  $+\infty$ . Thus  $|\log |f|| \leq 2 \log g$  in some neighbourhood of  $+\infty$ . The second assertion results from the fact that  $\log |f| - \log g$  tends to  $\log |c|$ , hence  $\log |f| - \log g \ll \log g$ .

## • Order relation in $\mathcal{E}$

In the set  $\mathcal{E}$  of functions of the type (2.1) the relation

$$R(f, g): \quad "f = g \quad \text{or} \quad f \ll g"$$

is a *total ordering*. This means that:

1. The relation is *reflexive*, i.e.  $R(f, f)$  is true for every function  $f \in \mathcal{E}$ .
2. The relation is *transitive*, i.e. if one has  $R(f, g)$  and  $R(g, h)$  for three functions in  $\mathcal{E}$ , then one also has  $R(f, h)$ ; this follows from (4.1.1).
3. For any two functions  $f, g$  in  $\mathcal{E}$ , either  $R(f, g)$  or  $R(g, f)$ , and the two relations hold simultaneously only if  $f = g$ . This follows from (2.2, (ii) and (iii)): the function  $f/g$  belongs to  $\mathcal{E}$ , and if it is not the constant 1, it tends to 0 or to  $+\infty$ .

It is tempting to try to represent graphically this order relation. If we confine ourselves to functions of the form  $x^\alpha$  ( $\alpha$  any real constant), this is easy, since from the definition the relation  $R(x^\alpha, x^\beta)$  is equivalent to  $\alpha \leq \beta$ . We thus have a satisfactory representation by associating the number  $\alpha$  with  $x^\alpha$ , which leads us to picture the  $x^\alpha$  as the points of a line (Fig. 5) where the origin corresponds to a "cut" between the functions which tend to  $+\infty$  and those which tend to 0. However, even if we only consider functions of the form  $x^\alpha (\log x)^\beta$ , the situation becomes much less intuitive. Indeed,  $(\log x)^\beta \ll x^\alpha$  for any  $\beta \in \mathbf{R}$  and any  $\alpha > 0$ , and in the same way  $(\log x)^\beta \gg x^{-\alpha}$  for any  $\alpha > 0$ . One must therefore imagine in the preceding graphic representation that in place of the origin there is a "hole", and that into this hole there is inserted a *whole line* representing all the  $(\log x)^\beta$  (Fig. 5). In fact there is a similar hole for each real number  $\lambda$ , since for *any*  $\beta$  the

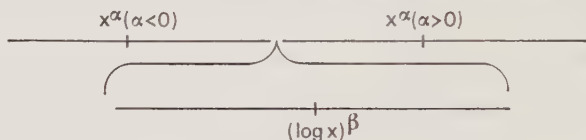


FIGURE 5

$x^\lambda (\log x)^\beta$  are inserted between all the  $x^{\lambda-\varepsilon}$  and all the  $x^{\lambda+\varepsilon}$ , for *each*  $\varepsilon > 0$  (Fig. 6). A more easily understandable representation is obtained by introducing the notion of "lexicographic order". Consider  $x^\alpha (\log x)^\beta$  as a "word"  $(\alpha, \beta)$  with two "letters"  $\alpha, \beta$ , and classify these words as in a dictionary, with the difference that the letters can take all

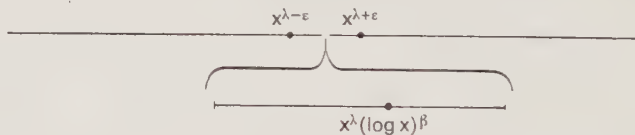


FIGURE 6

real values. Thus if  $(\alpha, \beta)$  and  $(\alpha', \beta')$  are compared, look initially at the first "letter", and if  $\alpha < \alpha'$ ,  $(\alpha, \beta) < (\alpha', \beta')$  whatever  $\beta$  and  $\beta'$  are. If, on the other hand,  $\alpha = \alpha'$ , look at the second "letter", and if  $\beta < \beta'$ , then  $(\alpha, \beta) < (\alpha', \beta')$ .

If the iterated logarithms  $(\log \log x)^\gamma$  are also introduced, one must in the same way classify in "lexicographic order" the "words"  $(\alpha, \beta, \gamma)$  with three "letters" in order to represent the order relation  $R(f, g)$  between functions of the form

$$x^\alpha (\log x)^\beta (\log \log x)^\gamma;$$

and so on.

The situation is still more complicated when the exponentials are introduced; for a fixed  $\gamma > 0$ , the functions  $e^{cx^\gamma}$  are all “after  $+\infty$ ” for  $c > 0$ , and “before  $-\infty$ ” for  $c < 0$  (Fig. 7). The classification of the functions of the form  $e^{cx^\gamma} x^\alpha (\log x)^\beta$  for a fixed  $\gamma > 0$



FIGURE 7

corresponds to the lexicographic order of the “words”  $(c, \alpha, \beta)$  with three “letters”. Moreover, if  $\gamma_1 > \gamma_2 > 0$ , the functions

$$\exp (cx^{\gamma_1})x^\alpha(\log x^\beta)$$

are “after” all the functions  $\exp (c'x^{\gamma_2})x^{\alpha'}(\log x)^{\beta'}$  for  $c > 0$ , and “before” all these functions for  $c < 0$  (Fig. 8).

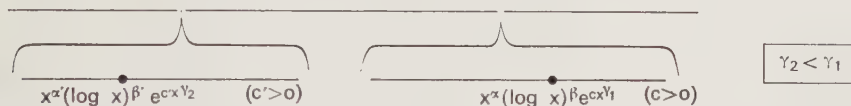


FIGURE 8

It is clear that these figures are destined only to help somewhat in conceiving an order relation much more complicated than the usual relation between numbers, and thus to guide one's calculations by preventing gross mistakes. The reader is advised to *translate* these remarks when, by a change of variable (no. 1), transforming from a neighbourhood of  $+\infty$  to a neighbourhood of 0 on the right, for example. Note that in the neighbourhood of 0, the order of greatness, for the powers  $x$ , is *opposite* to that of the exponents, whereas it remains the *same* for the  $|\log |x||^\beta$  and the  $e^{cx^{-\gamma}}$  ( $\gamma$  fixed *positive*).

## Asymptotic developments

(6.1) Suppose a *scale of comparison* is given, i.e. a set  $\mathcal{E}$  of functions satisfying the properties (2.2) in the neighbourhood of  $+\infty$ . To *compare* a complex function  $f$  to the functions of  $\mathcal{E}$  in the neighbourhood of  $+\infty$ , is in the first place to seek for a *principal part*  $c.g$  of  $f$  relative to  $\mathcal{E}$  (3.1). Since, by definition,  $c \neq 0$ , the existence of such a principal part implies that  $f$  *does not vanish* in a neighbourhood of  $+\infty$ . The “oscillating” functions such as  $\sin x$ ,  $x \sin x$ , etc. have, therefore, *no* principal part (cf. (7.6.)). However, it is possible for  $f$  not to vanish and still to have no principal part; the hypothesis  $f \sim c.g$  implies in fact (by (2.2)) that for each function  $g_1 \in \mathcal{E}$ , the ratio  $|f|/g_1$  tends to a finite or infinite limit, the function  $g$  being the *only* function  $g_1 \in \mathcal{E}$  for which this limit is  $\neq 0$  and  $\neq +\infty$ . A function such as  $1 + x^2 \sin^2 x$  or  $e^{x \sin^2 x}$ , which remains  $\geq 1$  does not satisfy the preceding conditions. Similarly, for the function  $e^{e^x}$  relative to the scale  $\mathcal{E}$  formed by the functions (2.1), the limit of  $e^{e^x}/g(x)$  being *always*  $+\infty$ . This function falls “beyond” all the functions of  $\mathcal{E}$ ; we can also give examples of functions which fall in a “hole” of  $\mathcal{E}$ , e.g.  $x^x$  or  $e^{\sqrt{\log x}}$ ; the comparison is then not possible since the scale  $\mathcal{E}$  is not large enough.

(6.2) When there is a principal part  $f \sim c_1 g_1$  of  $\mathcal{E}$ , it is *unique*, because of the preceding argument. We can then write (3.2.7)

$$f = c_1 g_1 + o(g_1).$$

To “approximate” this function in a more precise way, compare the function  $f - c_1 g_1$  to the functions of  $\mathcal{E}$ ; if this function has a principal part  $c_2 g_2$ , then  $g_2 = o(g_1)$  and

$$f = c_1 g_1 + c_2 g_2 + o(g_2).$$

In general, we shall call an *asymptotic* (or *limited*) *development with  $k$  terms* of the function  $f$  relative to  $\mathcal{E}$  (in the neighbourhood of  $+\infty$ ) a sum

$$(6.2.1) \quad c_1 g_1 + c_2 g_2 + \cdots + c_k g_k$$

where all the  $c_j$  are *non-zero* complex constants, all the  $g_j$  are functions of  $\mathcal{E}$  such that  $g_{j+1} = o(g_j)$  for  $1 \leq j \leq k-1$  and where

$$(6.2.2) \quad f = c_1 g_1 + c_2 g_2 + \cdots + c_k g_k + o(g_k).$$

The difference  $f - c_1 g_1 - c_2 g_2 - \cdots - c_k g_k$  is called the *remainder* of the development (6.2.1).

We also say that (6.2.1) is an asymptotic development *with precision  $g_k$* . If such a development exists, it is unique, since the first term  $c_1 g_1$  is the principal part of  $f$  and for  $2 \leq j \leq k$ ,  $c_j g_j$  is the principal part of  $f - c_1 g_1 - \cdots - c_{j-1} g_{j-1}$ . When an asymptotic development is restricted to its first  $j$  terms, the asymptotic development of  $f$  with precision  $g_j$  is obtained. When  $\mathcal{E}$  is replaced by a larger scale  $\mathcal{E}'$ , the development (if it exists) with  $k$  terms relative to  $\mathcal{E}$  of  $f$  is also its development with  $k$  terms relative to  $\mathcal{E}'$ .

All these conditions for the neighbourhood of  $+\infty$  naturally apply also to the neighbourhood (on the right or on the left) of every point  $x_0 \in \mathbf{R}$  and to the neighbourhood of  $-\infty$  with the appropriate changes of variable.

*Examples* (6.3). The best known examples of asymptotic developments are concerned precisely with the case of a neighbourhood of a point  $x_0 \in \mathbf{R}$  (on the right or on the left). These are the Taylor developments

$$(6.3.1) \quad f(x) = f(x_0) + f'(x_0)(x - x_0) + \cdots + \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k + o((x - x_0)^k)$$

valid for every complex function  $k$  times continuously differentiable in a neighbourhood of  $x_0$ . Observe that to have an asymptotic development in the sense defined in (6.2) we must retain in (6.3.1) only those terms whose coefficients are  $\neq 0$ ; if  $f$  and its derivatives of order  $\leq k$  are all zero at the point  $x_0$ , there is not a genuine asymptotic development, but only the relation  $f(x) = o((x - x_0)^k)$ . The following are some of the commonest Taylor developments (in the neighbourhood of 0):

$$(6.3.2) \quad (1 + x)^\mu = 1 + \binom{\mu}{1}x + \binom{\mu}{2}x^2 + \cdots + \binom{\mu}{k}x^k + o(x^k) \quad (\mu \text{ real } \neq 0)$$

$$(6.3.3) \quad e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^k}{k!} + o(x^k)$$

$$(6.3.4) \quad e^{ix} = 1 + \frac{ix}{1!} + \frac{i^2 x^2}{2!} + \cdots + \frac{i^k x^k}{k!} + o(x^k) \dagger$$

a relation equivalent to the two developments

$$(6.3.5) \quad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^k \frac{x^{2k}}{(2k)!} + o(x^{2k})$$

$$(6.3.6) \quad \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^k \frac{x^{2k+1}}{(2k+1)!} + o(x^{2k+1})$$

$$(6.3.7) \quad \log(1+x) = \frac{x}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{k-1} \frac{x^k}{k} + o(x^k).$$

(6.4) One should take care, in dealing with asymptotic developments, not to fall into two frequent errors. First, the notion of an asymptotic development *has nothing to do with the notion of a series*, in spite of the confusion systematically maintained by numerous books, which speak of “asymptotic series”. A series has an *infinite* number of terms, whereas, by definition, an asymptotic development has only a *finite* number of terms. To speak of the “convergence” of an asymptotic development, therefore, *does not make sense*. The confusion arises from the fact that in numerous cases (cf. Chap. VI) the Taylor development of a function in the neighbourhood of a point  $x_0 \in \mathbf{R}$  can be extended *arbitrarily far*, and hence we can pose the problem of the convergence of the Taylor series, and of the relation between its sum and the function with which one started. However, this problem has *no relation* to the study of the behaviour of the function in the neighbourhood of  $x_0$ , as we shall see. On the other hand, the existence of an asymptotic development with an arbitrarily large number of terms is a *very special* phenomenon. The function  $x^2 + x \sin x$  has an asymptotic development  $x^2 + o(x^2)$  in the neighbourhood of  $+\infty$  with *one term only*, and it is easy to give examples of developments which stop compulsorily after a given number of terms. Also, sometimes great effort is required to obtain *just one* term of an asymptotic development. For example, let  $\pi(x)$  be the number of prime numbers  $p \leq x$ ; it took one hundred years until 1896 to find a proof (Hadamard-de la Vallée-Poussin) that  $\pi(x) \sim \int_2^x dt/\log t$ , a conjecture which went as far back as Gauss. It is suspected that

$$\pi(x) = \int_2^x \frac{dt}{\log t} + O(x^{1/2+\varepsilon}) \quad \text{for } \varepsilon > 0 \quad (\text{Riemann's hypothesis}),$$

but all attempts made since 1866 have not yet succeeded in proving it.

A second, much grosser error, is to forget that an asymptotic development is *relative to the neighbourhood of a given point*. When a differentiable function is defined in a whole interval, it has an asymptotic development in the neighbourhood of *each* of the points of this interval, but the knowledge we have of one of these generally tells us *nothing* about the others. To know that, in the neighbourhood of  $x = 1$ , we have the limited development  $\log x = x - 1 - \frac{1}{2}(x-1)^2 + o((x-1)^2)$ , gives *no information* on the develop-

† We do not here assume known the general theory of the complex exponential (VI, 8.3); it will be sufficient to define  $e^{ix}$  by the formula  $e^{ix} = \cos x + i \sin x$ , assuming the theory of the trigonometric functions known; this implies  $e^{i(x+x')} = e^{ix} e^{ix'}$  for any real numbers  $x$  and  $x'$ .

ment of  $\log x$  in the neighbourhood of  $x = 3$ , nor in the neighbourhood of  $+\infty$ . A good rule to bear in mind is that *one does not develop the functions of the scale of comparison*: they are already developed! The confusions arise from the fact that a function such as  $e^x$  or  $\log x$ , defined in a whole interval  $[x_0, +\infty[$ , is a function of the scale *in the neighbourhood of the point*  $+\infty$ , but *not* in the neighbourhood of other points. It is legitimate to consider (6.3.3) as an asymptotic development of  $e^x$  in the neighbourhood of 0, but *grotesque* to think that this formula gives a limited development of  $e^x$  in the neighbourhood of  $+\infty$ !

One sometimes hesitates over the possibility of applying Taylor's development in the neighbourhood of a point, when it is not immediately apparent that the derivatives of the function are continuous in the neighbourhood of this point: for example, can we apply Taylor's development to the function  $(x \log |x|)/(1 + e^x)$  in the neighbourhood of 0? The calculation of successive derivatives being somewhat painful, it is recommended in such cases that the function be considered (if possible) as obtained by means of elementary operations (sum, product, power, etc.) from simpler functions, and that the methods in section 7 below be applied. Apart from some very simple cases, like the formulae (6.3.2) to (6.3.7), the Taylor development is most often a *theoretical rather than a practical tool*.

## 7. Calculus of asymptotic developments

(7.1) If we know the asymptotic developments of two functions  $f_1$  and  $f_2$  relative to the same comparison scale  $\mathcal{E}$ , in the neighbourhood of  $+\infty$  (or of any point of  $\mathbf{R}$ ), we can obtain the asymptotic development of the *sum*  $f_1 + f_2$  (resp. the *product*  $f_1 f_2$ ) by adding (resp. multiplying) the two given developments, then regrouping the terms thus obtained in the following manner. Let

$$\begin{aligned} f_1 &= a_1 g_1 + a_2 g_2 + \cdots + a_r g_r + o(g_r) \\ f_2 &= b_1 h_1 + b_2 h_2 + \cdots + b_s h_s + o(h_s) \end{aligned}$$

be the two given developments. For the sum, add the coefficients of like terms, classify these terms in decreasing order of dominance (Section 5), then suppress all terms in  $g_j$  such that  $g_j \ll h_s$ , and all terms in  $h_k$  such that  $h_k \ll g_r$ . The precision thus obtained is the "smaller" of the precisions of the two developments. For the product, using (2.2, (iii)), group together like terms  $a_j b_k \cdot g_j h_k$ , then suppress all of these terms which are negligible compared to  $g_1 h_s$  or  $h_1 g_r$ ; the precision thus obtained is, therefore, the "smaller" of the two precisions  $g_1 h_s$  and  $h_1 g_r$ .

(7.2) To obtain an asymptotic development of a power  $f^\mu$ , when a development (6.2.2) of  $f$  with a first coefficient  $c_1 > 0$  is known, put  $f = c_1 g_1 \cdot u$ , to derive an asymptotic development of  $u$

$$u = 1 + v = 1 + b_2 h_2 + \cdots + b_k h_k + o(h_k)$$

with  $b_j = c_j/c_1$ ,  $h_j = g_j/g_1$  for  $j > 1$ . Since  $f^\mu = c_1^\mu g_1^\mu \cdot u^\mu$ , the problem reduces to obtaining a development of  $u^\mu$ . Because of the definition,  $v$  tends to 0, hence one can write

$$u^\mu = 1 + \binom{\mu}{1} v + \binom{\mu}{2} v^2 + \cdots + \binom{\mu}{r} v^r + o(v^r)$$

for each integer  $r \geq 1$ , and  $o(v^r) = o(h_2^r)$ . Thus one is reduced to the two problems treated in (7.1) by limiting the precision to  $h_2^r$ , i.e. by suppressing all the terms negligible compared with  $h_2^r$ . When  $\mu$  is an integer (positive or negative) we can proceed in the same way suppressing the restriction that the complex number  $c_1$  be  $> 0$ .

*Example (7.3)* For the example considered at the end of (6.4), in the neighbourhood of 0,  $1 + e^x = 2 + x + \frac{1}{2}x^2 + o(x^2)$ , hence, by (7.2) applied with  $\mu = -1$ ,  $1/(1 + e^x) = \frac{1}{2} - \frac{1}{4}x + o(x^2)$  (by choosing  $r = 2$ ), and finally

$$\frac{x \log |x|}{1 + e^x} = \frac{x \log |x|}{2} - \frac{x^2 \log |x|}{4} + o(x^3 \log |x|).$$

The sole principal part shows that Taylor's formula is not applicable at the point  $x = 0$ .

(7.4) With the same notations, the method indicated in (7.2) applies in the same way to the development of  $F(u(x))$  provided  $F$  admits a Taylor development in the neighbourhood of 0 (the case treated in (7.2) being that of  $F(x) = (1 + x)^\mu$ ). Amongst other things, this permits us to give a development of  $\log f$ , when  $\log g_1$  admits a development relative to the scale  $\mathcal{E}$ , since we can write  $\log f = \log g_1 + \log c_1 + \log u$ , and then use the Taylor development (6.3.7) of  $\log(1 + x)$  in the neighbourhood of 0. This also permits a development of  $e^f$  when  $f = o(1)$  (which is equivalent to  $g_1 = o(1)$ ), because then

$$e^f = 1 + \frac{f}{1!} + \frac{f^2}{2!} + \cdots + \frac{f^r}{r!} + o(g_1^r) \quad \text{for } r \geq 1.$$

When we do not have  $f = o(1)$ , but when in the development (6.2.2) not all the  $g_j$  tend to  $+\infty$ , we consider the smallest index  $i$  such that  $g_i = o(1)$ , and write  $f = f_1 + f_2$  with

$$f_1 = c_1 g_1 + \cdots + c_{i-1} g_{i-1}, \quad f_2 = c_i g_i + \cdots + c_k g_k + o(g_k),$$

and so  $e^f = e^{f_1} e^{f_2}$ , and the preceding applies to  $e^{f_2}$ . As for the factor  $e^{f_1} = \prod_{j=1}^{i-1} e^{c_j g_j}$ , this will be a function of  $\mathcal{E}$  if each of the functions  $e^{g_j}$  belongs to  $\mathcal{E}$ .

The problem of the development of  $u^v$  reduces to the preceding problems by writing, as usual,  $u^v = e^{v \log u}$ .

*Examples (7.5.1)* Let us develop  $f(x) = (1 + x)^{1/x}$  in the neighbourhood of  $+\infty$ . Write  $f(x) = \exp((1/x) \log(1 + x))$ ; then

$$\log(1 + x) = \log x + \log\left(1 + \frac{1}{x}\right) = \log x + \frac{1}{x} - \frac{1}{2x^2} + o\left(\frac{1}{x^2}\right)$$

hence

$$\frac{1}{x} \log(1 + x) = \frac{\log x}{x} + \frac{1}{x^2} - \frac{1}{2x^3} + o\left(\frac{1}{x^3}\right).$$

Since  $u(x) = (1/x) \log(1+x) = o(1)$ , we can use the development (6.3.3) and obtain, stopping the development of  $e^u$  at the term in  $u^3$

$$\begin{aligned} f(x) &= 1 + \frac{u(x)}{1} + \frac{u(x)^2}{2} + \frac{u(x)^3}{6} + o\left(\frac{(\log x)^3}{x^3}\right) \\ &= 1 + \frac{\log x}{x} + \frac{1}{x^2} + \frac{(\log x)^2}{2x^2} + \frac{(\log x)^3}{6x^3} + o\left(\frac{(\log x)^3}{x^3}\right) \\ &= 1 + \frac{\log x}{x} + \frac{(\log x)^2}{2x^2} + \frac{1}{x^2} + \frac{(\log x)^3}{6x^3} + o\left(\frac{(\log x)^3}{x^3}\right). \end{aligned}$$

(7.5.2) Let us develop  $f(x) = x^{1/x} = \exp(x^{1/x} \log x)$  in the neighbourhood of  $+\infty$ . First

$$x^{1/x} \log x = \log x \cdot \exp\left(\frac{1}{x} \log x\right)$$

and since  $v(x) = (1/x) \log x = o(1)$ , we can use the development (6.3.3), stopping at the term in  $v^2$

$$x^{1/x} \log x = \log x + \frac{1}{x} (\log x)^2 + \frac{1}{2x^2} (\log x)^3 + o\left(\frac{(\log x)^3}{x^2}\right).$$

Here, the terms only start to tend to 0 at the second one; we thus write  $f(x) = e^{\log x} e^{w(x)} = x e^{w(x)}$  with  $w = o(1)$ , and confine ourselves to the terms in  $w^2$ , obtaining

$$f(x) = x + (\log x)^2 + \frac{1}{2x} (\log x)^4 + o\left(\frac{(\log x)^4}{x}\right).$$

Observe here that the terms start to tend to 0 only from the third one on. We shall later see examples of asymptotic developments of arbitrary length, but where all the terms tend to  $+\infty$  (10.10.2).

(7.6) In numerous applications (cf. Chapters IV, IX, XIII, XIV, XV) we are led into widening the notion of asymptotic development defined in (6.2). Consider a scale of comparison  $\mathcal{E}$  in the neighbourhood of  $+\infty$ , and on the other hand a set  $\mathcal{C}$  of *complex* functions defined in the neighbourhood of  $+\infty$  and satisfying the following conditions:

- (i) *Each function of  $\mathcal{C}$  is bounded in the neighbourhood of  $+\infty$ .*
- (ii) *No function of  $\mathcal{C}$ , other than the constant 0, tends to 0 as  $x$  tends to  $+\infty$ .*
- (iii) *The sum of two functions of  $\mathcal{C}$  belongs to  $\mathcal{C}$ , as well as the product of a function of  $\mathcal{C}$  with a complex constant.*

An example of such a set  $\mathcal{C}$  is given by the functions bounded in  $\mathbf{R}$  and periodic with a fixed period  $\tau > 0$ ; only condition (ii) needs verification. If  $c(x)$  is periodic with period  $\tau$  and if  $\lim_{x \rightarrow +\infty} c(x) = 0$ , then for each  $\varepsilon > 0$ , there exists  $x_0$  such that for  $x \geq x_0$ ,  $|c(x)| \leq \varepsilon$ ; we then also have  $|c(x)| \leq \varepsilon$  for  $0 \leq x \leq \tau$ , since there is an integer  $n$  such that  $n\tau > x_0$ , so  $x + n\tau > x_0$  for  $0 \leq x \leq \tau$  and  $c(x + n\tau) = c(x)$ . Since  $\varepsilon$  is arbitrary, we must have  $c(x) = 0$  in  $[0, \tau]$ , and so everywhere.

We shall say that a complex function  $f(x)$  defined in the neighbourhood of  $+\infty$  has

the (generalized) principal part  $c(x)g(x)$ , where  $c \in \mathcal{C}$ ,  $g \in \mathcal{E}$  and  $c$  is not identically zero in the neighbourhood of  $+\infty$ , if

$$(7.6.1) \quad f(x) - c(x)g(x) = o(g(x)).$$

Such a principal part, if it exists, is then *unique*. Indeed, suppose there exists another one  $c_1(x)g_1(x)$ ; we cannot have  $g = o(g_1)$ , for by (i) and (4.1.3) we would deduce

$$c(x)g(x) = o(g_1(x)) \quad \text{and} \quad f(x) - c(x)g(x) = o(g(x)) = o(g_1(x)),$$

so  $f = o(g_1)$ . But then also  $c_1(x)g_1(x) = o(g_1(x))$  and as  $g_1$  does not vanish in the neighbourhood of  $+\infty$ ,  $c_1(x) = o(1)$ . Now, because of (ii), this is possible only if  $c_1$  is *identically zero* in the neighbourhood of  $+\infty$ , contrary to the hypothesis. Similarly we cannot have  $g_1 = o(g)$ ; therefore  $g_1 = g$  (2.2). The relations

$$f(x) - c(x)g(x) = o(g(x)), \quad f(x) - c_1(x)g(x) = o(g(x))$$

imply (4.1.2) that  $(c(x) - c_1(x))g(x) = o(g(x))$ : hence, as above,  $c(x) - c_1(x) = o(1)$  and because of (ii) and (iii) this is possible only if  $c_1 = c$ .

With the notion of principal part with coefficients in  $\mathcal{C}$ , we are able to deduce in stages the notion of *asymptotic development with coefficients in  $\mathcal{C}$*  as in (6.2). The methods given in (7) to form the asymptotic developments of a sum or a product are still valid, provided that the product of two functions of  $\mathcal{C}$  still belongs to  $\mathcal{C}$ . We define in the same way the generalized asymptotic developments in the neighbourhood of any point of  $\mathbf{R}$ , or for sequences (functions defined in  $\mathbf{N}$ ).

### 3. Asymptotic developments of implicit functions

(8.1) We shall confine ourselves to the study, in the neighbourhood of  $+\infty$ , of the *inverse function* of a *strictly increasing* continuous function  $y \rightarrow G(y)$  defined in an interval  $[y_0, +\infty[$  and tending to  $+\infty$  with  $y$  (cf. Appendix). We shall admit the theorem which affirms the existence and uniqueness of a function  $x \rightarrow u(x)$  defined in the interval  $[x_0, +\infty[$ , where  $x_0 = G(y_0)$  and such that

$$(8.1.1) \quad G(u(x)) = x$$

in this interval (K-R, p. 52). Moreover, this function is continuous and strictly increasing and tends to  $+\infty$  with  $x$ . Knowing an asymptotic development of  $G$  in the neighbourhood of  $+\infty$ , we wish to find a development of  $u$ . We shall examine just one case, although one can frequently reduce to this case by appropriate changes of variable.

(8.2) *Let  $g$  be a function defined in a neighbourhood of  $+\infty$ , not equivalent to a non-zero constant, continuously differentiable and having the following properties:*

1.  $g'$  is monotonic and does not vanish in the neighbourhood of  $+\infty$  and  $g' = o(1)$ .
2. For each real constant  $c$ ,  $g'(y + cg(y)) \sim g'(y)$  as  $y$  tends to  $+\infty$ .

*Under these conditions, let  $u$  be the unique solution in the neighbourhood of  $+\infty$  of the equation*

$$(8.2.1) \quad u(x) - g(u(x)) = x.$$

Define a sequence of functions  $u_n$  as follows:

$$u_0(x) = x, \quad u_n(x) = x + g(u_{n-1}(x)) \quad \text{for } n \geq 1;$$

then the functions  $u_n(x)$  tend to  $+\infty$  with  $x$ , and

$$(8.2.2) \quad u(x) - u_n(x) \sim g(x)(g'(x))^n$$

in the neighbourhood of  $+\infty$ .

We observe that, by the mean value theorem,

$$u(x) - u_n(x) = g(u(x)) - g(u_{n-1}(x)) = g'(z)(u(x) - u_{n-1}(x))$$

where  $z$  is in the interval with endpoints  $u(x)$  and  $u_{n-1}(x)$ ; this leads naturally to the formula (8.2.2) and the following proof is just a development of this idea.

First note that for each  $\varepsilon > 0$  there exists  $y_1$  such that for  $y \geq y_1$ ,  $|g'(y)| \leq \varepsilon$ , hence by the mean-value theorem  $|g(y) - g(y_1)| \leq \varepsilon(y - y_1)$ , and therefore there exists  $y_2 \geq y_1$  such that for  $y \geq y_2$ ,  $|g(y)| \leq 2\varepsilon y$ ; i.e.  $g(y) = o(y)$  in the neighbourhood of  $+\infty$ . The function  $G(y) = y - g(y)$  has a derivative  $G'(y) = 1 - g'(y)$  which tends to 1 as  $y$  tends to  $+\infty$ , so  $G(y)$  is increasing and tends to  $+\infty$  in the neighbourhood of  $+\infty$ . The existence of  $u$ , therefore, follows from (8.1). We first prove (8.2.2) in the case  $n = 0$ , i.e.

$$(8.2.3) \quad u(x) - x \sim g(x).$$

If we put  $z = u(x)$ , then we have

$$z - x = g(z) = g(x) + (z - x)g'(t)$$

where  $t$  lies between  $x$  and  $z$ , applying the mean value theorem. As  $x$  tends to  $+\infty$ , so does  $z = u(x)$  and therefore so does  $t$ . Thus the hypothesis  $g' \ll 1$  gives

$$u(x) - x = g(x) + o(u(x) - x).$$

Since  $g$  is strictly monotonic in the neighbourhood of  $+\infty$  (therefore not identically zero), it has constant sign and the preceding relation implies (8.2.3) (cf. 3.2.7). Since  $g(x) = o(x)$ , this implies  $u(x) \sim x$ .

We now prove (8.2.2) for  $n \geq 1$ , by induction on  $n$ ; we show at the same time that  $u_n \gg 1$  and

$$(8.2.4) \quad u(x) - u_n(x) \ll u(x) - u_{n-1}(x).$$

This is a legitimate statement, since the induction hypothesis and the properties of  $g$  and  $g'$  imply that  $u - u_{n-1}$  has constant sign and is not identically zero in the neighbourhood of  $+\infty$ .

Put  $z_n = u_n(x)$ ; by the mean value theorem

$$(8.2.5) \quad z - z_n = g(z) - g(z_{n-1}) = (z - z_{n-1})g'(t_{n-1})$$

where  $t_{n-1}$  lies between  $z$  and  $z_{n-1}$ . By the induction hypothesis  $z_{n-1} = u_{n-1}(x)$  tends to  $+\infty$  with  $x$ , therefore so does  $t_{n-1}$ , and the hypothesis  $g' \ll 1$  implies (8.2.4). From (8.2.4) and the analogous relations for the integers  $< n$ , we deduce by transitivity

$$u(x) - u_n(x) \ll u(x) - x \sim g(x) \ll x \sim u(x)$$

hence  $u_n(x) \sim u(x)$  (3.2.7) and so  $u_n(x) \sim x \gg 1$ . Note now that the relation

$$u(x) - u_{n-1}(x) \ll u(x) - x$$

can also be written  $(u(x) - x) - (u_{n-1}(x) - x) \ll u(x) - x$ , hence (3.2.7)

$$(8.2.6) \quad u_{n-1}(x) - x \sim u(x) - x \sim g(x).$$

Since  $t_{n-1}$  lies between  $z$  and  $z_{n-1}$ , it follows from (8.2.6) that  $t_{n-1} - x \sim g(x)$ ; we shall deduce from this that  $g'(t_{n-1}) \sim g'(x)$ . Indeed, for each  $\varepsilon > 0$  there exists  $x_0 > 0$  such that for  $x \geq x_0$

$$x + (1 - \varepsilon)g(x) \leq t_{n-1} \leq x + (1 + \varepsilon)g(x).$$

But since  $g'$  is monotonic, this implies that  $g'(t_{n-1})$  is contained in the interval with end-points

$$g'(x + (1 - \varepsilon)g(x)) \quad \text{and} \quad g'(x + (1 + \varepsilon)g(x));$$

the assertion follows, since by (2) these two functions of  $x$  are equivalent to  $g'(x)$ . It follows, then, from (8.2.5) that  $u(x) - u_n(x) \sim (u(x) - u_{n-1}(x))g'(x)$ , and (8.2.2) is proved by induction.

(8.3) Amongst the most important functions  $g$  satisfying the hypotheses of (8.2) mention is made of  $g(x) = x^\alpha$  with  $\alpha < 1$  and  $\alpha \neq 0$ , and  $g(x) = \log x$  (cf. Problem 6).

*Examples* (8.4.1) The equation

$$(8.4.2) \quad y^5 + y = x$$

has, in a suitable neighbourhood of  $+\infty$ , one and only one real root  $v(x)$ , which tends to  $+\infty$  with  $x$ . This follows from (8.1) applied to the equation

$$(8.4.3) \quad z + z^{1/5} = x$$

which is deduced from (8.4.2) by the change of variable  $z = y^5$ . We can apply (8.2) with  $g(z) = -z^{1/5}$  and deduce, for  $n = 2$  for example, from (8.2.2)

$$(8.4.4) \quad u(x) - u_2(x) \sim -\frac{1}{25}x^{-7/5}.$$

On the other hand,  $u_2(x) = x - (x - x^{1/5})^{1/5}$  can be developed in the neighbourhood of  $+\infty$  with the help of section 7, which gives

$$u_2(x) = x - x^{1/5}(1 - x^{-4/5})^{1/5} = x - x^{1/5} + \frac{1}{5}x^{-3/5} + \frac{2}{25}x^{-7/5} + o(x^{-7/5}).$$

Hence, because of (8.4.4)

$$(8.4.5) \quad u(x) = x - x^{1/5} + \frac{1}{5}x^{-3/5} + \frac{1}{25}x^{-7/5} + o(x^{-7/5}).$$

We have  $v(x) = (u(x))^{1/5}$  and applying (7.2) finally obtain

$$(8.4.6) \quad v(x) = x^{1/5} - \frac{1}{5}x^{-3/5} - \frac{1}{25}x^{-7/5} + o(x^{-7/5}).$$

(8.5.1) As a second example consider the equation

$$(8.5.2) \quad \frac{y}{\log y} = x$$

where for  $y > e$ , the first member is an increasing continuous function of  $y$  which tends to  $+\infty$  with  $y$ . Writing  $z = \log y$ ,  $t = \log x$ , the equation is equivalent to

$$(8.5.3) \quad z - \log z = t$$

and we can apply (8.2) with  $g(z) = \log z$ . If  $u(t)$  is the solution of (8.5.3) which tends to  $+\infty$  with  $t$ , we then have

$$(8.5.4) \quad u(t) - u_2(t) \sim \frac{\log t}{t^2}$$

and the methods of section 7 give

$$(8.5.5) \quad \begin{aligned} u_2(t) &= t + \log(t + \log t) \\ &= t + \log t + \frac{\log t}{t} + \frac{(\log t)^2}{2t^2} + o\left(\frac{(\log t)^2}{t^2}\right). \end{aligned}$$

One deduces from (8.5.4) and (8.5.5) that  $u$  and  $u_2$  have the same development with the precision  $(\log t)^2/t^2$ , in other words if  $v(x)$  is the solution of (8.5.2) tending to  $+\infty$  with  $x$ , then

$$\log v(x) = \log x + \log \log x + \frac{\log \log x}{\log x} - \frac{(\log \log x)^2}{2(\log x)^2} + o\left(\frac{(\log \log x)^2}{(\log x)^2}\right).$$

The terms begin to tend to 0 in this development only from the third one on; applying (7.4) we obtain†

$$(8.5.6) \quad v(x) = x \log x + x \log \log x + o\left(x \frac{\log \log x}{\log x}\right).$$

*Remark (8.6)* Be careful to note that it is possible for  $G(y) \sim G_1(y)$  for two functions satisfying the conditions of (8.1) without the inverse functions being equivalent, as is shown by the example

$$G(y) = \log y \quad \text{and} \quad G_1(y) = \log y + 1.$$

## 9. Convergence of improper integrals

(9.1) Let  $f$  be a complex function defined in an interval  $[a, +\infty[$  and piecewise-continuous in every finite interval  $[a, b]$  (abbreviated by saying  $f$  is *piecewise-continuous* in  $[a, +\infty[$ ); for every  $x > a$ , the integral  $F(x) = \int_a^x f(t) dt$  is defined (0, 4.3). By definition,

$$(9.1.1) \quad \int_a^{+\infty} f(t) dt = \lim_{x \rightarrow +\infty} F(x) = \lim_{x \rightarrow +\infty} \int_a^x f(t) dt$$

when this limit exists, and is called the (improper) *integral of  $f$  in the infinite interval  $[a, +\infty[$* . It is evident that for every  $b > a$

$$\int_a^{+\infty} f(t) dt = \int_a^b f(t) dt + \int_b^{+\infty} f(t) dt$$

where the existence of one member implies the existence of the other. For each  $x \geq a$ ,

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† The scale  $\mathcal{E}$ , which we use here, contains the powers of  $\log \log x$ .

we call the integral  $\int_x^{+\infty} f(t) dt$  the *remainder* of the improper integral  $\int_a^{+\infty} f(t) dt$ ; it is a function of  $x$  which tends to 0 as  $x$  tends to  $+\infty$ .

When the integral  $\int_a^{+\infty} f(t) dt$  exists, it is also said to be *convergent*. The most important case is the one where the integral  $\int_a^{+\infty} |f(t)| dt$  of the *absolute value* of  $f$  is convergent; then  $f = f_1 - f_2 + if_3 - if_4$ , where the four functions  $f_k$  are  $\geq 0$  and  $|f_k| \leq f$  (we take  $f_1(t) = \Re f(t)$  if  $\Re f(t) > 0$ ,  $f_1(t) = 0$  otherwise and define  $f_3$  similarly). In view of the criterion (9.3) below, the integrals  $\int_a^{+\infty} f_k(t) dt$  converge, and therefore so does the integral  $\int_a^{+\infty} f(t) dt$ . In this case the integral  $\int_a^{+\infty} f(t) dt$  is said to be *absolutely convergent*, and  $|\int_a^{+\infty} f(t) dt| \leq \int_a^{+\infty} |f(t)| dt$  by passage to the limit in (I, 3.3.2).

(9.2) In this chapter we shall be concerned exclusively with the integrals of functions which *do not change sign* in the neighbourhood of  $+\infty$ . We can assume that the functions  $f$  are  $\geq 0$ . Then the primitive  $F(x) = \int_a^x f(t) dt$  is an increasing function of  $x$ , and therefore tends to a finite limit or to  $+\infty$  as  $x$  tends to  $+\infty$  (in the latter case we write  $\int_a^{+\infty} f(t) dt = +\infty$ ). The integral  $\int_a^{+\infty} f(t) dt$  is therefore convergent if, and only if,  $F(x)$  is *bounded above* in the neighbourhood of  $+\infty$ . We deduce immediately (I, 3.1.1) the *comparison principle* analogous to that for series (I, 2.2):

(9.3) Let  $f, g$  be two functions non-negative and piecewise-continuous in the neighbourhood of  $+\infty$ , and satisfying  $f \leq Ag$ , where  $A$  is a constant  $> 0$ . Then, if the integral  $\int_a^{+\infty} g(t) dt$  is convergent, so is  $\int_a^{+\infty} f(t) dt$  and

$$(9.3.1) \quad \int_a^{+\infty} f(t) dt \leq A \int_a^{+\infty} g(t) dt.$$

If the integral  $\int_a^{+\infty} f(t) dt$  is infinite, so is  $\int_a^{+\infty} g(t) dt$ .

We must, of course, take care not to apply this principle to the integrals of functions which *change sign* in every neighbourhood of  $+\infty$ .

(9.4) To apply the comparison principle, we shall compare the function  $f$ , whose integral we require, to the *functions of the scale*  $\mathcal{E}$  defined in (2.1). For some of these functions the convergence of their integrals can be decided immediately: indeed, for  $a > 1$

$$(9.4.1) \quad \begin{cases} \int_a^x t^\alpha dt = \frac{1}{\alpha+1} (x^{\alpha+1} - a^{\alpha+1}) & \text{when } \alpha \neq -1 \\ \int_a^x t^{-1} dt = \log x - \log a \end{cases}$$

$$(9.4.2) \quad \begin{cases} \int_a^x t^{-1} (\log t)^\beta dt = \frac{1}{\beta+1} ((\log x)^{\beta+1} - (\log a)^{\beta+1}) & \text{if } \beta \neq -1 \\ \int_a^x t^{-1} (\log t)^{-1} dt = \log \log x - \log \log a. \end{cases}$$

Therefore

- (9.5) (i) If  $f(x) = O(x^\alpha(\log x)^\beta)$  with  $\alpha < -1$ , or  $\alpha = -1$  and  $\beta < -1$ , the integral  $\int_a^{+\infty} f(t) dt$  is convergent.  
 (ii) If  $f(x) \geq x^\alpha(\log x)^\beta$  with  $\alpha > -1$ , or  $\alpha = -1$  and  $\beta \geq -1$ , the integral  $\int_a^{+\infty} f(t) dt$  is infinite.

In practice, at the level of this book, we apply ONLY THE FOLLOWING RULE: take the principal part c.g. of the function  $f$  to be integrated, and (in the order relation of (5)) look for where  $g$  is relative to the functions  $x^\alpha(\log x)^\beta$  (Fig. 9). More precisely, if in the function (2.1)

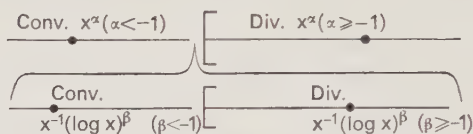


FIGURE 9

the exponential factor  $e^{P(x)}$  is not equal to 1, it is sufficient to consider this factor and drop  $x^\alpha(\log x)^\beta$ ; the integral is then convergent if  $c_1 < 0$ , infinite if  $c_1 > 0$ . If, on the other hand,  $P = 0$ , look first at the exponent  $\alpha$ : if  $\alpha < -1$ , the integral is convergent, if  $\alpha > -1$ , it is infinite. Lastly, if  $\alpha = -1$ , consider the exponent  $\beta$ : if  $\beta < -1$ , the integral is convergent, if  $\beta \geq -1$ , it is infinite. The partition is made at the "holes" corresponding to  $\alpha = -1$  and  $\beta = -1$  (analogous results would be obtained by introducing the iterated logarithms).

It is only when the function  $f$  has no principal part relative to the scale (2.1) (or when we see easily a majorization or minorization of  $f$  permitting us to apply (9.5) and to dispense with looking for the principal part), that the more precise rules (9.5) are used.

(9.6) Let, now,  $f$  be a complex function defined in an interval  $]a, b]$  half-open on the left and piecewise-continuous in every interval  $[a + h, b]$  with  $h > 0$  (we say that  $f$  is piecewise-continuous in  $]a, b]$ ); we can, then, for  $h > 0$ , define the function  $F(h) = \int_{a+h}^b f(t) dt$ . By definition

$$(9.6.1) \quad \int_a^b f(t) dt = \lim_{h \rightarrow 0} F(h) = F(0+) = \lim_{h \rightarrow 0} \int_{a+h}^b f(t) dt$$

when this limit exists, and is called the (improper) integral of  $f$  in the half-open interval  $]a, b]$ . One also says that the integral  $\int_a^b f(t) dt$  is convergent in the neighbourhood of  $a$ ; if  $\int_a^b |f(t)| dt$  exists, the integral  $\int_a^b f(t) dt$  is absolutely convergent in the neighbourhood of  $a$ . The change of variable  $t = a + (1/s)$  reduces the improper integral  $\int_a^b f(t) dt$  to the improper integral  $\int_{1/(b-a)}^{+\infty} \frac{f(a + (1/s))}{s^2} ds$ . The rule corresponding to (9.5) is then:

- (9.6.2) (i) If, in the neighbourhood of  $a$  on the right,  $f(x) = O((x - a)^\alpha |\log(x - a)|^\beta)$  with

$\alpha > -1$ , or  $\alpha = -1$  and  $\beta < -1$  (in particular if  $f = O(1)$ ), the integral  $\int_a^b f(t) dt$  is convergent in the neighbourhood of  $a$ .

- (ii) If  $f(x) \geq (x-a)^\alpha |\log(x-a)|^\beta$  with  $\alpha < -1$ , or  $\alpha = -1$  and  $\beta \geq -1$ , the integral  $\int_a^b f(t) dt$  is infinite.

In practice, we look for the principal part of  $f$  in the neighbourhood of  $a$ .

(9.7) We define in the same way the improper integrals in an infinite closed interval  $]-\infty, a]$ , or in an interval half-open on the right  $[b, a[$ ; the task of reducing these integrals to those considered in (9.1) by appropriate changes in variable is left to the reader. When a function  $f$  is defined in an *open* interval  $]a, b[$  ( $a$  and  $b$  finite or infinite) and is *piecewise-continuous* (i.e. piecewise-continuous in any bounded closed interval  $[\alpha, \beta] \subset ]a, b[$ ), the (improper) integral  $\int_a^b f(t) dt$  is, by definition, equal to  $\int_a^c f(t) dt + \int_c^b f(t) dt$ , for  $c$  such that  $a < c < b$ , assuming that *each* of the two improper integrals exists. It is immediate that this definition does not depend on the choice of  $c$ .

Note that when a piecewise-continuous function  $f$  is defined in a *bounded* interval  $[a, b]$ ,  $\int_a^b f(t) dt$  is often written in the form  $\int_{-\infty}^{+\infty} f(t) dt$ , by agreeing to *extend*  $f$  to the value 0 in the rest of  $\mathbf{R}$  outside  $[a, b]$ . The convergence of the integral is evident.

Note also that if  $u$  and  $v$  are primitives of piecewise-continuous functions in  $]a, b[$ , and if  $t \rightarrow u(t)v(t)$  has a limit on the right at the point  $a$  and a limit on the left at the point  $b$ , denoted (by abuse of language)

$$u(a+)v(a+) \quad \text{and} \quad u(b-)v(b-),$$

then, if one of the two improper integrals  $\int_a^b u(t)v'(t) dt$ ,  $\int_a^b v(t)u'(t) dt$  exists, so does the other, and we have the formula for integration by parts

$$(9.8) \quad \int_a^b u(t)v'(t) dt = u(b-)v(b-) - u(a+)v(a+) - \int_a^b v(t)u'(t) dt.$$

This indeed follows on passing to the limit in the same formula applied to a bounded closed interval contained in  $]a, b[$ .

Lastly, we define an improper integral  $\int_a^b \mathbf{f}(t) dt$  of a *vector* function  $\mathbf{f}$  in the same way as above; the existence of this integral is equivalent to the existence of the integral of each of the components  $f_j$  of  $\mathbf{f}$ , and then the vector  $\int_a^b \mathbf{f}(t) dt$  has components  $\int_a^b f_j(t) dt$ .

*Example (9.9) The gamma function.* Let us show that the integral

$$(9.9.1) \quad \Gamma(x) = \int_0^{+\infty} t^{x-1} e^{-t} dt$$

is convergent for every  $x > 0$ . We must prove separately that each of the integrals  $\int_0^1 t^{x-1} e^{-t} dt$  and  $\int_1^{+\infty} t^{x-1} e^{-t} dt$  is convergent. The first integral is improper only when  $0 < x < 1$ . For  $t$  near to 0,  $e^{-t} \sim 1$ , so  $t^{x-1} e^{-t} \sim t^{x-1}$ ; the convergence follows from the criterion (9.6). On the other hand the integral  $\int_1^{+\infty} t^{x-1} e^{-t} dt$  is convergent for

every real  $x$  by (9.5), since the integrand is a function of  $\mathcal{E}$  in which there is an exponential factor tending to 0.

The integral (9.9.1) is called Euler's *Gamma function*; it plays a large role in Analysis and we shall study it in detail later on (Chap. IV and IX).

## 10. Asymptotic development of a primitive

(10.1) The fundamental inequality (I, 3.1.1), on which the comparison principle is already based, yields, under the conditions of (9.1), *far more* than the existence of the limit as  $x$  tends to  $+\infty$  of  $F(x) = \int_a^x f(t) dt$ . If we know the "behaviour" of  $f$  in the neighbourhood of  $+\infty$ , we can obtain information about the behaviour of  $F$ .

(10.2) Let  $f, g$  be two piecewise-continuous complex functions in  $[a, +\infty[$  and suppose that  $g(x) > 0$  in this interval.

- (i) If the integral  $\int_a^{+\infty} g(t) dt$  is infinite, then in the neighbourhood of  $+\infty$ :
  - the relation  $f = O(g)$  implies  $\int_a^x f(t) dt = O\left(\int_a^x g(t) dt\right)$ ;
  - the relation  $f = o(g)$  implies  $\int_a^x f(t) dt = o\left(\int_a^x g(t) dt\right)$ ;
  - the relation  $f \sim c.g$  ( $c$  constant  $\neq 0$ ) implies  $\int_a^x f(t) dt \sim c \cdot \int_a^x g(t) dt$ .
- (ii) If the integral  $\int_a^{+\infty} g(t) dt$  is convergent, then in the neighbourhood of  $+\infty$ :
  - the relation  $f = O(g)$  implies  $\int_x^{+\infty} f(t) dt = O\left(\int_x^{+\infty} g(t) dt\right)$ ;
  - the relation  $f = o(g)$  implies  $\int_x^{+\infty} f(t) dt = o\left(\int_x^{+\infty} g(t) dt\right)$ ;
  - the relation  $f \sim c.g$  ( $c$  constant  $\neq 0$ ) implies  $\int_x^{+\infty} f(t) dt \sim c \cdot \int_x^{+\infty} g(t) dt$ .

(i) To prove the first assertion, note that, by hypothesis, there exists  $b > a$  such that for  $x \geq b$ ,  $|f(x)| \leq A.g(x)$  for a suitable constant  $A > 0$ . From the mean-value theorem (I, 3.3.2) we deduce that for  $x \geq b$ ,  $\left|\int_b^x f(t) dt\right| \leq A \cdot \int_b^x g(t) dt$ . Since  $\int_a^b g(t) dt > 0$ , there exists a constant  $B > A$  such that  $\left|\int_a^b f(t) dt\right| \leq B \cdot \int_a^b g(t) dt$ . We conclude from this that for  $x \geq b$ ,  $\left|\int_a^x f(t) dt\right| \leq B \cdot \int_a^x g(t) dt$ .

Secondly, suppose that  $f = o(g)$ . For each  $\varepsilon > 0$ , there exists  $x_0 > a$  (depending on  $\varepsilon$ ) such that for  $x \geq x_0$ ,  $|f(x)| \leq \varepsilon g(x)$ . We deduce from the theorem of the mean that for  $x \geq x_0$ ,

$$\left|\int_{x_0}^x f(t) dt\right| \leq \varepsilon \int_{x_0}^x g(t) dt \leq \varepsilon \int_a^x g(t) dt.$$

On the other hand, by hypothesis,  $\int_a^x g(t) dt$  tends to  $+\infty$  with  $x$ ; hence there exists  $x_1 > x_0$  such that for  $x \geq x_1$ ,  $\left|\int_a^{x_0} f(t) dt\right| \leq \varepsilon \int_a^x g(t) dt$ ; we deduce that for  $x \geq x_1$ ,  $\left|\int_a^x f(t) dt\right| \leq 2\varepsilon \int_a^x g(t) dt$ , which shows that

$$\int_a^x f(t) dt = o\left(\int_a^x g(t) dt\right).$$

The third assertion of (i) is deduced from the second by recalling that the relation  $f \sim c.g$  is equivalent to  $f - cg = o(g)$ .

The proofs of (ii) are analogous and simpler, and are left to the reader.

*Example (10.3)* The relation  $x^{-1} = o(x^{-1+\alpha})$  for  $\alpha > 0$  gives again, by integration, the known relation  $\log x = o(x^\alpha)$  for  $\alpha > 0$ , which was used in (2.2).

(10.4) When a function  $f$  admits a *principal part*  $c.g$  relative to the scale  $\mathcal{E}$  defined in (2.1), the study of the primitive  $\int_a^x f(t) dt$  in the neighbourhood of  $+\infty$  thus reduces to the analogous study for  $\int_a^x g(t) dt$ , which is often simpler. However, one must beware of thinking that for functions of  $\mathcal{E}$  there are *equalities* analogous to (9.4.1) or (9.4.2), which are quite *exceptional*. Thus although the primitive of a function of  $\mathcal{E}$  is *not* in general itself a function  $\mathcal{E}$  (nor even a finite linear combination of such functions), nevertheless the primitive often has a *principal part* which is a function of  $\mathcal{E}$ ; this follows from the relation (for the function  $g$  defined in (2.1))

$$(10.4.1) \quad \frac{g'(x)}{g(x)} = \frac{\alpha}{x} + \frac{\beta}{x \log x} + P'(x)$$

and from the following theorems.

Note that (10.4.1) also gives an asymptotic development of  $g'/g$  and therefore when  $g$  is not the constant 1,  $g'(x)$  has constant sign in the neighbourhood of  $+\infty$ .

(10.5) Let  $g$  be a continuously differentiable function  $> 0$  in the neighbourhood of  $+\infty$ , and suppose that  $g'(x)/g(x) \sim \mu/x$ , where  $\mu \neq 0$  and  $\mu \neq -1$ . Then:

(i) If  $\mu > -1$ ,  $g(x) \gg x^{\mu-\varepsilon}$  for each  $\varepsilon > 0$ ; the integral  $\int_a^{+\infty} g(t) dt$  is infinite and in the neighbourhood of  $+\infty$

$$(10.5.1) \quad \int_a^x g(t) dt \sim \frac{xg(x)}{\mu+1}.$$

(ii) If  $\mu < -1$ ,  $g(x) \ll x^{\mu+\varepsilon}$  for each  $\varepsilon > 0$ ; the integral  $\int_a^{+\infty} g(t) dt$  is convergent and

$$(10.5.2) \quad \int_x^{+\infty} g(t) dt \sim -\frac{xg(x)}{\mu+1}.$$

We confine ourselves to the proof of (i), the reasoning being analogous for (ii). The hypothesis implies  $\log g(x) \sim \mu \log x$  by virtue of (10.2); for a given  $\varepsilon > 0$ , we therefore have, in the neighbourhood of  $+\infty$ ,

$$\log g(x) \geq (\mu - \tfrac{1}{2}\varepsilon) \log x,$$

or again  $g(x) \geq x^{\mu-(\varepsilon/2)} \gg x^{\mu-\varepsilon}$ . Since  $\mu > -1$ , we can take  $\mu - \varepsilon > -1$  and the integral  $\int_a^{+\infty} g(t) dt$  is infinite because of (9.5). Integrating by parts

$$(10.5.3) \quad \int_a^x g(t) dt = xg(x) - ag(a) - \int_a^x tg'(t) dt$$

which can be written

$$(10.5.4) \quad \int_a^x (g(t) + tg'(t)) dt = xg(x) - ag(a).$$

But the hypothesis  $tg'(t) \sim \mu g(t)$  shows, taking (10.2) into account, that the first number of (10.5.4) is equivalent to

$$(\mu + 1) \int_a^x g(t) dt,$$

hence the formula (10.5.1). The formula (10.5.2) is proved similarly.

*Remark (10.6)* If  $g'(x)/g(x) = o(1/x)$ , we find in the same way:

$$|\log g(x)| \ll \log x$$

(10.2), which gives  $g(x) \gg x^{-\varepsilon}$  for every  $\varepsilon > 0$  (4.2.1), therefore the integral  $\int_a^{+\infty} g(t) dt$  is still infinite. Furthermore,  $xg'(x) \ll g(x)$  implies  $g(x) + xg'(x) \sim g(x)$ , and the reasoning of (10.5) shows that the formula (10.5.1) is still valid with  $\mu = 0$ .

Lastly, consider the case where  $g$  is  $> 0$  and continuously differentiable and  $|g'(x)/g(x)| \gg 1/x$ , and suppose that  $g'$  does not change sign in the neighbourhood of  $+\infty$ , which implies that  $h(x) = g(x)/g'(x)$  is defined in such a neighbourhood.

(10.7) *Under these conditions suppose, further, that  $h(x) = g(x)/g'(x)$  is continuously differentiable in the neighbourhood of  $+\infty$ , and satisfies  $h'(x) = o(1)$ .*

*Then:*

(i) *if  $g'(x) > 0$  in the neighbourhood of  $+\infty$ , the integral  $\int_a^{+\infty} g(t) dt$  is infinite, and*

$$(10.7.1) \quad \int_a^x g(t) dt \sim \frac{(g(x))^2}{g'(x)};$$

(ii) *if  $g'(x) < 0$  in the neighbourhood of  $+\infty$ , the integral  $\int_a^{+\infty} g(t) dt$  is convergent, and*

$$(10.7.2) \quad \int_a^{+\infty} g(t) dt \sim \frac{(g(x))^2}{|g'(x)|}.$$

First note that the relation  $h'(x) = o(1)$  implies  $h(x) = o(x)$  (10.2). If  $g'(x) > 0$  in the neighbourhood of  $+\infty$ ,  $g$  is increasing and  $> 0$  so the integral  $\int_a^{+\infty} g(t) dt$  is infinite. Moreover by (10.2)

$$|\log g(x)| \gg \log x$$

and, since  $g$  is increasing, this shows that  $g(x)$  tends to  $+\infty$  with  $x$ , therefore also  $\log g(x)$ , and we have by (4.2.1),  $g(x) \gg x^\alpha$  for every  $\alpha > 0$ . If, on the other hand,  $g'(x) < 0$  in the neighbourhood of  $+\infty$ , we obtain in the same way  $|\log 1/g(x)| \gg \log x$ , then  $g(x) \ll x^{-\alpha}$  for every  $\alpha > 0$ , hence the convergence of the integral  $\int_a^{+\infty} g(t) dt$  (9.5). In case (i) integration by parts gives

$$(10.7.3) \quad \int_a^x g(t) dt = \int_a^x h(t) g'(t) dt = h(x)g(x) - h(a)g(a) - \int_a^x g(t)h'(t) dt$$

or

$$(10.7.4) \quad \int_a^x (1 + h'(t))g(t) dt = h(x)g(x) - h(a)g(a).$$

Since, by hypothesis,  $(1 + h')g \sim g$ , the first member of (10.7.4) is equivalent to  $\int_a^x g(t) dt$  (10.2); since it tends to  $+\infty$  and  $h(a)g(a)$  is constant, the conclusion follows. Case (ii) is treated similarly.

*Examples* (10.8.1) In the neighbourhood of  $+\infty$  (with  $a > 1$ )

$$(10.8.2) \quad \int_a^x \frac{dt}{\log t} \sim \frac{x}{\log x}.$$

Indeed, here  $g'(x)/g(x) = -1/(x \log x)$  and we have the case considered in (10.6).

(10.8.3) In the neighbourhood of  $+\infty$

$$(10.8.4) \quad \int_x^{+\infty} e^{-t^2} dt \sim \frac{e^{-x^2}}{2x}.$$

Indeed, here  $g'(x)/g(x) = -2x$ , and we have the case considered in (10.7 (ii)).

(10.8.5) Consider the integral  $\int_a^x e^{\sqrt{\log t}}/(t \log t) dt$  (with  $a > 1$ ), which is infinite, since the integrand  $g(x) \gg 1/(x \log x)$  (9.5). Here we have

$$g'(x)/g(x) \sim -1/x,$$

and this is not one of our previous cases. However, it can be reduced to the case (10.7 (i)) by the change of variable  $u = \log t$ , which gives the integral  $\int_{\log a}^{\log x} e^{\sqrt{u}}/u du$ . Here

$$\frac{g'(x)}{g(x)} = \frac{1}{2\sqrt{x}} - \frac{1}{x} \sim \frac{1}{2\sqrt{x}},$$

hence by (10.7 (i))†

$$(10.8.6) \quad \int_a^x \frac{e^{\sqrt{\log t}}}{t \log t} dt \sim \frac{2e^{\sqrt{\log x}}}{\sqrt{\log x}},$$

(10.9) In any of the cases where (10.5), (10.6) or (10.7) is applied and when  $g \in \mathcal{E}$ , we can try to find not only a *principal part* but also an *asymptotic development* of  $\int_a^x g(t) dt$ . Suppose first that  $\int_a^{+\infty} g(t) dt$  is infinite; if, for example, we are under the conditions of (10.5 (i)), we can write by (10.5.4)

$$\int_a^x g(t) dt = \frac{xg(x)}{\mu + 1} - \frac{ag(a)}{\mu + 1} + \int_a^x g_1(t) dt$$

with  $g_1(x) = (1/(\mu + 1))(\mu g(x) - xg'(x)) \ll g(x)$  by hypothesis. If the integral  $\int_a^{+\infty} g_1(t) dt$  is infinite, the constant  $ag(a)/(\mu + 1)$  is negligible compared to  $\int_a^x g_1(t) dt$ , and we shall obtain a second term of the desired development by obtaining a *principal part* of  $\int_a^x g_1(t) dt$  with the help of (10.2) and one of the propositions (10.5), (10.6) or (10.7), if possible. If, on the other hand, the integral  $\int_a^{+\infty} g_1(t) dt$  is convergent, we must write

$$\int_a^x g(t) dt = \frac{xg(x)}{\mu + 1} + A - \int_x^{+\infty} g_1(t) dt$$

---

† Here the scale  $\mathcal{E}$  contains the functions  $\exp((\log x)^\gamma)$  for  $\gamma > 0$ .

where  $A$  is the constant  $-ag(a)/(\mu + 1) + \int_a^{+\infty} g_1(t) dt$  (whose approximate value can be calculated by the methods of computation of definite integrals). A principal part of  $\int_a^{+\infty} g_1(t) dt$  is then sought. The other cases are treated in the same way, taking into account the fact that in (10.7) the function  $g^2/g'$  does not necessarily belong to  $\mathcal{E}$ , and so it must be replaced by an asymptotic development, if there is one.

*Examples* (10.10.1) The method described in (10.9), applied to the example (10.8.1), gives

$$\int_a^x \frac{dt}{\log t} - \frac{x}{\log x} \sim \int_a^x \frac{dt}{(\log t)^2}$$

an integral to which (10.6) again applies. The method here can be continued as far as we please, and gives an asymptotic development to any finite number  $k$  of terms:

$$(10.10.2) \quad \int_a^x \frac{dt}{\log t} = \frac{x}{\log x} + \frac{1! x}{(\log x)^2} + \cdots + \frac{(k-1)! x}{(\log x)^k} + o\left(\frac{x}{(\log x)^k}\right).$$

Observe that *all* the terms of this development tend to  $+\infty$  with  $x$ .

(10.10.3) The example (10.8.3) can be treated in the same way, but it is simpler here to integrate by parts in a slightly different way.

$$\int_x^{+\infty} e^{-t^2} dt = \frac{e^{-x^2}}{2x} - \frac{1}{2} \int_x^{+\infty} \frac{e^{-t^2}}{t^2} dt$$

then

$$\int_x^{+\infty} \frac{e^{-t^2}}{t^2} dt = \frac{e^{-x^2}}{2x^3} - \frac{3}{2} \int_x^{+\infty} \frac{e^{-t^2}}{t^4} dt$$

and we can continue as long as we please, which gives

$$(10.10.4) \quad \int_x^{+\infty} e^{-t^2} dt = e^{-x^2} \left( \frac{1}{2x} - \frac{1}{4x^3} + \cdots + (-1)^k \frac{1 \cdot 3 \cdot \dots \cdot (2k-1)}{2^{k+1} x^{2k+1}} + o\left(\frac{1}{x^{2k+1}}\right) \right).$$

(10.11) Returning to the study of a primitive  $\int_a^x f(t) dt$ , when  $f$  does not belong to  $\mathcal{E}$ , but admits a principal part  $c.g$  ( $c \neq 0, g \in \mathcal{E}$ ), we begin by seeking an asymptotic development of  $c \cdot \int_a^x g(t) dt$ , then proceed in the same way for  $f - cg$  and so on, and add the asymptotic developments obtained (following the rules of (7.1)).

(10.12) *Example.* An asymptotic development of

$$(10.12.1) \quad \int_a^x \frac{e^t}{t^2 + 1} dt$$

is required.

We have the asymptotic development

$$\frac{e^x}{x^2 + 1} = \frac{e^x}{x^2} - \frac{e^x}{x^4} + o\left(\frac{e^x}{x^4}\right),$$

then by successive integration by parts

$$(10.12.2) \quad \int_a^x \frac{e^t dt}{t^2} = \frac{e^x}{x^2} + 2 \frac{e^x}{x^3} + 6 \frac{e^x}{x^4} + o\left(\frac{e^x}{x^4}\right)$$

$$(10.12.3) \quad \int_a^x \frac{e^t dt}{t^4} = \frac{e^x}{x^4} + o\left(\frac{e^x}{x^4}\right).$$

Finally, taking into account (10.12.3) and (10.2)

$$(10.12.4) \quad \int_a^x \frac{e^t dt}{t^2 + 1} = \frac{e^x}{x^2} + 2 \frac{e^x}{x^3} + 5 \frac{e^x}{x^4} + o\left(\frac{e^x}{x^4}\right).$$

*Remark (10.13)* In contrast to the situation for the *primitive* of a function, one cannot in general deduce any information about the *derivative* of a function  $f$ , when a principal part of  $f$  is known. For example, if  $f(x) = x + \sin x^2$ , then  $f(x) \sim x$ , but  $f'(x) = 2x \cos x^2$  is not even bounded and oscillates in every neighbourhood of  $+\infty$ . This illustrates the simplicity of integration compared to differentiation, mentioned in a general way in (I, 3.7).

## I. Convergence of series and asymptotic development of partial sums

(11.1) We confine ourselves to the study of series whose general term  $u_n$  *does not change sign* in the neighbourhood of  $+\infty$ ; we may suppose that  $u_n > 0$  for every  $n \geq 0$  (as long as we do no numerical calculations). At the level of this book, the convergence of a series with terms  $> 0$  is determined by the following rule: *seek a principal part*  $n \rightarrow cg(n)$  of the sequence  $n \rightarrow u_n$ , where  $c \neq 0$  and  $g \in \mathcal{E}$ , and the series  $(u_n)$  converges if, and only if, the series  $(g(n))$  converges. Indeed, this is an obvious consequence of the comparison principle (I, 2.2), since there exists  $n_0$  such that, for  $n \geq n_0$ ,  $u_n \leq \frac{3}{2}cg(n)$  and  $u_n \geq \frac{1}{2}cg(n)$ . Usually the function  $n \rightarrow u_n$  will be the *restriction* to  $\mathbf{N}$  of a function  $x \rightarrow u(x)$  defined in a neighbourhood  $[a, +\infty[$  of  $+\infty$ , and it will be sufficient to apply to this function the methods already described in this chapter.

Warning is given against the injudicious use of certain “cookery recipes” which go under the name of “Cauchy’s rule” and “d’Alembert’s rule”, and which strictly speaking apply only to power series, of which more later (Chap. VI).

(11.2) We are thus led to the series with general term  $g(n)$ , where  $g \in \mathcal{E}$ . It has already been observed (10.4) that the functions of  $\mathcal{E}$  are continuously differentiable and monotonic in the neighbourhood of  $+\infty$ . As we shall see, these properties alone enable us to reduce the study of the series with general term  $g(n)$  to that of the *primitive*  $\int_a^x g(t)dt$ .

(11.3) *Let  $g$  be a function  $> 0$  and decreasing in the neighbourhood of  $+\infty$ . Then the series with general term  $g(n)$  converges if, and only if, the integral  $\int_a^{+\infty} g(t)dt$  is convergent.*

It may be supposed that  $g$  is decreasing for  $x \geq n_0$  and a piecewise-continuous function  $f(x)$ , for  $x \geq n_0$ , may be defined in the following way: for each integer  $n \geq n_0$  put  $f(x) = g(n+1)$  for  $n \leq x < n+1$  (Fig. 10). It is clear that

$$(11.3.1) \quad g(x+1) \leq f(x) \leq g(x) \quad \text{for } x \geq n_0$$

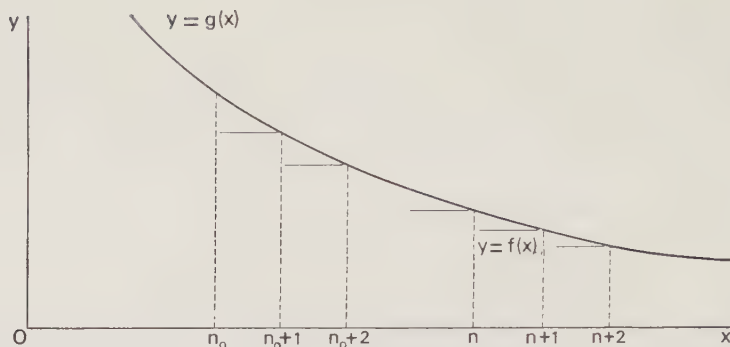


FIGURE 10

and the conclusion follows from the relation

$$(11.3.2) \quad \sum_{k=n_0+1}^n g(k) = \int_{n_0}^{n-1} f(t) dt$$

and the monotone limit theorem.

From a practical point of view, the convergence of the series with general term  $c.g(n)$  ( $c \neq 0$ ,  $g \in \mathcal{E}$ ) is thus determined by looking at where the function  $g(x)$  is situated compared with the functions  $x^\alpha (\log x)^\beta$  in the neighbourhood of  $+\infty$ , exactly as in (9.4).

(11.4) With more precise information about the derivative of  $g$ , one can, as we shall see, obtain a principal part for the partial sum

$$(11.4.1) \quad s_n = g(0) + g(1) + \cdots + g(n)$$

when the series is divergent, and for the remainder

$$(11.4.2) \quad r_n = g(n+1) + g(n+2) + \cdots,$$

when the series is convergent, by application of the methods of no. 10 to the primitive of  $g$ .

(11.5) Let  $g$  be a continuously differentiable function  $> 0$  in the neighbourhood of  $+\infty$ , and suppose that  $g'(x)/g(x) \sim \mu$  (with  $\mu \neq 0$ ).

Then:

(i) if the integral  $\int_a^{+\infty} g(t) dt$  is infinite (which is always the case when  $\mu > 0$ ),

$$(11.5.1) \quad s_n \sim \frac{\mu}{1 - e^{-\mu}} \int_a^n g(t) dt;$$

(ii) if the integral  $\int_a^{+\infty} g(t) dt$  is convergent,

$$(11.5.2) \quad r_n \sim \frac{\mu}{1 - e^{-\mu}} \int_n^{+\infty} g(t) dt.$$

Let  $v_n = \int_{n-1}^n g(t) dt$ ; let  $u$  and  $v$  be functions defined in the neighbourhood of  $+\infty$  in the following way: for every integer  $n > a$  and for  $n-1 \leq x < n$  put  $u(x) = g(n)$ ,  $v(x) = v_n$ . Since for each integer  $n_0 > a$  we have  $\sum_{k=n_0+1}^n g(k) = \int_{n_0}^n u(t) dt$  and  $\int_{n_0}^n g(t) dt = \int_{n_0}^n v(t) dt$ , by (10.2) the proposition will be proved, if the relation

$$(11.5.3) \quad \int_{n-1}^n g(t) dt \sim \frac{1 - e^{-\mu}}{\mu} g(n)$$

is proved.

Now, we can write  $g(t) = e^{\mu t} h(t)$  and the hypothesis is equivalent to saying

$$\frac{h'(x)}{h(x)} = o(1).$$

We have

$$(11.5.4) \quad \begin{aligned} \int_{n-1}^n g(t) dt &= \int_{n-1}^n e^{\mu t} h(t) dt \\ &= \frac{1 - e^{-\mu}}{\mu} g(n) + \int_{n-1}^n e^{\mu t} (h(t) - h(n)) dt. \end{aligned}$$

For each  $\varepsilon > 0$ , there exists an integer  $n_1$  such that

$$\left| \frac{h'(x)}{h(x)} \right| \leq \varepsilon \quad \text{for } x > n_1;$$

the theorem of the mean thus shows that for  $n > n_1$  and for  $n-1 \leq t \leq n$

$$-\varepsilon \leq \log \frac{h(t)}{h(n)} \leq \varepsilon$$

and hence

$$(1 - e^\varepsilon)h(n) \leq (e^{-\varepsilon} - 1)h(n) \leq h(t) - h(n) \leq (e^\varepsilon - 1)h(n)$$

or

$$|h(t) - h(n)| \leq (e^\varepsilon - 1)h(n)$$

from which we obtain, by the theorem of the mean,

$$\begin{aligned} \left| \int_{n-1}^n e^{\mu t} (h(t) - h(n)) dt \right| &\leq (e^\varepsilon - 1) e^{\mu n} h(n) = (e^\varepsilon - 1) g(n) \quad \text{if } \mu > 0 \\ \left| \int_{n-1}^n e^{\mu t} (h(t) - h(n)) dt \right| &\leq (e^\varepsilon - 1) e^{\mu(n-1)} h(n) = e^\mu (e^\varepsilon - 1) g(n) \quad \text{if } \mu < 0 \end{aligned}$$

and since  $e^\varepsilon - 1$  is arbitrarily small with  $\varepsilon$ , this proves (11.5.3), taking into account (11.5.4).

*Remark (11.6)* If  $g'(x)/g(x) = o(1)$ , the preceding reasoning remains valid replacing  $h$  by  $g$  and  $(1 - e^{-\mu})/\mu$  by 1. Therefore, the formulae (11.5.1) and (11.5.2) remain valid in this case replacing the factor  $\mu/(1 - e^{-\mu})$  by 1.

(11.7) Let  $g$  be a continuously differentiable, monotonic function  $> 0$  in the neighbourhood of  $+\infty$ , and suppose that  $|g'(x)/g(x)| \gg 1$  (i.e.  $\lim_{x \rightarrow +\infty} |g'(x)/g(x)| = +\infty$ ).

Then:

(i) if  $g$  is increasing in the neighbourhood of  $+\infty$ ,

$$(11.7.1) \quad s_n \sim g(n);$$

(ii) if  $g$  is decreasing in the neighbourhood of  $+\infty$ ,

$$(11.7.2) \quad r_n \sim g(n+1).$$

The hypothesis implies (10.2) that  $|\log g(x)| \gg x$ ; if  $g$  is increasing, this shows that  $\log g(x)$  tends to  $+\infty$  with  $x$ , and by (4.2.1)  $g(x) \gg e^x$ . On the other hand, if  $g$  is decreasing, the same reasoning on  $1/g$  gives  $g(x) \ll e^{-x}$ , and hence in this case the convergence of the integral  $\int_a^{+\infty} g(t) dt$  and so of the series with general term  $g(n)$ . Furthermore, if for  $x > n_0$ ,  $g$  is increasing, we have, with the same notations as in the proof of (11.5)

$$\begin{aligned} \sum_{k=n_0+1}^{n-1} g(k) &= \int_{n_0}^{n-1} u(t) dt \leq \int_{n_0+1}^n g(t) dt \ll \int_{n_0+1}^n g'(t) dt \\ &= g(n) - g(n_0+1) \end{aligned}$$

taking (10.2) into account. Since  $g(n)$  tends to  $+\infty$ , this implies

$$s_{n-1} = o(g(n))$$

and since  $s_n = s_{n-1} + g(n)$  this proves (11.7.1). We reason similarly for (11.7.2), showing that  $r_{n+1} = o(g(n+1))$ .

(11.8) Under the conditions of (11.7), if  $g$  is increasing, we can write for each integer  $k > 0$

$$s_n = g(n) + g(n-1) + \cdots + g(n-k) + o(g(n-k)).$$

If we have an asymptotic development for each of the functions  $g(n-j)$  ( $0 \leq j \leq k$ ), we can deduce an asymptotic development of  $s_n$ ; similarly, if  $g$  is decreasing

$$r_n = g(n+1) + g(n+2) + \cdots + g(n+k) + o(g(n+k))$$

and it is sufficient to find asymptotic developments of the  $g(n+j)$ . For example, suppose that  $g(x) = x^x$ ; starting from the formula

$$(n-1) \log(n-1) = (n-1) \log n - 1 + \frac{1}{2n} + o\left(\frac{1}{n}\right)$$

we obtain (7.4)

$$(n-1)^{n-1} = \frac{1}{e} n^{n-1} + \frac{1}{2e} n^{n-2} + o(n^{n-2})$$

and in the same way

$$(n-2)^{n-2} = \frac{1}{e^2} n^{n-2} + o(n^{n-2})$$

hence the development with three terms†

$$s_n = n^n + \frac{1}{e} n^{n-1} + \left(\frac{1}{2e} + \frac{1}{e^2}\right) n^{n-2} + o(n^{n-2}).$$

(11.9) Let us return now to the study of the series with general term  $u_n > 0$  for which we do not necessarily have  $u_n = g(n)$  for some function  $g \in \mathcal{E}$ . If  $(v_n)$  is a second series with terms  $> 0$  and if, for  $x \geq 1$ , we set  $u(x) = u_n$  for  $n \leq x < n+1$ ,  $v(x) = v_n$  for  $n \leq x < n+1$ , then the relation  $u_n \leq v_n$  for  $n \geq n_0$  is equivalent to  $u(x) \leq v(x)$  for  $x \geq n_0$ . It follows immediately that each of the relations  $u_n = O(v_n)$ ,  $u_n = o(v_n)$ ,  $u_n \sim c \cdot v_n$  ( $c \neq 0$ ) is respectively equivalent to  $u(x) = O(v(x))$ ,  $u(x) = o(v(x))$ ,  $u(x) \sim c \cdot v(x)$  in the neighbourhood of  $+\infty$ . Furthermore, since the function  $x \rightarrow \int_1^x u(t) dt$  is increasing, the convergence of the series with general term  $u_n$  is equivalent to the convergence of the integral  $\int_1^{+\infty} u(t) dt$ . Applying (10.2) to the functions  $u$  and  $v$  therefore:

(11.10) Let  $u_n, v_n$  be the general terms of two series with terms  $> 0$ .

(i) If the series with general term  $v_n$  is convergent and if  $u_n = o(v_n)$  (resp.  $u_n \sim c \cdot v_n$ ), then

$$\sum_{k=n}^{\infty} u_k = o\left(\sum_{k=n}^{\infty} v_k\right) \quad \left(\text{resp. } \sum_{k=n}^{\infty} u_k \sim c \cdot \sum_{k=n}^{\infty} v_k\right).$$

(ii) If  $\sum_{n=1}^{\infty} v_n = +\infty$ , and if  $u_n = o(v_n)$  (resp.  $u_n \sim c \cdot v_n$ ), then

$$\sum_{k=1}^n u_k = o\left(\sum_{k=1}^n v_k\right) \quad \left(\text{resp. } \sum_{k=1}^n u_k \sim c \cdot \sum_{k=1}^n v_k\right).$$

(11.11) With the same notations, if  $v_n = g(n)$ , where  $g \in \mathcal{E}$ , then the existence of a principal part  $u_n \sim c \cdot g(n)$  ( $c \neq 0$ ) of  $u_n$  reduces the problem of obtaining a principal part of  $\sum_{k=n}^{\infty} u_k$  or of  $\sum_{k=1}^n u_k$  (according to whether the series with general term  $u_n$  is convergent or not) to the same problem for  $g(n)$ , i.e. to the problem treated in (11.5), (11.6), (11.7).

If, for example,  $\sum_{k=1}^{\infty} u_n = +\infty$ , when we have obtained a principal part  $\sum_{k=1}^n u_k \sim c \cdot f(n)$ , where  $f \in \mathcal{E}$ , we may propose to extend the asymptotic development: to do this, consider the series with general term  $w_n = u_n - c(f(n) - f(n-1))$  (where  $f(0)$  is replaced by 0). By hypothesis,  $\sum_{k=1}^n w_k = o(f(n))$ ; if  $w_n$  does not change sign in the neighbourhood of  $+\infty$ , we can seek a principal part of  $w_n$  and proceed as with  $u_n$ . However, there are two cases to distinguish according to whether the series with general term  $w_n$  converges or not; in the latter case we shall have a principal part  $\sum_{k=1}^n w_k \sim c_1 \cdot f_1(n)$  with  $f_1 \in \mathcal{E}$ ,  $f_1 = o(f)$  and  $f_1 \gg 1$ , and the asymptotic development will then be

$$\sum_{k=1}^n u_k = c \cdot f(n) + c_1 \cdot f_1(n) + o(f_1(n)).$$

† The scale  $\mathcal{E}$  contains here the functions  $\exp(x^\alpha (\log x)^\beta)$  for  $\alpha > 0$  and  $\beta > 0$ .

On the other hand, if the series with general term  $w_n$  is convergent, the difference  $\sum_{k=1}^n u_k - c.f(n)$  tends to a *constant*  $S$  (in general  $\neq 0$ ) equal to  $\sum_{n=1}^{\infty} w_n$ ; the method indicated above then gives a principal part

$$\sum_{k=n+1}^{\infty} w_k \sim c_1.f_1(n) \quad \text{with } f_1 = o(1),$$

and there is thus (if  $S \neq 0$ ) an asymptotic development with three terms

$$\sum_{k=1}^n u_k = c.f(n) + S - c_1.f_1(n) + o(f_1(n)).$$

One can, of course, continue the method. An important case will be examined later (Chap. IX) when we can immediately write asymptotic developments of *arbitrary* length for the partial sums  $\sum_{k=1}^n g(k)$ .

*Examples.* (11.12.1) The asymptotic development

$$(11.12.2) \quad 1 + \frac{1}{2} + \cdots + \frac{1}{n} = \log n + \gamma + \frac{1}{2n} + o\left(\frac{1}{n}\right)$$

will be derived where  $\gamma$  is a constant called *Euler's constant*, whose value to 9 places of decimals is

$$(11.12.3) \quad = 0.577\,215\,644\dots$$

Apply our method in the case where  $u_n = g(n)$  with  $g(x) = 1/x$ ; then

$$\frac{g'(x)}{g(x)} = -\frac{1}{x};$$

we have the case treated in (11.6) and the principal part is therefore the same as that of  $\int_1^n g(t) dt$ , i.e.  $\log n$ . Next consider the series with general term

$$w_n = \frac{1}{n} - (\log n - \log(n-1)) \sim -\frac{1}{2n^2}$$

which is convergent; its remainder has the same principal part as that of the series with general term  $-1/2n^2$ . To evaluate it, note that we are still in the case (11.6) and that  $\int_x^{+\infty} dt/t^2 = 1/x$ . Hence the formula (11.12.2).

(11.12.4) We have the principal part

$$(11.12.5) \quad 2^2 \log 2 + 2^3 \log 3 + \cdots + 2^n \log n \sim 2^{n+1} \log n.$$

Here  $g(x) = 2^x \log x$ , hence  $\frac{g'(x)}{g(x)} = \log 2 + \frac{1}{x \log x} \sim \log 2$ , so we are in the

case (11.5) with  $\mu = \log 2$ . On the other hand, by virtue of (10.7), we have  $\int_1^x 2^t \log t dt \sim 2^x \log x / \log 2$ , hence (11.12.5).

(11.12.6) We have the asymptotic development†

$$(11.12.7) \quad n! \sim A n^{n+1/2} e^{-n} \left( 1 + \frac{1}{12n} + o\left(\frac{1}{n}\right) \right)$$

where  $A$  is a constant  $> 0$  (we shall see in Chap. IV that  $A = \sqrt{2\pi}$ ). Write  $n! = e^{\log n!}$  and find a development of  $\sum_{k=1}^n \log k$ . Here  $g(x) = \log x$ , so we are in the case (11.6) and therefore have the principal part  $\sum_{k=1}^n \log k \sim n \log n$ . Applying repeatedly the method described in (11.11), one obtains

$$\sum_{k=1}^n \log k = n \log n - n + \frac{1}{2} \log n + B + \frac{1}{12n} + o\left(\frac{1}{n}\right)$$

where  $B$  is a constant; since  $1/12n$  is the first term of the development which tends to 0, the required development is obtained by the method of (7.4) from the preceding one.

## APPENDIX

### Newton's polygon and Puiseux developments

#### I. Possible forms of the branches of an algebraic curve

The implicit functions problem consists, as we know (K-R, p. 111), in the study of the properties of those real functions  $u(x)$  of a real variable, which for every value of  $x$  in a neighbourhood of a point  $x_0$  satisfy a given equation

$$(1.1) \quad F(x, u(x)) = 0.$$

Suppose  $F$  defined in a neighbourhood of a point  $(x_0, y_0) \in \mathbf{R}^2$  and  $F(x_0, y_0) = 0$ . We then know that if  $F$  is continuous and admits continuous partial derivatives in a neighbourhood of  $(x_0, y_0)$  and if  $\frac{\partial F}{\partial y}(x_0, y_0) \neq 0$ , then there exists one, and only one, function  $u(x)$  continuous and differentiable in a neighbourhood of  $x_0$ , which satisfies (1.1) with  $u(x_0) = y_0$ .

If we do not assume that  $\frac{\partial F}{\partial y}(x_0, y_0) \neq 0$ , the problem is far more complex and we shall only study a few special cases. To simplify matters let us assume that  $x_0 = y_0 = 0$ ; it will also be necessary to confine ourselves to values of  $x$  and  $y$  of a determined *sign* (the example  $y^2 - x = 0$  shows the necessity of this limitation). Suppose therefore that the function  $F(x, y)$  is defined for  $0 \leq x \leq a$ ,  $0 \leq y \leq a$  and can be written in the form

$$F(x, y) = \sum_j c_j x^{\alpha_j} y^{\beta_j} (1 + \varphi_j(x, y))$$

with the following hypotheses satisfied:

1. The numbers  $\alpha_j, \beta_j$  are real and  $\geq 0$ , the pairs  $(\alpha_j, \beta_j)$  are distinct, no pair  $(\alpha_j, \beta_j)$  is equal to  $(0, 0)$ , and  $\inf \alpha_j = \inf \beta_j = 0$  (in other words  $F(0, 0) = 0$  and one cannot factorize out of  $F$  any power of  $x$  or of  $y$  to eliminate trivial solutions).

† The scale  $\mathcal{E}$  contains here the functions  $\exp(x^\alpha (\log x)^\beta)$  for  $\alpha > 0$  and  $\beta > 0$ .

2. The functions  $\varphi_j$  are continuous, admit continuous partial derivatives  $\partial\varphi_j/\partial x$ ,  $\partial\varphi_j/\partial y$  for  $0 \leq x \leq a$ ,  $0 \leq y \leq a$ , and satisfy  $\varphi_j(0, 0) = 0$ .
3. The numbers  $c_j$  are real and  $\neq 0$ .

Under these conditions we shall show that a principal part can be determined in the neighbourhood of  $x = 0$  for every continuous solution  $u$  of (1.1) satisfying  $u(0) = 0$ .

(1.2) Suppose that  $u$  is continuous for  $0 \leq x \leq b < a$ , that  $u(x) > 0$  for  $x > 0$  and that  $u(0) = 0$ . Then for each number  $\mu > 0$  the quotient  $u(x)/x^\mu$  tends to a finite or infinite limit as  $x$  tends to 0 through positive values.

If the result is false, there exist two sequences  $(x'_n)$ ,  $(x''_n)$  tending to 0 through positive values such that the sequences

$$(u(x'_n)/x'^\mu_n), \quad (u(x''_n)/x''^\mu_n)$$

tend respectively to distinct limits  $h'$ ,  $h'' \geq 0$  (one being possibly infinite; see [FA] 3.16.4). Assuming  $h' < h''$ , choose  $t$  such that  $h' < t < h''$ . Then there exists a sequence  $(z_n)$  of distinct numbers tending to 0 such that

$$(1.2.1) \quad u(z_n) = t \cdot z_n^\mu$$

In fact, by hypothesis there is an integer  $n_1$  such that

$$u(x'_{n_1}) < tx'^\mu_{n_1},$$

then an integer  $n_2 > n_1$  such that  $x''_{n_2} < x'_{n_1}$  and  $u(x''_{n_2}) > tx''^\mu_{n_2}$ , then an integer  $n_3 > n_2$  such that  $x'_{n_3} < x''_{n_2}$  and  $u(x'_{n_3}) < tx'_{n_3}^\mu$ , and so on. By (0, 3.3) there is, therefore, a sequence of numbers  $(z_n)$  such that

$$x'_{n_1} > z_1 > x''_{n_2} > z_2 > x'_{n_3} > z_3 > x''_{n_4} > \dots$$

satisfying (1.2.1). Now, we can write

$$(1.2.2) \quad F(x, tx^\mu) = \sum_j c_j x^{\alpha_j + \mu\beta_j} t^{\beta_j} (1 + \varphi_j(x, tx^\mu)).$$

Let  $\gamma > 0$  be the smallest of the numbers  $\alpha_j + \mu\beta_j$ ; then the coefficient of  $x^\gamma$  in (1.2.2) is

$$(1.2.3) \quad \psi(t) = \sum_j c_j t^{\beta_j}$$

where the summation is over those indices  $j$  such that  $\alpha_j + \mu\beta_j = \gamma$ . If  $\psi(t) \neq 0$ , then the properties of the  $\varphi_j$  imply that

$$F(x, tx^\mu) \sim \psi(t)x^\mu$$

as  $x$  tends to 0. This contradicts the existence of an infinity of roots  $z_n$  of the equation  $F(x, tx^\mu) = 0$ , the roots belonging to the interval  $]0, b[$  and tending to 0. Therefore,  $\psi(t) = 0$  for every  $t$ ,  $h' < t < h''$ . However, since the  $\beta_j$  in (1.2.3) are distinct and the  $c_j$  are  $\neq 0$ , this is impossible, as can be seen by an induction argument on the number of exponents. The proposition is trivial when there is just one exponent, otherwise we divide by the power  $t^{\beta_j}$  having the smallest exponent, which we may even suppose is zero; from the relation  $\psi(t) = 0$  for  $h' < t < h''$  we deduce  $\psi'(t) = 0$  in this interval, and we obtain a relation with the same form and less terms, hence the result.

(1.3) Under the hypotheses of (1.2) let us examine the possibility that the ratio  $u(x)/x^\mu$  tends to a finite, non-zero limit  $t$  as  $x$  tends to 0. With the same notations as (1.2), we must necessarily have  $\psi(t) = 0$ , as otherwise  $F(x, u(x)) \sim \psi(t)x^\gamma$  as  $x$  tends to 0, which is absurd since

$F(x, u(x))$  is identically zero. Since we suppose  $t \neq 0$ ,  $\psi(t) = 0$  only if at least two pairs  $(\alpha_j, \beta_j)$  satisfy  $\alpha_j + \mu\beta_j = \gamma$ ; in other words the number  $\mu$  must satisfy the following two conditions:

$$(1.3.1) \quad \alpha_j + \mu\beta_j = \alpha_k + \mu\beta_k$$

for at least two distinct pairs  $(\alpha_j, \beta_j)$  and  $(\alpha_k, \beta_k)$ ;

$$(1.3.2) \quad \alpha_j + \mu\beta_j \leq \alpha_i + \mu\beta_i$$

for every other pair  $(\alpha_i, \beta_i)$ .

It is convenient to represent these conditions graphically, considering the points  $(\alpha_j, \beta_j)$  in the plane  $\mathbf{R}^2$ . To say that  $\mu$  satisfies the preceding conditions signifies that the line whose equation is  $u + \mu v = \gamma$ , contains at least two of the points  $(\alpha_j, \beta_j)$  and that all the other points  $(\alpha_j, \beta_j)$  are above this line. All of these lines are obtained in the following manner: begin with the point  $(0, \beta_j)$  corresponding to the smallest value of the  $\beta_j$  among those pairs  $(\alpha_j, \beta_j)$  that have  $\alpha_j = 0$ ; let  $\sigma_0$  be this smallest value. Then consider the lines  $u + \mu(v - \sigma_0) = 0$  passing through the point  $(0, \sigma_0)$ , and examine the quotients  $\alpha_j/(\sigma_0 - \beta_j)$  amongst those pairs  $(\alpha_j, \beta_j)$  having  $\alpha_j > 0$  and  $\beta_j < \sigma_0$ ; if  $\mu_1$  is the smallest of these quotients, the line  $u + \mu_1(v - \sigma_0) = 0$  is the first to be considered. Let  $(\rho_1, \sigma_1)$  be the pair  $(\alpha_j, \beta_j)$  situated on this line for which  $\alpha_j$  is as large as possible. Now consider the lines  $u - \rho_1 + \mu(v - \sigma_1) = 0$  passing through this point, and examine the quotients  $(\alpha_j - \rho_1)/(\sigma_1 - \beta_j)$  amongst those pairs  $(\alpha_j, \beta_j)$  satisfying  $\alpha_j > \rho_1$ ,  $\beta_j < \sigma_1$ ; if  $\mu_2$  is the smallest such quotient, the line  $u + \mu_2 v = \rho_1 + \mu_2 \sigma_1$  is our second one. Let  $(\rho_2, \sigma_2)$  be the pair  $(\alpha_j, \beta_j)$  situated on this line for which  $\alpha_j$  is as large as possible. The third operation will consist in examining the quotients  $(\alpha_j - \rho_2)/(\sigma_2 - \beta_j)$  for  $\alpha_j > \rho_2$ ,  $\beta_j < \sigma_2$  and taking the smallest quotient  $\mu_3$ . We continue in this way, and the operation is completed when we obtain a point  $(\rho_r, 0)$  where  $\rho_r$  is the smallest of the numbers  $\alpha_j$  occurring amongst the pairs  $(\alpha_j, \beta_j)$  with  $\beta_j = 0$ . A convex polygonal curve is thus obtained with  $r$  sides, which is called the *Newton Polygon* of the function  $F$  in the neighbourhood of  $(0, 0)$  (Fig. 11).

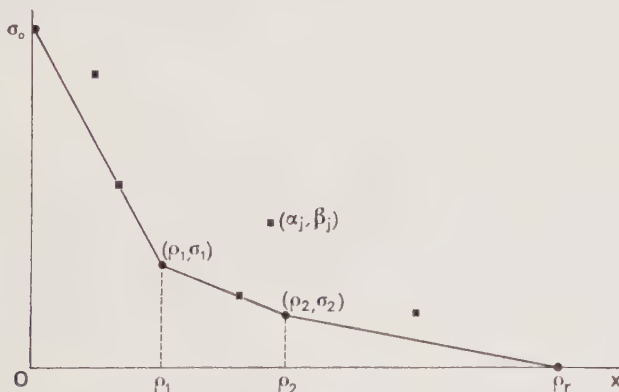


FIGURE 11

(1.4) If  $\mu_1, \mu_2, \dots, \mu_r$  are the distinct numbers  $> 0$  determined in (1.3), then, under the hypotheses of (1.2),  $u(x)/x^{\mu_h}$  tends to a finite, non-zero limit for one index  $h$ .

It follows from the definitions that  $\mu_1 < \mu_2 < \dots < \mu_r$ . We first prove that  $u(x)/x^{\mu_1}$  tends to a finite limit. Because of the properties of the  $\varphi_j$ , it may be supposed that for two

distinct indices  $i, j$  we do not have simultaneously  $\alpha_i \leq \alpha_j$  and  $\beta_i \leq \beta_j$  (otherwise the term  $c_j c_i^{-1} x^{\alpha_j - \alpha_i} y^{\beta_j - \beta_i} (1 + \varphi_j(x, y))$  would be absorbed into  $\varphi_i(x, y)$ ). We may suppose that the sequence  $(\beta_j)$  is decreasing, so  $\beta_1 = \sigma_0$  and  $\beta_j < \beta_1$  for  $j > 1$ . Equation (1.1) can then be written in the form

$$c_1(u(x))^{\beta_1}(1 + \varphi_1(x, u(x))) + \sum_h c_h x^{\alpha_h}(u(x))^{\beta_h}(1 + \varphi_h(x, u(x))) \\ = - \sum_j c_j x^{\alpha_j}(u(x))^{\beta_j}(1 + \varphi_j(x, u(x)))$$

where we have taken in the first member those indices  $h$  satisfying  $\alpha_h + \mu_1 \beta_h = \mu_1 \beta_1$  and in the second member those indices  $j$  such that  $\alpha_j + \mu_1 \beta_j > \mu_1 \beta_1$ . Putting  $t(x) = u(x)/x^{\mu_1}$ , we obtain

$$(1.4.1) \quad c_1(1 + \varphi_1(x, u(x))) + \sum_h c_h (t(x))^{\beta_h - \beta_1}(1 + \varphi_h(x, u(x))) \\ = - \sum_j c_j x^{\alpha_j + \mu_1 \beta_j - \mu_1 \beta_1} (t(x))^{\beta_j - \beta_1} (1 + \varphi_j(x, u(x))).$$

If  $t(x)$  tends to  $+\infty$  as  $x$  tends to 0, because  $\beta_h - \beta_1 < 0$  and  $\beta_j - \beta_1 < 0$ , all the terms of this equation tend to 0 except the first, which tends to  $c_1 \neq 0$ . This is absurd.

The reasoning proves (1.4) when  $r = 1$ : indeed, in this case  $\beta_h = 0$  for the largest  $l$  of the indices  $h$  occurring in the first member of (1.4.1) (such that  $\beta_l = \sigma_1$ ). By multiplying the two members of (1.4.1) by  $(t(x))^{\beta_1}$ , we see that if  $t(x)$  tends to 0, we again obtain an equation where the first member tends to  $c_l \neq 0$  and the second to 0, which is absurd.

To complete the proof of (1.4) in the general case, reason by induction on  $r$ . If  $t(x)$  tends to a non-zero limit, the proposition is established. Otherwise,  $t(x)$  tends to 0; we have then the relation between  $x$  and  $t(x)$

$$\sum_j c_j x^{\alpha_j + \mu_1 \beta_j - \mu_1 \beta_1} (t(x))^{\beta_j}(1 + \varphi_j(x, t(x)x^{\mu_1})) = 0$$

i.e. a relation of the same type as (1.1). However, this time the terms not containing a power of  $x$  are

$$c_1 y^{\beta_1}(1 + \varphi_1(x, y)) + \cdots + c_l y^{\beta_l}(1 + \varphi_l(x, y))$$

which can be written

$$c_l y^{\beta_l}(1 + \psi_l(x, y))$$

where  $\psi_l$  tends to 0. It is immediately apparent that the construction of (1.3) on this relation yields a "Newton Polygon" of  $r - 1$  sides corresponding to the numbers  $\mu_2 - \mu_1, \mu_3 - \mu_1, \dots, \mu_r - \mu_1$ . The induction hypothesis shows that the quotient  $t(x)/x^{\mu_h - \mu_1}$  tends to a finite limit  $\neq 0$  for some  $h$  satisfying  $2 \leq h \leq r - 1$ , which completes the proof of (1.4).

## 2. Existence of the branches

The induction reasoning of (1.4) shows that if we wish to know whether, for each of the numbers  $\mu_h$ , there exists a "branch" of the curve (1.1) defined by  $y = u(x)$  satisfying  $u(x) \sim \alpha x^{\mu_h}$  for a constant  $\alpha \neq 0$ , one only need consider the case  $r = 1$ . It then follows from (1.4.1) that the coefficient  $\alpha$  must be the root of the equation

$$(2.1) \quad c_1 t^{\beta_1} + \sum_h c_h t^{\beta_h} = 0$$

and since we confined ourselves to the case where  $x \geq 0$  and  $y \geq 0$ , it is necessary that this equation *have roots*  $> 0$ . The first member of (2.1) being indefinitely differentiable for  $t > 0$ , can be written, for such a root  $t_0 > 0$ , in the form  $(t - t_0)^q g(t)$  where  $g$  is continuous for  $t > 0$  and  $g(t_0) \neq 0$ . Let us set  $y = zx^{\mu_1}$  in (1.1); from the above the equation is written

$$(2.2) \quad (z - t_0)^q = x^\lambda G(x, z)$$

where  $\lambda > 0$  and  $G$  is a continuous function for  $0 \leq x \leq a$ ,  $|z - t_0| \leq d$  (for some  $d > 0$ ). Furthermore, the hypotheses of differentiability made on the  $\varphi_j$  imply that  $\partial G / \partial z$  is continuous in the neighbourhood of the point  $(0, t_0)$ .

This being so, if  $q$  is *even* and  $G(0, t_0) < 0$ , there is no function  $v(x) = z$  continuous and tending to  $t_0$  as  $x$  tends to 0 satisfying (2.2). If, on the other hand,  $G(0, t_0) \geq 0$ , the equation (2.2) can be written in the form

$$z - t_0 = \pm x^{\lambda/q} (G(x, z))^{1/q}$$

and we can apply to each of these two equations the implicit functions theorem recalled in no. 1; there are thus *two branches* of distinct curves such that  $u(x) \sim t_0 x^{\mu_1}$ .

If  $q$  is *odd*, the equation (2.2) is again written

$$z - t_0 = x^{\lambda/q} (G(x, z))^{1/q}$$

and we always obtain *one, and only one, branch* of the curve such that  $u(x) \sim t_0 x^{\mu_1}$ .

Having thus obtained the principal part of a function  $u(x)$  satisfying (1.1), one can, of course, seek an asymptotic development of several terms by setting  $u(x) = t_0 x^{\mu_1} (1 + u_1(x))$  and treating similarly the relation between  $x$  and  $u_1(x)$  so obtained.

*Example (2.3)* Consider the relation

$$(2.3.1) \quad y^3 - 2xy^2 + x^2y + x^5 - x^4y = 0.$$

The Newton Polygon has here two sides corresponding to  $\mu_1 = 1, \mu_2 = 3$ ; the equation (2.1) in  $t$  for  $\mu_1$  is  $t(t^2 - 2t + 1) = 0$ , which has just one positive root  $t_0 = 1$  and for which  $q = 2$ . The equation corresponding to (2.2) is

$$(z - 1)^2 = x^4 \frac{z - 1}{z}$$

which has two roots  $v(x) = 1, v(x) = 1 + x^4 + o(x^4)$ . For  $\mu_2$  the equation in  $t$  is  $t + 1 = 0$ , which has no positive root. We have, therefore, two "branches" which are  $> 0$  for  $x > 0$  and which are respectively

$$u_1(x) = x, \quad u_2(x) = x + x^5 + o(x^5).$$

Note that in a case like the one considered here, the function  $F(x, y)$  is defined for  $x$  and  $y$  of *any sign*; the same method can thus be applied replacing  $x$  or  $y$  by  $-x$  or  $-y$ ; in this case a third "branch" of (2.3.1) is obtained

$$u_3(x) = -x^3$$

the three functions  $u_1, u_2, u_3$  being defined for  $x$  of any sign near to 0.

The asymptotic developments obtained in this way when  $F$  is a *polynomial* in  $x$  and  $y$  are called *Puiseux developments*.

### 3. Generalizations

Newton's method can be extended to the case where

$$F(x, y) = \sum_j c_j g_j(x) y^{\beta_j} (1 + \varphi_j(x, y))$$

with the same hypotheses as before on the  $\varphi_j$ , the  $g_j$  being here *functions of a scale of comparison*  $\mathcal{E}$  (III, 2) in the neighbourhood of  $x = 0$ . We also assume that  $\inf \beta_j = 0$  and that one of the functions  $g_j$  is a non-zero *constant*, the others *tending to 0* with  $x$ ; further, as in the proof of (1.4), one may assume that the two relations  $\beta_j < \beta_i$  and  $g_i = O(g_j)$  do not hold simultaneously (otherwise absorb  $y^{\beta_i - \beta_j} g_i(x)/g_j(x)$  into  $\varphi_i(x, y)$ ). It can then be assumed that the sequence  $(\beta_j)$  is decreasing; in this case  $g_1(x)$  is necessarily a non-zero constant and  $g_j(x)$  tends to 0 with  $x$  for  $j > 1$ .

Consider then the functions  $(g_j(x))^{1/(\beta_1 - \beta_j)}$ . There exists an index  $m$  such that

$$(g_j(x))^{1/(\beta_1 - \beta_j)} = O((g_m(x))^{1/(\beta_1 - \beta_m)})$$

for every  $j$ ; set

$$u(x) = t(x)(g_m(x))^{1/(\beta_1 - \beta_m)}$$

and the relation (1.1) takes the form

$$\begin{aligned} b_1(t(x))^{\beta_1}(1 + \psi_1(x, u(x))) + \sum_h b_h(t(x))^{\beta_h}(1 + \psi_h(x, u(x))) \\ = - \sum_j b_j(t(x))^{\beta_j} g_j(x) (g_m(x))^{(\beta_j - \beta_1)/(\beta_1 - \beta_m)} (1 + \psi_j(x, u(x))) \end{aligned}$$

where the  $b_i$  are real constants  $\neq 0$ , the  $\psi_i$  are functions having the same properties as the  $\varphi_i$ . The first member takes in those indices  $h$  such that the ratio

$$(g_h(x))^{1/(\beta_1 - \beta_h)} / (g_m(x))^{1/(\beta_1 - \beta_m)}$$

tends to a finite, non-zero limit as  $x$  tends to 0; the second member takes in those indices  $j$  for which the analogous ratio tends to 0. Reasoning as in section 1, and taking into account the fact that  $\beta_1 > \beta_i$  for  $i > 1$ , the function  $t(x)$  tends to a *finite limit* as  $x$  tends to 0. If this limit is not zero, it satisfies the equation

$$b_1 t^{\beta_1} + \sum_h b_h t^{\beta_h} = 0.$$

Otherwise the relation between  $x$  and  $t(x)$  is of the same form as that between  $x$  and  $u(x)$ , but with one fewer term in the equation, and one can then proceed by induction.

For example, consider the relation

$$y^7 + y^4 x \log x + y^2 x^2 - e^{-1/x} = 0.$$

We obtain three "branches" with principal parts

$$\begin{aligned} u_1(x) &\sim \left(x \log \frac{1}{x}\right)^{1/3}, & u_2(x) &\sim \left(\frac{x}{\log 1/x}\right)^{1/2}, \\ u_3(x) &\sim \frac{1}{x} e^{-1/2x}. \end{aligned}$$

## PROBLEMS

1. Show that in the neighbourhood of  $+\infty$ , the function  $f(x) = (x \cos^2 x + \sin^2 x) e^{x^2}$  is monotonic and tends to  $+\infty$ , but that neither the ratio  $f(x)/x^{1/2} e^{x^2}$  nor its reciprocal is bounded.

2. Let  $\varphi$  be a function  $> 0$  defined and increasing for  $x > 0$ .

(a) Show that, if the function  $\log \varphi(x)/\log x$  is increasing, the relation  $f \ll g$  between two functions  $> 0$  implies  $\varphi \circ f \ll \varphi \circ g$ .

(b) Show that, if the function  $\log \varphi(x)/\log x$  is decreasing, the relation  $f \sim g$  between two functions  $> 0$  implies  $\varphi \circ f \sim \varphi \circ g$ .

3. Set  $f(x) = 1/x$  for  $x > 0$ . Define an increasing function  $g$ , continuously differentiable in the neighbourhood of  $+\infty$ , such that  $f' \ll g'$ , but such that  $(1/g)' \ll (1/f)'$  does not hold. (Take  $g'(x) = 1$  except in sufficiently small intervals with centres at the integers  $n > 0$ , in which  $g'$  takes very large values.)

4. In the neighbourhood of  $+\infty$ , the function  $\frac{\sin x}{\sqrt{x}} + \frac{\sin^2 x}{x}$  has for generalized principal part (7.6) the function  $\frac{\sin x}{\sqrt{x}}$ ; but the integral  $\int_1^{+\infty} \frac{\sin t}{\sqrt{t}} dt$  is convergent whereas the integral  $\int_1^{+\infty} \left( \frac{\sin t}{\sqrt{t}} + \frac{\sin^2 t}{t} \right) dt$  is not convergent.

5. Define inductively for  $n$  an integer  $\geq 1$  and  $x > 0$ .

$$e_1(x) = e^x, \quad e_n(x) = e_{n-1}(e^x);$$

$e_n$  is an increasing function such that  $e_{n-1}(x^\mu) \ll e_n(x)$  for every  $\mu > 0$ ; note that the inverse function  $l_n(x)$  is defined in the interval  $[e_n(0), +\infty[$ , and in the neighbourhood of  $+\infty$   $l_n(x) \ll (l_{n-1}(x))^\mu$  for every  $\mu > 0$ .

(a) Show that there exists a function  $f$  increasing, continuous and  $> 0$  defined for  $x \geq 0$ , such that  $f(2x) = 2^{f(x)}$  for every  $x \geq 0$  (define  $f$  inductively in the intervals  $[2^n, 2^{n+1}]$ ). Show that, in the neighbourhood of  $+\infty$ ,  $f(x) \gg e_n(x)$  for every integer  $n$ .

(b) Show that, if  $g$  is the inverse function of  $f$ , then  $1 \ll g(x) \ll l_n(x)$  in the neighbourhood of  $+\infty$ , for every integer  $n$ .

6. Let  $f$  and  $g$  be two continuously differentiable functions  $> 0$  in the neighbourhood of  $+\infty$ .

(a) Suppose  $f'$  monotone and  $\neq 0$  in the neighbourhood of  $+\infty$ . Show that, if  $g \ll f/f'$ , then  $f(x + g(x)) \sim f(x)$  in the neighbourhood of  $+\infty$  (use the mean value theorem). Show that this result cannot be improved, by proving that it is no longer true if we suppose only that  $g = O(f/f')$ .

(b) Suppose further that  $g(x) = o(x)$ . Show that we then have  $f(x - g(x)) \sim f(x)$  in each of the following cases:

1.  $|f'/f|$  is increasing in the neighbourhood of  $+\infty$ .

2.  $f'(x)/f(x) = O(1/x)$  in the neighbourhood of  $+\infty$ .

3. The function  $h(x) = |f(x)/f'(x)|$  satisfies  $1 \ll h(x) \ll x$ , and furthermore is differentiable and that such  $h'(x)/h(x) = O(1/x)$  (observe that in this case  $h(x - g(x)) \sim h(x)$ ).

Give an example where  $g = o(f/f')$ ,  $g(x) = O(x)$ , and where

$$f(x - g(x)) = o(f(x))$$

in the neighbourhood of  $+\infty$ .

7. Find asymptotic developments of the various "branches"  $u(x)$  of the functions defined by the equation

$$x(u(x))^5 + u(x) + 1 = 0$$

in the neighbourhood of  $x = 0$  and in the neighbourhood of  $+\infty$ .

8. Find asymptotic developments of the functions defined by

$$\begin{aligned} 2e^{u(x)-x} + u(x) + x &= 0 && \text{in the neighbourhood of } -\infty; \\ x^{u(x)} - (u(x))^{2x} &= 0 && \text{in the neighbourhood of } +\infty. \end{aligned}$$

9. Let  $f$  and  $g$  be two real functions defined in a neighbourhood of  $+\infty$ . Suppose the following conditions satisfied:

1. The roots of the equation  $f(x) = 0$  form an increasing sequence  $(x_n)$  tending to  $+\infty$ .
2. There exist constants  $a > 0$ ,  $b \geq 0$ ,  $c > 0$  such that  $|f'(x_n)| \geq a$ ,  $|f''(x)| \leq b$  for every  $n$  and for every  $x$  satisfying  $|x - x_n| \leq c$ .
3. The functions  $g$ ,  $g'$  and  $g''$  tend to 0 as  $x$  tends to  $+\infty$ .

Set  $\Phi_n(u) = g(x_n + u) - g(x_n) - f(x_n + u) + f'(x_n)u$ , and define for  $n \geq 1$  and  $m \geq 0$  the numbers  $u_{mn}$  by the following conditions

$$\begin{aligned} u_{0n} &= 0 \\ f'(x_n)u_{mn} &= g(x_n) + \Phi_n(u_{m-1,n}) \quad \text{for } m \geq 1. \end{aligned}$$

Show that as soon as  $n$  is large enough, as  $m$  tends to  $+\infty$ , the sequence  $(u_{mn})_{m \geq 0}$  tends to a limit  $u_n$  such that

$$f(x_n + u_n) = g(x_n + u_n)$$

and, moreover, as  $n$  tends to  $+\infty$ ,

$$u_n - u_{mn} = o(u_{mn} - u_{m-1,n})$$

for each fixed value of  $m$ .

By applying these results, find asymptotic developments of the roots  $x = z_n$  of the following equations:

$$\tan x = ax \quad (a > 0) \quad \text{with} \quad z_n \sim (2n+1)\frac{\pi}{2}$$

$$\sin x = \frac{1}{\log x} \quad \text{with} \quad z_n \sim n\pi.$$

How can we treat in the same way the equation

$$\sin^2 x = \frac{1}{\log x}$$

where the above condition 2 is no longer satisfied?

10. For each real number  $t > 0$  denote by  $[t]$  the largest integer  $n \leq t$ . Show that the function  $f(x) = 1/x - [1/x]$  admits an improper integral in the interval  $]0, 1]$  and that

$$\int_0^1 \left( \frac{1}{t} - \left[ \frac{1}{t} \right] \right) dt = 1 - \gamma \quad (\gamma \text{ Euler's constant}).$$

11. Let  $f(x)$  be a function monotone in the interval  $]0, 1]$ , and such that the improper integral  $\int_0^1 t^a f(t) dt$  is convergent. Show that

$$\lim_{x \rightarrow 0} x^{a+1} f(x) = 0$$

(majorize and minorize the integral  $\int_x^{2x} t^a f(t) dt$ ).

Deduce from this that if  $f$  is monotone in the interval  $[1, +\infty[$  and if the improper integral  $\int_1^{+\infty} t^a f(t) dt$  is convergent, then  $\lim_{x \rightarrow +\infty} x^{a+1} f(x) = 0$ .

12. Study the convergence of the improper integrals

$$\int_{\pi}^{+\infty} \frac{dx}{x^a (\sin x)^{p/q}} \quad (p \text{ integer } > 0, q \text{ odd integer } > 0)$$

$$\int_{\pi}^{+\infty} \frac{dx}{x^a |\sin x|^b}, \quad \int_0^{+\infty} \frac{x^c dx}{1 + x^a |\sin x|^b}$$

( $a, b, c$  real).

13. For which values of the real constants  $a, b$ , is the series with general term

$$u_n = \log n + a \log(n+1) + b \log(n+2)$$

convergent?

14. Find an asymptotic development of

$$(p_1 a_1^n + p_2 a_2^n + \cdots + p_k a_k^n)^{1/n}$$

as  $n$  tends to  $+\infty$ , the  $p_j$  and  $a_j$  being numbers  $> 0$ .

15. Find an asymptotic development of

$$\frac{1}{n^2 + 1} + \frac{1}{n^2 + 2} + \cdots + \frac{1}{kn^2 - 1} + \frac{1}{kn^2} \quad (k \text{ integer } > 1).$$

16. Let  $f$  and  $g$  be two functions  $> 0$ , continuously differentiable for  $x > 0$  and such that the integral  $\int_1^{+\infty} f(t) dt$  is infinite. Set  $F(x) = \int_1^x f(t) dt$ .

(a) Show that, if  $F(x)g'(x) = o(f(x))$  (which implies  $g(x) = o(\log F(x))$ ), then we have a generalized principal part

$$\int_1^x f(t) e^{tg(t)} dt \sim F(x) e^{tg(x)}$$

(integrate by parts).

(b) Show that, if  $F(x)g'(x) \sim c \cdot f(x)$  ( $c$  real constant  $\neq 0$ ) (which implies  $g(x) \sim c \cdot \log F(x)$ ), then

$$\int_1^x f(t) e^{tg(t)} dt \sim \frac{1}{1+ic} F(x) e^{tg(x)} \quad (\text{analogous method}).$$

(c) Suppose  $f$  and  $g$  indefinitely differentiable, and  $|f(x)/g'(x)|$  increasing and tending to  $+\infty$  with  $x$ . Suppose further that

$$\frac{d}{dx} \left( \frac{f(x)}{g'(x)} \right) = f(x)h(x)$$

with  $h(x) = o(1)$  and  $h'(x) = o(1)$  (which implies  $|F(x)g'(x)| \gg f(x)$  and  $g(x) \gg \log F(x)$ ). Show that we then have

$$\int_1^x f(t)e^{tg(t)} dt \sim \frac{f(x)}{ig'(x)} e^{tg(x)}$$

(integrate several times by parts).

Formulate the analogous results for  $\int_x^{+\infty} f(t)e^{tg(t)} dt$  when the integral  $\int_1^{+\infty} f(t)dt$  is convergent (in the case (c), we must suppose  $|f(x)/g'(x)|$  decreasing and tending to 0 with  $1/x$ ).

As an example, consider the integrals  $\int_1^{+\infty} t^\alpha e^{it^\beta} dt$ .

17. With the same hypotheses as in problem 16, case (a) or case (b), suppose further that  $f'(x) = o(f(x))$  and  $g'(x) = o(1)$ . Show then that

$$\begin{aligned} \sum_{k=1}^n f(k)e^{ig(k)} &\sim F(n)e^{ig(n)} \quad \text{in case (a).} \\ \sum_{k=1}^n f(k)e^{ig(k)} &\sim \frac{1}{1+ic} F(n)e^{ig(n)} \quad \text{in case (b).} \end{aligned}$$

As an example, consider the sums  $\sum_{k=1}^n k^\alpha e^{icf(\log k)\beta}$  where  $\alpha > 0$  and  $0 < \beta \leq 1$ .

18. Let  $f$  be a real function twice differentiable in an open interval  $I \subset \mathbf{R}$  and such that  $f''(x) \geq \lambda > 0$  in  $I$ . Show that, for any two points  $a < b$  in  $I$ ,

$$\left| \int_a^b e^{if(t)} dt \right| \leq \frac{8}{\sqrt{\lambda}}.$$

(Reduce to the case where  $f'(x) \geq 0$  in  $[a, b]$ ; partition the interval  $[a, b]$  at the point  $a + \frac{2}{\sqrt{\lambda}}$  if  $b - a > 2/\sqrt{\lambda}$ , and integrate by parts in the interval  $\left[a + \frac{2}{\sqrt{\lambda}}, b\right]$ .)

19. Let  $(r_n)_{n \geq 1}$  be an increasing sequence of numbers  $> 0$  tending to  $+\infty$ . For each  $r \geq 0$ , call  $N(r)$  the number of indices  $n$  such that  $r_n \leq r$ ;  $N$  is a step function in  $[0, +\infty[$ .

(a) For every continuous function  $f$ , primitive of a piecewise-continuous function  $f'$  in  $[0, +\infty[$ , we have

$$\sum_{r_0 \leq r} f(r_n) = N(r)f(r) - \int_0^r N(t)f'(t) dt.$$

(b) We say that a function  $L(r)$  defined for  $r \geq 0$  and taking values  $> 0$  is *slowly monotone*, if it is monotone and if  $\lim_{r \rightarrow +\infty} L(2r)/L(r) = 1$ .

For example, the products of functions of the form  $(\log r)^\alpha$ ,  $(\log \log r)^\alpha$ , etc. are slowly monotone. If  $L$  is slowly monotone, then

$$\lim_{r \rightarrow +\infty} L(cr)/L(r) = 1$$

for every number  $c > 0$ , and  $\lim_{r \rightarrow +\infty} \log L(r)/\log r = 0$  (consider the values  $r = 2^m$  for  $m$  an integer).

(c) We say that the sequence  $(r_n)$  is *regular* if  $N(r) \sim r^\lambda L(r)$ , where  $\lambda$  is a number  $> 0$  and  $L$  a slowly monotone function. Then  $\log N(r) \sim \lambda \log r$ , and  $\lambda$  is the infimum of the set of numbers  $\sigma > 0$  such that the series with general term  $r_n^{-\sigma}$  is convergent.  $\lambda$  is the *exponent of convergence* of the sequence  $(r_n)$ .

(d) Let  $c$  be a number  $> 0$ ,  $f$  a piecewise-continuous function in the interval  $[0, c]$ . Show that

$$\lim_{r \rightarrow +\infty} \frac{1}{N(r)} \sum_{r_n \leq cr} f\left(\frac{r_n}{r}\right) = \int_0^{c^\lambda} f(t^{1/\lambda}) dt$$

(prove this first for a step function, then approximate  $f$  uniformly (Chap. V) by a step function in  $[0, c]$ ).

20. Let  $(a_n)_{n \geq 1}$  be a sequence of numbers  $\geq 0$ . Set

$$t_n = \sqrt{a_1 + \sqrt{a_2 + \cdots + \sqrt{a_n}}}.$$

(a) Show that, if all the  $a_n$  are equal to 1, the sequence  $(t_n)$  tends to  $\frac{1}{2}(1 + \sqrt{5})$ . (Observe that the sequence  $(t_n)$  is increasing and that  $t_n^2 = 1 + t_{n-1} > t_{n-1}^2$ .)

(b) Show that, if  $a_n < e^{2^n}$  for every  $n$ , the sequence  $(t_n)$  is convergent (use (a)). On the other hand, if there exists  $\beta > 2$  such that for infinitely many  $n$ ,  $a_n > e^{\beta^n}$ , show that for all these values of  $n$ ,  $t_n \geq \exp((\beta/2)^n)$ .

21. Let  $f$  be an increasing function in an interval  $[0, b]$  admitting in the neighbourhood of 0 an asymptotic development of the form

$$f(x) = x - ax^\alpha + o(x^\alpha) \quad \text{where } \alpha > 1 \text{ and } a > 0.$$

(a) Show that there exist two real numbers  $\lambda < 0$  and  $c > 0$ , well determined and such that

$$f(cn^\lambda) - c(n+1)^\lambda = o(n^{\lambda-1}).$$

(b) Show that there exists  $b' < b$  such that, for each  $x_0 \in ]0, b'[,$  the sequence defined by  $x_1 = f(x_0), \dots, x_{n+1} = f(x_n), \dots,$  is decreasing and tends to 0, and that

$$x_n \sim cn^\lambda.$$

(If  $c' < c < c''$ , note that, for sufficiently large  $n$ ,

$$f(c'n^\lambda) > c'(n+1)^\lambda, \quad f(c''n^\lambda) < c''(n+1)^\lambda.$$

Note further that for each  $n$ , there exists  $p > 0$  such that  $x_n > c'(n+p)^\lambda$  and deduce that for each  $m > 0$ ,  $x_{n+m} > c'(n+p+m)^\lambda$ . Also, for each  $n$ , there exists  $q$  such that  $x_q < c''n^\lambda$ ; deduce that for each  $m > 0$ ,  $x_{q+m} < c''(n+m)^\lambda$ .)

(c) In particular, for each  $x > 0$ , set  $\sin_1 x = \sin x$ ,  $\sin_n x = \sin(\sin_{n-1} x)$  for  $n \geq 2$ ; show that

$$\sin_n x \sim \sqrt{\frac{3}{n}}.$$

22. Let  $f$  be a continuous function  $\geq 0$  in an open, unbounded set  $D \subset \mathbf{R}^2$ . For each  $N$  let  $Q_N$  be the square  $|x| < N, |y| < N$  in  $\mathbf{R}^2$ . We say that the integral  $\iint_D f(x, y) dx dy$  is *convergent* if as  $N$  tends to  $+\infty$ , the integral  $\iint_{D \cap Q_N} f(x, y) dx dy$  tends to a limit (which is then, by definition, the value of  $\iint_D f(x, y) dx dy$ ). For this it is sufficient that the set of these integrals is majorized in  $\mathbf{R}$ .

(a) Show that, if  $f(x, y) \leq C(x^2 + y^2)^\alpha$  with  $C > 0$  and  $\alpha < -1$  in  $D$ , the integral  $\iint_D f(x, y) dx dy$  is convergent. Deduce that if  $D$  is the quadrant  $x > 0, y > 0$ , and if  $f(x, y) = (a + bx + cy)^{-s}$  with  $a > 0, b > 0, c > 0$  and  $s > 2$ , the integral  $\iint_D f(x, y) dx dy$  is convergent.

(b)  $D$  being still the quadrant  $x > 0, y > 0$ , suppose that  $f$  is twice continuously differentiable in  $D$  and suppose that the integrals

$$\iint_D f(x, y) \, dx \, dy, \quad \iint_D \left| \frac{\partial f}{\partial x} \right| \, dx \, dy, \quad \iint_D \left| \frac{\partial f}{\partial y} \right| \, dx \, dy, \quad \iint_D \left| \frac{\partial^2 f}{\partial x \partial y} \right| \, dx \, dy$$

are all convergent. Show that the double series  $\sum_{m \geq 1, n \geq 1} f(m, n)$  is convergent. (Evaluate the difference

$$f(m, n) - \int_m^{m+1} dx \int_n^{n+1} f(x, y) \, dy$$

with the help of integration by parts with respect to  $x$  and with respect to  $y$ .)

Deduce that, under the same conditions as (a), the double series  $\sum_{m \geq 1, n \geq 1} (a + bm + cn)^{-s}$  is convergent. Give another proof of this by majorizing the partial sums

$$\sum_{m=1}^N \left( \sum_{n=1}^N (a + bm + cn)^{-s} \right).$$

23. Let  $h$  be an integer  $\geq 1$ . Show that, as  $n$  tends to  $+\infty$ , there exist two constants  $a > 0$ ,  $b > 0$  such that

$$an^{(4h-1)/2} \leq \sum_{k=1}^{n-1} \frac{\sin^2(\pi k^h/n)}{\sin^2(\pi k/n)} \leq bn^{(4h-1)/2}.$$

(Observe that the terms of the sum corresponding to  $k$  and  $n - k$  are equal, and consider separately the sums for

$$1 \leq k \leq \left[ \left( \frac{n}{2} \right)^{1/2h} \right] \quad \text{and} \quad \left[ \left( \frac{n}{2} \right)^{1/2h} \right] \leq k \leq \frac{n}{2}.$$

For the first sum, use the majorization

$$\left| \frac{\sin p\theta}{\sin \theta} \right| \leq p$$

valid for  $\theta \leq \pi/2p$ .)

24. Let  $x$  be a real number  $> 0$ .

(a) If  $N_1(x)$  is the number of integers  $n \leq x$  which have the form  $a^m$ , with  $a$  an integer  $> 1$ ,  $m$  an integer  $> 1$ , show that in the neighbourhood of  $+\infty$ ,

$$N_1(x) = O(x^{1/2} \log x).$$

(Observe that  $2^m \leq x$  and  $a^2 \leq x$ .)

(b) If  $N_2(x)$  is the number of integers  $n \leq x$  such that  $n = a^b + c^d$  where  $a, b, c, d$  are integers  $> 1$  and  $b$  and  $d$  are not both equal to 2, show that, in the neighbourhood of  $+\infty$ ,

$$N_2(x) = O(x^{5/6} (\log x)^2).$$

(Observe that we may assume that  $b \geq 2, d \geq 3$ , so  $c^3 \leq x$ .)

(c) Deduce from (a) and (b) that if  $N(x)$  is the number of integers  $n \leq x$  which have the form  $a^b + c^d$ , with  $a \geq 0, c \geq 0, b \geq 2, d \geq 2$ , then for each  $\varepsilon > 0$ ,  $N(x) \leq (\frac{3}{4} + \varepsilon)x$  in the neighbourhood of  $+\infty$ . (Observe that an integer of the form  $a^2 + b^2$  cannot have the form  $4k + 3$ ,  $k$  and integer.)

25. For each number  $\alpha > 0$ , prove the formula

$$1^{\alpha n} + 2^{\alpha n} + \dots + n^{\alpha n} \sim \frac{n^{\alpha n}}{1 - e^{-\alpha}}$$

as  $n$  tends to  $+\infty$ . (Majorize the sum  $\sum_{k=1}^{n-1} \left| \left(1 - \frac{k}{n}\right)^{\alpha n} - e^{-k\alpha} \right|$  by decomposing it into two parts at the value  $[n^\lambda]$  of  $k$ ,  $\lambda$  being suitably chosen.)

26. For each number  $\alpha > 0$ , prove the formula

$$(1!)^{-\alpha/n} + (2!)^{-\alpha/n} + \dots + (n!)^{-\alpha/n} \sim \frac{1}{\alpha} \frac{n}{\log n}$$

as  $n$  tends to  $+\infty$  (compare each term  $(p!)^{-\alpha/n}$  to  $n^{-\alpha p/n}$ , by decomposing the sum into two parts at the value  $[en]$  of  $p$ ,  $\varepsilon$  being an arbitrary number  $> 0$ ).

27. Let  $f$  be a piecewise-continuous function in the open interval  $]0, 1[$ , such that the improper integral  $\int_0^1 f(t) dt$  is convergent and such that there are two points  $a, b$ ,  $0 < a < b < 1$ , with  $f$  monotone in each of the intervals  $]0, a]$  and  $[b, 1[$ . Show that

$$f\left(\frac{1}{n}\right) + f\left(\frac{2}{n}\right) + \dots + f\left(\frac{n-1}{n}\right) \sim cn$$

where  $c = \int_0^1 f(t) dt$  (if  $c \neq 0$ ; otherwise the first member is  $o(n)$ ). Deduce from this a new proof of the relation

$$1^{\alpha-1} + 2^{\alpha-1} + \dots + n^{\alpha-1} \sim \frac{1}{\alpha} n^\alpha$$

for  $\alpha > 0$ .

28. Let  $f$  be a piecewise-continuous function in the open interval  $]0, +\infty[$ , such that the improper integral  $\int_0^{+\infty} f(t) dt$  is convergent and such that there are two points  $a, b$ ,  $0 < a < b$ , with  $f$  monotone in each of the intervals  $]0, a]$  and  $[b, +\infty[$ . Show that

$$\sum_{n=1}^{\infty} f(n\xi) \sim c/\xi$$

as  $\xi$  tends to 0 through positive values, where  $c = \int_0^{+\infty} f(t) dt$ .

29. Let  $f$  be a piecewise-continuous function in the closed interval  $0 \leq x < +\infty$ . Show that, if the integral  $\int_0^{+\infty} f(t)e^{-at} dt$  is convergent, then so is the integral  $\int_0^{+\infty} f(t)e^{-xt} dt$  for every  $x > a$ . (Use the fact that the function  $F(t) = \int_0^t f(s)e^{-as} ds$  is bounded.)

30. In the power series

$$e^{n\alpha} = 1 + \frac{n\alpha}{1!} + \dots + \frac{(n\alpha)^m}{m!} + \dots$$

where  $\alpha > 0$  and  $n$  is an integer  $> 0$ , set

$$(*) \quad P_n(\alpha) = \sum_{m=0}^n \frac{(n\alpha)^m}{m!}, \quad R_n(\alpha) = \sum_{m=n+1}^{\infty} \frac{(n\alpha)^m}{m!}$$

so that  $P_n(\alpha) + R_n(\alpha) = e^{n\alpha}$ .

(a) Show that, if  $0 < \alpha < 1$ , then as  $n$  tends to  $+\infty$ ,

$$P_n(\alpha) \sim e^{n\alpha}, \quad R_n(\alpha) \sim \frac{\alpha}{1-\alpha} \frac{(e\alpha)^n}{\sqrt{2\pi n}},$$

and for  $\alpha > 1$

$$P_n(\alpha) \sim \frac{\alpha}{\alpha-1} \frac{(e\alpha)^n}{\sqrt{2\pi n}}, \quad R_n(\alpha) \sim e^{n\alpha}.$$

(b) Show that as  $n$  tends to  $+\infty$ ,

$$P_n(1) \sim \frac{1}{2}e^n \quad \text{and} \quad R_n(1) \sim \frac{1}{2}e^n.$$

(Prove that  $(P_n(1) - R_n(1))/(P_n(1) + R_n(1))$  tends to 0 with  $1/n$ ; decompose the numerator sum into two parts, by considering, in (\*), the values  $n - [n^\lambda]$  and  $n + [n^\lambda]$  of  $m$ ,  $\lambda$  being some number  $< 1$ .)

# Integrals depending on a parameter

## 1. Introduction

We often meet functions of a real variable  $t$  which have the form

$$(1.1) \quad I(t) = \int_a^b F(x, t) \, dx$$

where for each  $t$  in the neighbourhood of a point  $t_0 \in \mathbf{R}$  (or of  $+\infty$ ), the integral exists (eventually as an *improper* integral (III, 9)). We propose to find a *principal part* (or a generalized principal part (III, 7.6)) of  $I(t)$  in the neighbourhood of  $t_0$ . We shall examine two cases where, with rather strict hypotheses on  $F$ , we are able to solve this problem; it happens that these hypotheses on  $F$  are often satisfied in applications.

The basic idea is to obtain a situation where  $t$  tends to  $+\infty$ , where  $a$  is a finite number and where for *every fixed*  $\delta > 0$ , the integral  $\int_{a+\delta}^b F(x, t) \, dx$  is *negligible* compared to the integral  $\int_a^{a+\delta} F(x, t) \, dx$ . For each  $t$ , we then replace the function  $x \rightarrow F(x, t)$  by its *principal part*  $x \rightarrow G(x, t)$  (assumed to exist) *in the neighbourhood of*  $a$ . If this function is sufficiently simple for the integral  $t \rightarrow \int_a^{a+\delta} G(x, t) \, dx$  to have a principal part easy to estimate (for  $t$  in the neighbourhood of  $+\infty$ ), it is *plausible* that the latter is also the principal part of  $I(t)$ . The proof of this fact calls for some care in the majorizations, because of the introduction of the parameter  $\delta$ , which must be “small” but which must not be made to “tend to 0” thoughtlessly.

## 2. Laplace's method

(2.1) Begin by examining a simple special case, the integral

$$(2.1.1) \quad J(t) = \int_0^b x^\alpha e^{t(a-cx^\beta)} \, dx$$

where  $a, c, \alpha, \beta$  are real,  $\alpha > -1, \beta > 0, c > 0, b$  can be a finite number  $> 0$  or  $+\infty$  and  $t$  tends to  $+\infty$ . If  $-1 < \alpha < 0$ , the integral is improper in the neighbourhood of 0, and

if  $b = +\infty$ , it is improper in the neighbourhood of  $+\infty$ ; but in both cases it is evidently *convergent*, by virtue of the hypotheses  $\alpha > -1$ ,  $c > 0$  and  $\beta > 0$ , for every  $t > 0$  (III, 9). The integral is

$$e^{at} \int_0^b x^\alpha e^{-ctx^\beta} dx$$

and after the change of variable  $u = ctx^\beta$  becomes

$$(2.1.2) \quad \frac{e^{at}}{\beta(ct)^{(\alpha+1)/\beta}} \int_0^{cb^\beta t} u^{((\alpha+1)/\beta)-1} e^{-u} du$$

where the upper limit of integration must be replaced by  $+\infty$  when  $b = +\infty$ . In all cases, the integral in (2.1.2) tends to  $\Gamma((\alpha+1)/\beta)$  as  $t$  tends to  $+\infty$  (III, 9.9), hence the *principal part*

$$(2.1.3) \quad J(t) \sim \frac{1}{\beta} \Gamma\left(\frac{\alpha+1}{\beta}\right) e^{at}(ct)^{-(\alpha+1)/\beta}.$$

It is quite easy to obtain a principal part of the remainder of the integral in (2.1.2) (when  $b$  is finite) by applying the criterion (III, 10.7.2); we obtain

$$(2.1.4) \quad J(t) = \frac{1}{\beta} \Gamma\left(\frac{\alpha+1}{\beta}\right) e^{at}(ct)^{-(\alpha+1)/\beta} (1 + O(t^{((\alpha+1)/\beta)-1} e^{-cb^\beta t})).$$

(2.2) *Laplace's method* is applied to integrals of the form

$$(2.2.1) \quad I(t) = \int_0^{+\infty} g(x) e^{th(x)} dx$$

where  $t$  tends to  $+\infty$ , and  $h(x)$  is assumed to have a *finite maximum* attained at the point 0 (and only at this point). If further, in the neighbourhood of  $x = 0$ , there are asymptotic developments

$$(2.2.2) \quad g(x) \sim Ax^\alpha, \quad h(x) = a - cx^\beta + o(x^\beta)$$

where  $A \neq 0$  and where  $\alpha$ ,  $\beta$  and  $c$  satisfy the conditions of (2.1), and if we substitute into the integral (2.2.1) the principal part of  $g$  and the first two terms of the development of  $h$  (instead of  $g$  and  $h$  respectively), we reduce to the integral  $AJ(t)$ , whose principal part is known (2.1.3). Under conditions, which we shall make precise, this process *does genuinely give the principal part of  $I(t)$* .

(2.3) *The functions  $g$  and  $h$  are assumed real, piecewise-continuous in the open interval  $]0, +\infty[$ , and satisfying the following conditions:*

1. *The improper integral  $\int_0^{+\infty} |g(x)| e^{h(x)} dx$  is convergent.*
2. *There is an interval  $]0, \delta_0[$  such that, for  $0 < \delta < \delta_0$ ,  $h(x) \leq h(\delta)$  for every  $x \geq \delta$ .*
3. *In the neighbourhood of 0, there are asymptotic developments (2.2.2), with  $\alpha > -1$ ,  $\beta > 0$ ,  $c > 0$ .*

*Under these conditions we have the principal part*

$$(2.3.1) \quad I(t) \sim \frac{A}{\beta} \Gamma\left(\frac{\alpha+1}{\beta}\right) e^{at}(ct)^{-(\alpha+1)/\beta}.$$

Note that the case of the integrals  $\int_0^b g(x) e^{th(x)} dx$  with  $b > 0$  finite, reduces immediately to the case (2.2.1) by taking  $g(x) = 0$  and  $h(x) = a - 1$  for  $x > b$ .

Multiplying by  $e^{-at}$ , it may be supposed that  $a = 0$ , and by changing the sign if necessary, that  $A > 0$ . Let

$$\varphi(t) = \frac{A}{\beta} \Gamma\left(\frac{\alpha+1}{\beta}\right) (ct)^{-(\alpha+1)/\beta}.$$

We shall prove that, for each  $\varepsilon > 0$ :

1. A number  $\delta \in [0, \delta_0]$  and a number  $t_1 > 0$  may be determined such that

$$(2.3.2) \quad \left(1 - \frac{\varepsilon}{2}\right) \varphi(t) \leq \int_0^\delta g(x) e^{th(x)} dx \leq \left(1 + \frac{\varepsilon}{2}\right) \varphi(t)$$

for every  $t \geq t_1$ .

2.  $\delta$  and  $t_1$  being thus fixed, a number  $t_2 > 0$  may be determined such that, for  $t \geq t_2$ ,

$$(2.3.3) \quad \int_\delta^{+\infty} |g(x)| e^{th(x)} dx \leq \frac{\varepsilon}{2} \varphi(t).$$

It is concluded from this that, for  $t \geq \sup(t_1, t_2)$ ,

$$(2.3.4) \quad (1 - \varepsilon) \varphi(t) \leq I(t) \leq (1 + \varepsilon) \varphi(t)$$

which will prove the theorem.

1. First choose arbitrarily a number  $\lambda$  such that  $0 < \lambda < 1$ ; by hypothesis, there exists a number  $\delta(\lambda) > 0$  such that, for  $0 < x < \delta(\lambda)$ ,

$$(2.3.5) \quad \begin{cases} A(1 - \lambda)x^\alpha \leq g(x) \leq A(1 + \lambda)x^\alpha & \text{and} \\ -c(1 + \lambda)x^\beta \leq h(x) \leq -c(1 - \lambda)x^\beta \end{cases}$$

from which

$$A(1 - \lambda) \int_0^{\delta(\lambda)} x^\alpha e^{-c(1 + \lambda)tx^\beta} dx \leq \int_0^{\delta(\lambda)} g(x) e^{th(x)} dx \leq A(1 + \lambda) \int_0^{\delta(\lambda)} x^\alpha e^{-c(1 - \lambda)tx^\beta} dx.$$

When  $\delta(\lambda)$  has been chosen, the first and third integrals, by virtue of (2.1.3), have respective *principal parts* (for  $t$  near to  $+\infty$ )

$$\frac{A(1 - \lambda)}{\beta} \Gamma\left(\frac{\alpha+1}{\beta}\right) (c(1 + \lambda)t)^{-(\alpha+1)/\beta}$$

and

$$\frac{A(1 + \lambda)}{\beta} \Gamma\left(\frac{\alpha+1}{\beta}\right) (c(1 - \lambda)t)^{-(\alpha+1)/\beta}$$

This being so, let us first choose  $\lambda$  so that

$$(1 - \lambda)(1 + \lambda)^{-(\alpha+1)/\beta} \geq \sqrt{1 - \frac{\varepsilon}{2}}$$

and

$$(1 + \lambda)(1 - \lambda)^{-(\alpha+1)/\beta} \leq \sqrt{1 + \frac{\varepsilon}{2}}$$

which is possible because of the continuity of the functions of  $\lambda$  which occur.

Let us then take  $\delta = \delta(\lambda)$  such that  $0 < \delta < \delta_0$  and such that (2.3.5) is satisfied. By definition, there then exists  $t_1 > 0$  such that, for  $t \geq t_1$ ,

$$A \int_0^\delta x^\alpha e^{-c(1+\lambda)t x^\beta} dx \geq (1 - \lambda)(1 + \lambda)^{-(\alpha+1)/\beta} \sqrt{1 - \frac{\varepsilon}{2}} \varphi(t)$$

and

$$A \int_0^\delta x^\alpha e^{-c(1-\lambda)t x^\beta} dx \leq (1 + \lambda)(1 - \lambda)^{-(\alpha+1)/\beta} \sqrt{1 + \frac{\varepsilon}{2}} \varphi(t).$$

For these choices of  $\delta$  and  $t_1$ , it is clear that we have (2.3.2) for every  $t \geq t_1$ .

2. Put  $h(\delta) = -\mu < 0$ . By hypothesis,  $h(x) + \mu \leq 0$  for  $x \geq \delta$ , hence for each  $t > 1$ ,  $t(h(x) + \mu) \leq h(x) + \mu$ , which can also be written

$$th(x) \leq -(t - 1)\mu + h(x).$$

Therefore

$$\left| \int_\delta^{+\infty} g(x) e^{th(x)} dx \right| \leq \int_\delta^{+\infty} |g(x)| e^{th(x)} dx \leq B e^{-(t-1)\mu}$$

where  $B = \int_\delta^{+\infty} |g(x)| e^{h(x)} dx$ , finite by hypothesis. Since  $\mu > 0$ ,  $\varphi(t) \gg e^{-\mu t}$  for  $t$  near  $+\infty$ , and hence there exists a number  $t_2$  for which the inequality (2.3.3) holds as soon as  $t \geq t_2$ .  
Q.E.D.

(2.4) Let us return to the case considered in (2.3) of integrals of the form

$$\int_{-\infty}^{+\infty} g(x) e^{th(x)} dx$$

where  $g, h$  are piecewise-continuously differentiable and  $h'$  changes sign at most finitely often. Suppose further that in the neighbourhood of each of the points where  $h$  has a local maximum there are (after translation of the considered point to 0) asymptotic developments of the form (2.2.2). Then decompose the interval of integration into subintervals, using the points where  $h'$  changes sign, so that in each subinterval  $h'$  has constant sign; a linear change of variable then reduces these integrals to the type (2.2.1). For instance:

(2.5) Suppose that in the interval  $]a, b[$  (bounded or not) the functions  $g$  and  $h$  are twice continuously differentiable, that the integral  $\int_a^b |g(x)| e^{h(x)} dx$  is defined and that  $h'$  changes sign at only one point  $c$ ,  $a < c < b$ , where furthermore  $h$  has a maximum,  $g(c) \neq 0$  and  $h''(c) < 0$ . Then for  $t$  near  $+\infty$ ,

$$(2.5.1) \quad \int_a^b g(x) e^{th(x)} dx \sim \Gamma\left(\frac{1}{2}\right) g(c) e^{th(c)} \sqrt{\frac{2}{-th''(c)}}$$

(we show a little later (3.7.7) that  $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ ).

*Example (2.6)* We propose to find a principal part of

$$(2.6.1) \quad \int_0^{+\infty} x^{-x} e^{tx} dx = \int_0^{+\infty} e^{tx - x \log x} dx$$

for  $t$  near  $+\infty$ . This improper integral is convergent for every  $t > 0$ , since, as  $x$  tends to  $+\infty$ ,  $(t+1)x - x \log x$  tends to  $-\infty$ , so  $e^{tx - x \log x} = o(e^{-x})$ , and the function  $e^{tx - x \log x}$  is continuous at  $x = 0$ . The integral is not of the type (2.2.1), but can be reduced to this by a change of variable. We look, in fact, at where the function  $f(x) = tx - x \log x$  attains its maximum; since  $f'(x) = t - 1 - \log x$ , it is seen that this is at the point  $x = e^{t-1}$ . Then set  $x = u e^{t-1}$  and the integral becomes

$$s \int_0^{+\infty} e^{sh(u)} du$$

where  $s = e^{t-1}$  and  $h(u) = u(1 - \log u)$ , which attains its unique maximum at the point  $u = 1$  with  $h(1) = 1$ ,  $h''(1) = -1$ . Applying (2.5) then gives

$$(2.6.2) \quad \int_0^{+\infty} x^{-x} e^{tx} dx \sim \sqrt{2\pi} e^{\frac{1}{2}(t-1) + e^{t-1}}.$$

### 3. Eulerian integrals

(3.1) We have already met the *Eulerian integral of the second kind* or *gamma function*

$$(3.1.1) \quad \Gamma(x) = \int_0^{+\infty} t^{x-1} e^{-t} dt$$

and shown that it is convergent for  $x > 0$ ; and of course  $\Gamma(x) > 0$  for  $x > 0$ . The integral

$$(3.1.2) \quad \mathbf{B}(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt$$

is called the *Eulerian integral of the first kind* or *beta function*. This integral (improper for  $x < 1$  or  $y < 1$ ) is convergent for  $x > 0$  and  $y > 0$ ; since the change of variable  $t' = 1 - t$  shows that

$$(3.1.3) \quad \mathbf{B}(y, x) = \mathbf{B}(x, y)$$

it is sufficient to prove the convergence at the point 0, which follows immediately as  $t^{x-1}(1-t)^{y-1} \sim t^{x-1}$  in the neighbourhood of  $t = 0$  (III, 9).

(3.2) For  $x > 0$ , the gamma function satisfies the functional equation

$$(3.2.1) \quad \Gamma(x+1) = x\Gamma(x).$$

In particular for every integer  $n > 0$

$$(3.2.2) \quad \Gamma(n+1) = n!$$

(in other words the gamma function “extrapolates” for every real  $x > 0$  the sequence of *factorials*).

Indeed, an integration by parts gives, for  $x > 0$ ,

$$\int_0^{+\infty} t^x e^{-t} dt = -t^x e^{-t} \Big|_0^{+\infty} + x \int_0^{+\infty} t^{x-1} e^{-t} dt$$

hence (3.2.1). Since  $\Gamma(1) = \int_0^{+\infty} e^{-t} dt = 1$ , we deduce (3.2.2) by induction on  $n$ .

Because of this fundamental property of the gamma function, the notation  $x!$  instead of  $\Gamma(x+1)$  for  $x > -1$  is sometimes used.

(3.3) Numerous integrals met in the calculus can be transformed into Eulerian integrals. We have already proved in (2.1) the formula

$$(3.3.1) \quad \int_0^{+\infty} t^\alpha e^{-ct^\beta} dt = \frac{1}{\beta c^{(\alpha+1)/\beta}} \Gamma\left(\frac{\alpha+1}{\beta}\right)$$

for  $\alpha > -1$ ,  $\beta > 0$ ,  $c > 0$ , hence in particular, taking into account (3.2.1)

$$(3.3.2) \quad \int_0^{+\infty} e^{-tx} dt = \Gamma\left(1 + \frac{1}{x}\right) \quad \text{for } x > 0.$$

On the other hand, the change of variable  $t = \sin^2 \theta$  in (3.1.2) gives

$$(3.3.3) \quad \int_0^{\pi/2} \sin^{2x-1} \theta \cos^{2y-1} \theta d\theta = \frac{1}{2} B(x, y)$$

for  $x > 0$ ,  $y > 0$ .

(3.4) For every real number  $c$ , the function  $\left(1 - \frac{t}{n}\right)^{c+n}$  has the limit  $e^{-t}$  as  $n$  tends to  $+\infty$ ; this makes *plausible* the following formula, which we now prove

$$(3.4.1) \quad \Gamma(x) = \lim_{n \rightarrow \infty} \int_0^n t^{x-1} \left(1 - \frac{t}{n}\right)^{c+n} dt \quad \text{for } x > 0.$$

Choose  $\varepsilon > 0$ ; by definition of an improper integral (III, 9), if  $a > 0$  is sufficiently small and  $A > 0$  is sufficiently large.

$$(3.4.2) \quad \left| \Gamma(x) - \int_a^A t^{x-1} e^{-t} dt \right| \leq \varepsilon.$$

To prove (3.4.1), assume that  $n > A$  and decompose the interval of integration into three subintervals  $[0, a]$ ,  $[a, A]$  and  $[A, n]$ ; one may also assume that  $c + A > 0$ ; in the interval  $[0, a]$ ,  $0 \leq 1 - \frac{t}{n} \leq 1$ , hence

$$(3.4.3) \quad \int_0^a t^{x-1} \left(1 - \frac{t}{n}\right)^{c+n} dt \leq \int_0^a t^{x-1} dt = \frac{1}{x} a^x \leq \varepsilon$$

provided  $a$  has been chosen sufficiently small. To majorize the integral over  $[A, n]$ , one shows that the function  $\varphi_n(t) = \left(1 - \frac{t}{n}\right)^{c+n} e^{-t}$  is decreasing in this interval; indeed

$$(3.4.4) \quad \frac{\varphi'_n(t)}{\varphi_n(t)} = -\frac{c+t}{n-t}$$

hence our assertion. From this is obtained the majorization

$$\left(1 - \frac{t}{n}\right)^{c+n} \leq \varphi_n(A) e^{-t}$$

and since  $\varphi_n(A)$  tends to 1, there exists  $n_0$  such that for  $n \geq n_0$

$$(3.4.5) \quad \int_A^n t^{x-1} \left(1 - \frac{t}{n}\right)^{c+n} dt \leq 2 \int_A^n t^{x-1} e^{-t} dt \leq 2\varepsilon$$

taking into account (3.4.2). It remains to majorize

$$\left| \int_a^A t^{x-1} \left[ e^{-t} - \left(1 - \frac{t}{n}\right)^{c+n} \right] dt \right|.$$

We have

$$e^{-t} - \left(1 - \frac{t}{n}\right)^{c+n} = e^{-t} \left[ 1 - e^{(c+n) \log \left(1 - \frac{t}{n}\right) + t} \right].$$

For  $0 \leq u < \frac{1}{2}$ , Taylor's formula gives

$$|\log(1 - u) + u| \leq 2u^2$$

hence, for each  $t$  satisfying  $a \leq t \leq A$ , and for  $n > 2A$ ,

$$\left| (c+n) \log \left(1 - \frac{t}{n}\right) + t \right| \leq 2(c+n) \frac{t^2}{n^2} + \frac{ct}{n} \leq \frac{C}{n}$$

where  $C$  depends neither on  $t$  nor on  $n > 2A$ . Hence

$$(3.4.6) \quad \left| \int_a^A t^{x-1} \left[ e^{-t} - \left(1 - \frac{t}{n}\right)^{c+n} \right] dt \right| \leq (1 - e^{-C/n}) \int_a^A t^{x-1} e^{-t} dt$$

and therefore there exists  $n_1 > \sup(n_0, 2A)$  such that for  $n \geq n_1$ , the second member is  $\leq \varepsilon$ . From (3.4.2), (3.4.3) and (3.4.5)

$$\left| \Gamma(x) - \int_0^n t^{x-1} \left(1 - \frac{t}{n}\right)^{c+n} dt \right| \leq 5\varepsilon$$

as soon as  $n \geq n_1$ .

Q.E.D.

(3.5) We use (3.4.1) to prove the following two *fundamental formulae* due to Euler:

$$(3.5.1) \quad \Gamma(x) = \lim_{n \rightarrow \infty} \frac{n^x \cdot n!}{x(x+1) \dots (x+n)}$$

$$(3.5.2) \quad \mathbf{B}(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

for  $x > 0, y > 0$ .

(3.6) Begin by proving the following lemma:

$$(3.6.1) \quad \mathbf{B}(x+1, y) = \frac{x}{x+y} \mathbf{B}(x, y).$$

To do this, integrate by parts by writing

$$\begin{aligned} \mathbf{B}(x+1, y) &= \int_0^1 t^x (1-t)^{y-1} dt = \int_0^1 \left( \frac{t}{1-t} \right)^x (1-t)^{x+y-1} dt \\ &= -\frac{(1-t)^{x+y}}{x+y} \left( \frac{t}{1-t} \right)^x \Big|_0^1 + \frac{x}{x+y} \int_0^1 (1-t)^{x+y} \left( \frac{t}{1-t} \right)^{x-1} \frac{dt}{(1-t)^2} \\ &= \frac{x}{x+y} \int_0^1 (1-t)^{y-1} t^{x-1} dt \end{aligned}$$

which proves (3.6.1). Taking into account the symmetry of the beta function (3.1.3), we obtain, for  $n > 0$ ,

$$\mathbf{B}(x, n+1) = \frac{n}{x+n} \mathbf{B}(x, n) = \frac{n}{x} \mathbf{B}(x+1, n)$$

then, by induction on  $n$ ,

$$(3.6.2) \quad \mathbf{B}(x, n+1) = \frac{n!}{x(x+1) \dots (x+n-1)} \mathbf{B}(x+n, 1).$$

But

$$\mathbf{B}(x+n, 1) = \int_0^1 t^{x+n-1} dt = \frac{1}{x+n} t^{x+n} \Big|_0^1 = \frac{1}{x+n}$$

therefore

$$\mathbf{B}(x, n+1) = \frac{n!}{x(x+1) \dots (x+n)}.$$

On the other hand, the formula (3.4.1) gives, for  $c = 0$ , by the change of variable  $t = nu$

$$(3.6.3) \quad \Gamma(x) = \lim_{n \rightarrow \infty} n^x \mathbf{B}(x, n+1)$$

and replacing  $\mathbf{B}(x, n+1)$  by its value (3.6.2) the Euler formula (3.5.1) is obtained.

(3.7) From the formula (3.6.1), we obtain, by induction on  $n$

$$(3.7.1) \quad \mathbf{B}(x, y) = \frac{(x+y)(x+y+1) \dots (x+y+n)}{x(x+1) \dots (x+n)} \mathbf{B}(x+n+1, y)$$

and hence

$$(3.7.2) \quad \mathbf{B}(x, y) = \lim_{n \rightarrow \infty} \frac{(x+y)(x+y+1) \dots (x+y+n)}{x(x+1) \dots (x+n)} \mathbf{B}(x+n+1, y).$$

We look for a *principal part*, as  $n$  tends to  $+\infty$ , of the integral

$$\mathbf{B}(x+n+1, y) = \int_0^1 t^{x+n} (1-t)^{y-1} dt = \int_0^1 t^{y-1} (1-t)^{x+n} dt = \int_0^1 g(t) e^{nh(t)} dt$$

where we have set

$$g(t) = t^{y-1}(1-t)^x \quad \text{and} \quad h(t) = \log(1-t).$$

The function  $h$  being strictly decreasing in  $[0, 1[$ , we have the conditions for the application of Laplace's method (2.3), since in the neighbourhood of  $t = 0$

$$g(t) \sim t^{y-1} \quad (y > 0) \quad \text{and} \quad h(t) \sim -t$$

hence

$$(3.7.3) \quad B(x+y+1, y) \sim \frac{1}{n^y} \Gamma(y).$$

But the Euler formula (3.5.1) gives the equivalences (for  $n$  in the neighbourhood of  $+\infty$ )

$$(3.7.4) \quad x(x+1)\dots(x+n) \sim \frac{n^x n!}{\Gamma(x)}$$

$$(3.7.5) \quad (x+y)(x+y+1)\dots(x+y+n) \sim \frac{n^{x+y} n!}{\Gamma(x+y)}$$

and since  $n^{x+y} = n^x n^y$ , on substituting into (3.7.2) the Euler formula (3.5.2) is obtained.

The integral (3.3.3) can thus be expressed with the aid of the gamma function

$$(3.7.6) \quad \int_0^{\pi/2} \sin^{2x-1} \theta \cos^{2y-1} \theta \, d\theta = \frac{1}{2} \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

for  $x > 0, y > 0$ . By putting  $x = y = \frac{1}{2}$  in this formula, we obtain, since  $\Gamma(1) = 1$ ,

$$(3.7.7) \quad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

Hence, using the functional equation (3.2.1),

$$\Gamma\left(n + \frac{1}{2}\right) = \left(n - \frac{1}{2}\right)\left(n - \frac{3}{2}\right)\dots\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}$$

for every integer  $n > 0$ , in other words

$$(3.7.8) \quad \Gamma\left(n + \frac{1}{2}\right) = \frac{1 \cdot 3 \cdot 5 \dots (2n-1)}{2^n} \sqrt{\pi}.$$

In particular, taking  $x = 2$  in (3.3.2)

$$(3.7.9) \quad \int_0^{+\infty} e^{-t^2} dt = \frac{1}{2}\sqrt{\pi}.$$

(3.8) We now determine precisely the principal part (III, 11.12.7) by proving *Stirling's Formula*

$$(3.8.1) \quad \Gamma(x) \sim \sqrt{2\pi} x^{x-\frac{1}{2}} e^{-x}$$

for all  $x$  in the neighbourhood of  $+\infty$ .

We apply Laplace's method to the integral

$$\Gamma(x+1) = \int_0^{+\infty} e^{x \log t - t} dt.$$

The integrand, as a function of  $t$ , has the logarithmic derivative  $\frac{x}{t} - 1$  and therefore attains a unique maximum in the interval  $]0, +\infty[$  at  $t = x$ . Changing the variable by  $u = t/x$

$$\Gamma(x+1) = x^{x+1} \int_0^{+\infty} e^{xh(u)} du$$

where the function  $h(u) = \log u - u$  attains its unique maximum at  $u = 1$ , with  $h(1) = -1$  and  $h''(1) = -1$ . The formula (2.5.1) is thus applicable and gives

$$(3.8.2) \quad \Gamma(x+1) \sim \sqrt{2\pi} x^{x+\frac{1}{2}} e^{-x}.$$

To obtain (3.8.1), it is sufficient to use the functional equation (3.2.1).

From this can be deduced principal parts of "functions of large numbers" frequently appearing in Analysis. For example, from (3.8.1) and (3.7.4), for each real number  $a > 0$

$$(3.8.3) \quad a(a+1)\dots(a+n) \sim \frac{\sqrt{2\pi}}{\Gamma(a)} n^{n+a+\frac{1}{2}} e^{-n}.$$

Taking into account (3.2.1)

$$\frac{\Gamma(n+a+1)}{\Gamma(n+1)} = \frac{a(a+1)\dots(a+n)}{n!} \Gamma(a)$$

and therefore

$$(3.8.4) \quad \frac{\Gamma(n+a)}{\Gamma(n)} \sim n^a.$$

Consider now, for a fixed real number  $k > 1$ , the binomial coefficient  $\binom{kn}{n}$  for  $n$  an integer tending to  $+\infty$ . We can write

$$\binom{kn}{n} = \frac{kn(kn-1)\dots(kn-n+1)}{n!} = \frac{\Gamma(kn+1)}{\Gamma(n+1)\Gamma((k-1)n+1)}$$

hence, by virtue of (3.8.2)

$$(3.8.5) \quad \binom{kn}{n} \sim \sqrt{\frac{k}{2\pi(k-1)n}} \left( \frac{k^k}{(k-1)^{k-1}} \right)^n.$$

In particular, in the development of  $(1+x)^{2n}$  by means of the binomial formula, the

largest of the coefficients is  $\binom{2n}{n}$ , the coefficient of  $x^n$ . It follows from (3.8.5) with  $k = 2$  that, for  $n$  tending to  $+\infty$ ,

$$(3.8.6) \quad \binom{2n}{n} \sim \frac{2^{2n}}{\sqrt{\pi n}}.$$

This will be taken up again and deepened in the study of the gamma function in Chap. IX.

#### 4. Stationary phase method

(4.1) Integrals of a quite different nature are now studied:

$$(4.1.1) \quad I(t) = \int_a^b g(x) e^{it h(x)} dx$$

where  $g$  and  $h$  are *real* functions, which may be supposed to simplify matters, as *infinitely differentiable* in the *open* interval  $]a, b[$ . The integral can be improper and is assumed convergent (but *not*, in general, absolutely convergent) for all sufficiently large values of  $t$ , and we have to determine its behaviour as  $t$  tends to  $+\infty$ . By changing the variable,  $]a, b[$  can be assumed *bounded*.

We begin with a lemma giving a somewhat rough majorization:

(4.2) *Suppose that in the bounded open interval  $]a, b[$ ,  $g$  and  $h$  satisfy the following conditions:*

1.  *$h$  tends to a finite limit at each of the points  $a$  and  $b$ , and  $h'$  does not vanish in the open interval  $]a, b[$ .*
2. *At each of the points  $a, b$ ,  $g/h'$  has a finite limit, and either  $\frac{d}{dx} \left( \frac{g(x)}{h'(x)} \right)$  has a finite limit or it has constant sign in the neighbourhood of that point.*

*Under these conditions the integral (4.1.1) is convergent for each real  $t > 0$ , and  $I(t) = O(1/t)$  in the neighbourhood of  $+\infty$ .*

For each sufficiently small number  $\delta > 0$ , integrating by parts

$$(4.2.1) \quad \begin{aligned} \int_{a+\delta}^{b-\delta} g(x) e^{it h(x)} dx &= \frac{1}{it} \int_{a+\delta}^{b-\delta} \frac{g(x)}{h'(x)} \frac{d}{dx} (e^{it h(x)}) dx \\ &= \frac{1}{it} \left[ \frac{g(b-\delta)}{h'(b-\delta)} e^{it h(b-\delta)} - \frac{g(a+\delta)}{h'(a+\delta)} e^{it h(a+\delta)} \right] \\ &\quad - \frac{1}{it} \int_{a+\delta}^{b-\delta} e^{it h(x)} \frac{d}{dx} \left( \frac{g(x)}{h'(x)} \right) dx. \end{aligned}$$

In order to study the integral in the third member of (4.2.1) we may confine our attention to the corresponding integrals over the intervals  $[a + \delta, c]$  and  $[d, b - \delta]$ , where  $c$  and  $d$  are *fixed*: indeed, since  $|e^{it h(x)}| = 1$ , the integral  $\int_c^d e^{it h(x)} \frac{d}{dx} \left( \frac{g(x)}{h'(x)} \right) dx$  is majorized in absolute value by a number *independent of  $t$* . Similarly, if  $\frac{d}{dx} \left( \frac{g(x)}{h'(x)} \right)$

tends to a finite limit at  $a$  (resp.  $b$ ), the integral over  $[a + \delta, c]$  (resp.  $[d, b - \delta]$ ) is bounded in absolute value by a number *independent of  $t$  and  $\delta$* . Suppose then that  $\frac{d}{dx} \left( \frac{g(x)}{h'(x)} \right)$  does not change sign in  $]a, c]$  for example. Then the integral

$$(4.2.2) \quad \int_{a+\delta}^c e^{t h(x)} \frac{d}{dx} \left( \frac{g(x)}{h'(x)} \right) dx$$

is majorized in absolute value by a number independent of  $t$  and  $\delta$ , and the improper integral  $\int_a^c e^{t h(x)} \frac{d}{dx} \left( \frac{g(x)}{h'(x)} \right) dx$  is *absolutely convergent*. Indeed, by virtue of the hypothesis on the sign of  $\frac{d}{dx} \left( \frac{g(x)}{h'(x)} \right)$

$$\begin{aligned} \int_{a+\delta}^c \left| e^{t h(x)} \frac{d}{dx} \left( \frac{g(x)}{h'(x)} \right) \right| dx &\leq \left| \int_{a+\delta}^c \frac{d}{dx} \left( \frac{g(x)}{h'(x)} \right) dx \right| \\ &= \left| \frac{g(c)}{h'(c)} - \frac{g(a+\delta)}{h'(a+\delta)} \right| \end{aligned}$$

and the last term tends to a finite limit as  $\delta$  tends to 0. We reason similarly for the integral over  $[d, b - \delta]$ . Thus in the formula (4.2.1) we can let  $\delta$  tend to 0, and it can thus be seen that there exists a number  $A$  independent of  $t$  such that the absolute value of  $I(t)$  is majorized by  $A/t$ .

(4.3) *Suppose that the conditions of (4.2) are fulfilled. (By an abuse of language denote by  $g(a)/h'(a)$  and  $g(b)/h'(b)$  the limits of  $g/h'$  at  $a$  and  $b$  respectively.) Then the integral  $I(t)$  has a generalized principal part (III, 7.6) in the neighbourhood of  $+\infty$*

$$(4.3.1) \quad I(t) \sim \frac{1}{it} \left[ \frac{g(b)}{h'(b)} e^{t h(b)} - \frac{g(a)}{h'(a)} e^{t h(a)} \right]$$

*provided the second member is not identically zero.*

It is required to prove that if

$$(4.3.2) \quad g_1(x) = \frac{d}{dx} \left( \frac{g(x)}{h'(x)} \right)$$

then

$$(4.3.3) \quad \int_a^b g_1(x) e^{t h(x)} dx = o(1) \quad \text{for } t \text{ near } +\infty.$$

Choose  $\varepsilon > 0$ ; since as seen above the integral (4.3.3) is *absolutely convergent*, a number  $\delta > 0$  can be found such that

$$\int_a^{a+\delta} |g_1(x)| dx \leq \varepsilon \quad \text{and} \quad \int_{b-\delta}^b |g_1(x)| dx \leq \varepsilon.$$

The number  $\delta$  being thus *fixed*, the result of (4.2) can be applied to the integral  $\int_{a+\delta}^{b-\delta} g_1(x) e^{t h(x)} dx$ , since  $g$  and  $h$  are indefinitely differentiable in a neighbourhood

of each of the points  $a + \delta$ ,  $b - \delta$  in  $]a, b[$  and  $h'$  does not vanish at these points. Hence there exists  $t_0$  such that for  $t \geq t_0$ ,

$$\left| \int_{a+\delta}^{b-\delta} g_1(x) e^{ith(x)} dx \right| \leq \varepsilon.$$

This proves that for each  $t \geq t_0$ ,  $\left| \int_a^b g_1(x) e^{ith(x)} dx \right| \leq 3\varepsilon$ , and completes the proof of (4.3).

(4.4) When the conditions of (4.2) are fulfilled, but the second member of (4.3.1) is *identically zero*, we have, from the above

$$I(t) = \int_a^b g(x) e^{ith(x)} dx = -\frac{1}{it} \int_a^b g_1(x) e^{ith(x)} dx = o(1/t)$$

and to proceed further we return to the problem with  $g$  replaced by  $g_1$ . This case will not be studied further.

(4.5) The most interesting case, in practice, occurs when the function  $h'$ , although still *not vanishing* in the *open* interval  $]a, b[$ , *tends to 0* at the endpoints of this interval, so that  $g/h'$  *tends to  $\pm\infty$*  at this point. By, if necessary, decomposing the interval  $[a, b]$  into two subintervals, it may be supposed that this happens at just *one* of the endpoints, for example at the point  $a$ . It will then be seen that in the most important case,  $I(t)$  has a principal part which dominates  $1/t$  and which depends only on the behaviour of  $g$  and  $h$  in the *neighbourhood of  $a$* . Intuitively, one can say that when  $t$  is very large, the point  $g(x)e^{ith(x)}$  “rotates very quickly” around 0 in the plane  $\mathbf{C}$  as  $x$  varies from  $a$  to  $b$ , and that the effect of this is to make the integral very small. If  $h'$  vanishes at a point, the “speed” of this rotation is very much “slowed” in the neighbourhood of this point (the “phase”  $th(x)$  is “stationary”) and the part of the integral corresponding to this neighbourhood gives a contribution which dominates the rest.

To make this precise, we begin by treating a special case:

(4.6) *Let  $\alpha, \beta, a, c$  be real numbers such that  $c \neq 0$ ,  $0 < \alpha + 1 < \beta$ . Then as  $t$  tends to  $+\infty$ , we have (for every  $b > 0$ )*

$$(4.6.1) \quad \int_0^b x^\alpha e^{it(a+cx^\beta)} dx \sim A \frac{e^{iat}}{\beta(|c|t)^{(\alpha+1)/\beta}}$$

where  $A$  is the value of the convergent integral

$$(4.6.2) \quad \int_0^{+\infty} u^{((\alpha+1)/\beta)-1} e^{tu} du \quad \text{if } c > 0$$

$$(4.6.3) \quad \int_0^{+\infty} u^{((\alpha+1)/\beta)-1} e^{-tu} du \quad \text{if } c < 0.$$

(It will be seen in Chap. VIII that for  $0 < \lambda < 1$

$$(4.6.4) \quad \begin{cases} \int_0^{+\infty} u^{\lambda-1} e^{tu} du = e^{\frac{1}{2}\lambda\pi i} \Gamma(\lambda) \\ \int_0^{+\infty} u^{\lambda-1} e^{-tu} du = e^{-\frac{1}{2}\lambda\pi i} \Gamma(\lambda). \end{cases}$$

If  $c > 0$ , the change of variable  $u = ct x^\beta$  is made in the integral (4.6.1) so obtaining

$$\frac{e^{iat}}{\beta(ct)^{(\alpha+1)/\beta}} \int_0^{cb^\beta t} u^{(\alpha+1)/\beta-1} e^{iu} du$$

and we have to prove that, for  $0 < \lambda < 1$ , the improper integral  $\int_0^{+\infty} u^{\lambda-1} e^{iu} du$  is convergent. Now the integral converges absolutely in the neighbourhood of 0, since  $u^{\lambda-1} e^{iu} \sim u^{\lambda-1}$  in this neighbourhood (III, 9). On the other hand, by integrating by parts, one can write

$$(4.6.5) \quad \int_1^N u^{\lambda-1} e^{iu} du = \frac{1}{i} u^{\lambda-1} e^{iu} \Big|_1^N - \frac{(\lambda-1)}{i} \int_1^N u^{\lambda-2} e^{iu} du$$

and since  $|u^{\lambda-2} e^{iu}| \leq u^{\lambda-2}$  with  $\lambda-2 < 1$ , the integral in the second member is absolutely convergent (III, 9). The case  $c < 0$  is treated in the same way.

Note that the formula (4.6.5) gives further a principal part

$$(4.6.6) \quad \int_N^{+\infty} u^{\lambda-1} e^{\pm iu} du \sim -\frac{1}{i} N^{\lambda-1} e^{\pm iN} = O(N^{\lambda-1})$$

for the remainder of the integrals (4.6.4).

This being so, it will be seen that, with conditions which are usually fulfilled in practice, the principal part of (4.4.1), as  $g/h'$  tends to  $\pm\infty$  at the points  $a$ , is again obtained by substituting for  $g$  and  $h$  their asymptotic developments in the neighbourhood of  $a$ , exactly as in (2.3). Clearly one can reduce to the case where  $a = 0$ .

(4.7) Suppose that the real functions  $g, h$  satisfy in the half-open interval  $]0, b]$  the following conditions:

1.  $g$  and  $h$  are indefinitely differentiable and  $h'$  does not vanish.
2. In a neighbourhood of 0,

$$(4.7.1) \quad g(x) = Cx^\alpha(1 + \theta(x)), \quad h(x) = a + cx^\beta(1 + f(x))$$

where  $\alpha, \beta, a, c, C$  are real constants such that  $c \neq 0, C \neq 0, 0 < \alpha + 1 < \beta$ , and  $\theta$  and  $f$  are continuous and indefinitely differentiable in the closed interval  $[0, b]$  with  $\theta(0) = f(0) = 0$ .

Under these conditions, in the neighbourhood of  $+\infty$

$$(4.7.2) \quad \int_0^b g(x) e^{ith(x)} dx \sim AC \frac{e^{iat}}{\beta(|c|t)^{(\alpha+1)/\beta}}$$

where the constant  $A$  has the same value as in (4.6).

Suppose for example  $c > 0$ . Consider the function

$$\varphi(x) = x(c + cf(x))^{1/\beta}$$

which is continuous and indefinitely differentiable in an interval  $[0, \delta]$  contained in  $[0, b]$ , and satisfies  $\varphi(0) = 0, \varphi'(0) = c^{1/\beta} > 0$ . It can thus be assumed that  $\delta$  has been chosen sufficiently small so that  $\varphi$  is strictly increasing in  $[0, \delta]$ .

Let  $\psi(u)$  be its *inverse function*, continuous, indefinitely differentiable and strictly increasing in the interval  $[0, \varphi(\delta)]$  and satisfying  $\psi(0) = 0$ ,  $\psi'(0) = c^{-1/\beta}$ , so that

$$(4.7.3) \quad g(\psi(u)) = Cc^{-\alpha/\beta}u^\alpha + O(u^{\alpha+1}).$$

Decompose the interval  $[0, b]$  into  $[0, \delta]$  and  $[\delta, b]$ , and in the first interval change the variable by  $u = \varphi(x)$ ; so obtaining

$$(4.7.4) \quad \int_0^\delta g(x)e^{ith(x)} dx = Cc^{-(\alpha+1)/\beta} \int_0^{\varphi(\delta)} u^\alpha e^{it(a+u^\beta)} du + \int_0^{\varphi(\delta)} g_1(u)e^{it h_1(u)} du$$

where  $g_1(u) = g(\psi(u))\psi'(u) - Cc^{-(\alpha+1)/\beta}u^\alpha = O(u^{\alpha+1})$  by virtue of (4.7.3), and

$$h_1(u) = a + u^\beta.$$

Since, in the neighbourhood of  $u = 0$ ,  $g_1(u)/h_1'(u) = O(u^{\alpha-\beta+2})$ , we see, by virtue of (4.2), that if  $\alpha + 2 \geq \beta$ , the second integral in (4.7.4) is  $O(1/t)$ . Since, on the other hand,  $h'$  does not vanish in the *closed* interval  $[\delta, b]$ , we have also, from (4.2)  $\int_\delta^b g(x)e^{ith(x)} dx = O(1/t)$ , hence (4.7.2) in this case by virtue of (4.6). If  $\alpha + 2 < \beta$ , the same reasoning as above can be applied to the integral  $\int_0^{\varphi(\delta)} g_1(u)e^{it h_1(u)} du$  replacing  $\alpha$  by  $\alpha + 1$ , since

$$g_1(u) = C_1 u^{\alpha+1} (1 + \theta_1(u))$$

where  $\theta_1$  is indefinitely differentiable in  $[0, \varphi(\delta)]$ . The reasoning is continued by induction until an integer  $n$  is reached such that  $\alpha + n \geq \beta$ , where (4.2) can be applied and the proposition is thus completely proved.

Here again, the most frequent case of the application of (4.7) is the following:

(4.8) *Suppose that in a bounded closed interval  $[a, b]$  the real functions  $g, h$  are indefinitely differentiable, and  $h'$  vanishes at just one point  $c$  of this interval, with  $a < c < b$ , and we have  $g(c) \neq 0$ ,  $h''(c) \neq 0$ . Then for  $t$  the neighbourhood of  $+\infty$*

$$(4.8.1) \quad \begin{cases} \int_a^b g(x)e^{ith(x)} dx = \left(\frac{\pi}{2th''(c)}\right)^{1/2} g(c)e^{it h(c) + i\pi/4} + O\left(\frac{1}{t}\right) & \text{if } h''(c) > 0 \\ \int_a^b g(x)e^{ith(x)} dx = \left(\frac{\pi}{-2th''(c)}\right)^{1/2} g(c)e^{it h(c) - i\pi/4} + O\left(\frac{1}{t}\right) & \text{if } h''(c) < 0 \end{cases}$$

The fact that the remainder is  $O(1/t)$  follows easily from (4.6.6).

By taking the real and imaginary parts of the integral (4.1.1), the preceding results allow a determination of the behaviour of the functions

$$\int_a^b g(x) \cos(th(x)) dx \quad \text{and} \quad \int_a^b g(x) \sin(th(x)) dx$$

as  $t$  tends to  $+\infty$  (assuming the correct hypotheses made on  $g$  and  $h$ ).

## PROBLEMS

1. Let  $\alpha > 0$ ; show that as  $t$  tends to  $+\infty$

$$\int_0^{+\infty} x^{-\alpha x} t^x dx \sim \sqrt{\frac{2\pi}{e\alpha}} t^{1/2\alpha} \exp(e^{-1}\alpha t^{1/\alpha})$$

2. Suppose that  $0 < \alpha < 1$ ; show that as  $t$  tends to 0 through positive values

$$\int_0^{+\infty} \exp\left(\frac{x^\alpha}{\alpha} - tx\right) dx \sim \sqrt{\frac{2\pi}{1-\alpha}} t^{(\alpha-2)/2(1-\alpha)} \exp\left(\frac{1-\alpha}{\alpha} t^{-\alpha/(1-\alpha)}\right).$$

3. (a) The real functions  $g, h$ , piecewise-continuous in  $]0, +\infty[$ , satisfy conditions 1 and 2 of (2.3); furthermore suppose that in an interval  $]0, \delta[$  of  $\mathbf{R}$

$$h(x) = a - cx \quad \text{with } c > 0$$

$$(*) \quad g(x) = \sum_{j=1}^r A_j x^{\alpha_j} + o(x^{\alpha_r}) \quad \text{with } -1 < \alpha_1 < \alpha_2 < \dots < \alpha_r,$$

the  $A_j$  being  $\neq 0$ . Show that

$$I(t) = e^{at} \left( \sum_{j=1}^r A_j \Gamma(\alpha_j + 1) (ct)^{-(\alpha_j+1)} + o(t^{-(\alpha_r+1)}) \right).$$

(b) Suppose that  $g$  and  $h$  satisfy conditions 1 and 2 of (2.3) and that in an interval  $]0, \delta[$  of  $\mathbf{R}$ ,  $h$  is continuous and decreasing with an asymptotic development in the neighbourhood of 0

$$h(x) = a - c_1 x^{\beta_1} - c_2 x^{\beta_2} - \dots - c_s x^{\beta_s} + o(x^{\beta_s})$$

where  $c > 0, 0 < \beta_1 < \beta_2 < \dots < \beta_s$ . Suppose further that  $h'(x)$  has an asymptotic development in the neighbourhood of 0, obtained by differentiating the preceding one term by term. Then  $y = \varphi(x) = (a - h(x))^{1/\beta_1}$  has an inverse function  $x = \psi(y)$  in the neighbourhood of 0 possessing an asymptotic development  $\psi(y) = c_1^{-1/\beta_1} y + c_2' y^{\gamma_2} + \dots$  whose derivative has an asymptotic development obtained by differentiating term by term that of  $\psi(y)$ . Then

$$\int_0^\delta g(x) e^{th(x)} dx = \int_0^{\varphi(\delta)} \psi'(y) g(\psi(y)) e^{t(a-y^{\beta_1})} dy.$$

Deduce with the help of (a) an asymptotic development of  $I(t)$  by supposing that  $g$  admits an asymptotic development (\*). Apply to the function  $\Gamma$ , to the integral (2.6.1) and to the integrals of problems 1 and 2.

4. With the hypotheses of problem 3 (b), find an asymptotic development of the integral

$$\int_0^{\omega(t)} g(x) e^{th(x)} dx$$

given that  $\omega(t)$  tends to 0 with  $1/t$  and knowing an asymptotic development of  $\omega(t)$  in the neighbourhood of  $+\infty$ . In particular:

1. Show that

$$\frac{1}{n!} \int_0^{n+\alpha\sqrt{n}+\beta} e^{-x} x^n dx = a + \frac{b}{\sqrt{n}} + o\left(\frac{1}{\sqrt{n}}\right)$$

where the constants  $a$  and  $b$  are functions of  $\alpha$  and  $\beta$ .

2. Show that, if  $\xi$  is the unique real root of the equation  $\xi e^{1+\xi} = 1$ , then

$$\frac{1}{n!} \int_0^{\xi n + \alpha \log n + \beta} e^x x^n dx \sim b n^a$$

where the constants  $a$  and  $b$  are functions of  $\alpha, \beta$  and  $\xi$ .

5. (a) Let  $f$  be a function  $> 0$  continuously differentiable in  $[0, +\infty[$  and such that  $f'(x)/f(x) \sim \alpha/x$  in the neighbourhood of  $+\infty$ , where  $\alpha > 0$ . Show that as  $t$  tends to 0

$$\int_0^{+\infty} e^{-tx} f(x) dx \sim \frac{\Gamma(\alpha + 1)}{t} f\left(\frac{1}{t}\right).$$

Extend to the case where  $f'(x)/f(x) = o(1/x)$ . (Decompose the integral over the subintervals  $[0, \lambda/t]$  and  $[\lambda/t, +\infty[$ , where  $\lambda$  is independent of  $t$  but arbitrarily small, and observe using the mean value theorem that  $f(\lambda/t)/f(1/t)$  tends to  $\lambda^\alpha$  as  $t$  tends to 0.)

(b) Find the principal part of

$$\int_1^{+\infty} e^{-tx + (\log x)^2} dx$$

as  $t$  tends to 0 (consider the point  $x = \xi$  where the function attains its maximum).

6. For which values of the constants  $\alpha > 0, \beta > 0$ , are the integrals

$$\int_0^{+\infty} e^{xt - x^{\alpha} t^{\beta}} dx, \quad \int_0^{+\infty} e^{-xt + x^{\alpha} t^{\beta}} dx, \quad \int_0^{+\infty} e^{-xt - x^{\alpha} t^{\beta}} dx$$

convergent for every  $t > 0$ ? Find in this case their principal parts as  $t$  tends to 0 or to  $+\infty$ .

7. For which values of the real constants  $\alpha$  and  $\beta$  is the integral

$$\int_0^{+\infty} x^{\alpha} |\cos x|^{x^{\beta}} dx$$

convergent? (To study the integral in the neighbourhood of  $+\infty$ , majorize and minorize all the integrals  $\int_{n\pi}^{(n+1)\pi} |\cos x|^{x^{\beta}} dx$  with the aid of the values of the function  $\mathbf{B}(u, v)$ .)

8. (a) Show that the function  $\Gamma$  is the only function  $f(x)$  satisfying the equation  $f(x+1) = xf(x)$  and  $f(1) = 1$  and which is such that the function  $g(x) = (e/x)^x f(x)$  is decreasing for  $x > 0$ .

(b) Show that, for a fixed  $y > 0$ , the function  $x \rightarrow \mathbf{B}(x, y)$  is the only decreasing function  $f$  for  $x > 0$ , such that

$$f(1) = 1/y \quad \text{and} \quad f(x+1) = \frac{x}{x+y} f(x).$$

9. Let  $g$  and  $h$  be two continuous functions  $> 0$  in an interval  $[0, a]$ . Suppose that in the neighbourhood of 0,  $g(x) \sim Ax^{\alpha}$ ,  $h(x) \sim Bx^{\beta}$  where  $B > 0, \alpha > -1, \beta > 0$ , and let  $\mu$  be a number  $\geq (\alpha + 1)/\beta$ . Show that as  $t$  tends to 0 through positive values

$$\begin{aligned} \int_0^a \frac{g(x) dx}{(h(x) + t)^{\mu}} &\sim Ct^{((\alpha+1)/\beta) - \mu} \quad \text{if } \frac{\alpha+1}{\beta} < \mu \\ \int_0^a \frac{g(x) dx}{(h(x) + t)^{\mu}} &\sim C \log \frac{1}{t} \quad \text{if } \frac{\alpha+1}{\beta} = \mu \end{aligned}$$

where  $C$  is a constant, which can be expressed with the aid of  $A, B, \alpha, \beta, \mu$  and of the function  $\Gamma$ . (Prove, by the same method as in (2.3), that  $g$  and  $h$  can be replaced by their principal parts.) As an example, deduce that as  $k$  tends to 1 in  $]0, 1[$ ,

$$\int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} \sim \frac{1}{2} \log \frac{1}{1-k}.$$

10. Show that for  $x > 0$  and  $y > 0$

$$B(x, y) = \frac{1}{x} + \int_0^1 t^x \frac{(1-t)^{y-1} - 1}{t} dt.$$

By considering on the one hand an asymptotic development of  $t^x$ , and on the other the Taylor developments of  $\Gamma(x+1)$  and  $\Gamma(x+y)$  in the neighbourhood of  $x=0$ , show that

$$\Gamma'(1) - \frac{\Gamma'(y)}{\Gamma(y)} = \int_0^1 \frac{(1-t)^{y-1} - 1}{t} dt$$

(Gauss's integral) (we shall see in Chap. IX that  $\Gamma'(1) = -\gamma$ , where  $\gamma$  is Euler's constant).

11. Give an example in which the conditions of (4.2) are satisfied, except the hypothesis that  $h$  tends to a finite limit at the points  $a$  and  $b$ , and in which the integral (4.1.1) is not convergent for  $t > 0$  (take  $g = h'$ ).

12. Determine the values of the real numbers  $\alpha, \beta, \gamma$  for which the integral

$$\int_0^{+\infty} \frac{x^\alpha e^{tx}}{(1+x^\beta)^\gamma} dx$$

is convergent for every  $t > 0$ , and when it is so, find its principal part as  $t$  tends to  $+\infty$ . Do the same for the integral

$$\int_0^{+\infty} \frac{x^\alpha e^{tx} dx}{(x+t)^\beta}.$$

13. Let  $f$  be a function  $>0$ , continuously differentiable in  $[0, +\infty[$  and such that  $f'(x)/f(x) \sim (\lambda-1)/x$  in the neighbourhood of  $+\infty$ , where  $0 < \lambda < 1$ . Show that as  $t$  tends to 0 through positive values

$$\int_0^{+\infty} e^{tx} f(x) dx \sim e^{\frac{1}{2}\lambda\pi i} \frac{\Gamma(\lambda)}{t} f\left(\frac{1}{t}\right).$$

(Proceed by decomposition of the interval into  $[0, A]$  and  $[A, +\infty[$  as in problem 5(a), taking  $A$  arbitrarily large.)

14. As  $t$  tends to  $+\infty$ , find the principal part of each of the integrals

$$\int_a^{+\infty} \frac{e^{t(x+(t/x))}}{x} dx, \quad \int_a^{+\infty} \frac{e^{t(x-(t/x))}}{x} dx \quad (a > 0)$$

(for the first, put  $x = \sqrt{tu}$ , and use the method of stationary phase).

15. Let  $h$  be a continuous, increasing function  $>0$  tending to  $+\infty$  with  $x$ . Show that as  $t$  tends to 0

$$\sum_{n=1}^{\infty} e^{-th(n)} \sim \int_1^{+\infty} e^{-th(x)} dx,$$

provided the integral is convergent for  $t > 0$ .

In particular

$$\sum_{n=1}^{\infty} e^{-tn^\alpha} \sim \frac{1}{\alpha} \Gamma\left(\frac{1}{\alpha}\right) t^{-1/\alpha} \quad \text{for } \alpha > 0$$

$$\sum_{n=1}^{\infty} \log(1 - e^{-nt}) \sim \frac{\pi^2}{6t}.$$

16. Let  $f$  be a function  $> 0$ , continuously differentiable in  $[0, +\infty[$  and such that  $f'(x)/f(x) \sim \alpha/x$  in the neighbourhood of  $+\infty$ , where  $\alpha > 0$ . Show that as  $t$  tends to 0 (cf. problem 5)

$$\sum_{n=1}^{\infty} f(n) e^{-tn} \sim \int_0^{\infty} e^{-tx} f(x) dx \sim \frac{\Gamma(\alpha + 1)}{t} f\left(\frac{1}{t}\right).$$

Treat in the same way the sums

$$\sum_{n=1}^{\infty} f(n) e^{-tn^{\beta}} \quad \text{with } \beta > 1, \quad \sum_{n=1}^{\infty} e^{n^{\gamma} - tn}, \quad \text{with } 0 < \gamma < 1.$$

17. Let  $f$  be a function  $> 0$ , decreasing and continuously differentiable for  $x \geq 0$ , and such that  $f'(x)/f(x) \gg 1/x$ . Show that as  $t$  tends to  $+\infty$

$$\sum_{n=0}^{\infty} e^{tn - f(n)} \sim \int_0^{\infty} e^{tx - f(x)} dx.$$

18. (a) With the help of Stirling's formula, show that the maximum in the interval  $[0, +\infty[$  of the function

$$f_n(x) = \left| e^{-2x} - e^{-x} \sum_{k=0}^n (-1)^k \frac{x^k}{k!} \right|$$

tends to 0 with  $1/n$ .

(b) Deduce, by induction on the integer  $p > 2$ , that for each  $\varepsilon > 0$ , there exists a polynomial  $P(x)$  such that

$$|e^{-px} - e^{-x} P(x)| \leq \varepsilon$$

for every  $x \geq 0$  (replace  $x$  by  $px/2$  in (a) and apply the induction hypothesis to  $e^{-(p-1)x/2}$ ).

19. Let  $k$  be an integer  $> 0$ ; put  $x = k + \frac{1}{2}$  and consider the series (dependent on  $k$ )

$$S(k) = \sum_{n=1}^{\infty} \frac{1}{n^{1+(1/\log x)} |\log x/n|}.$$

Show that as  $k$  tends to  $+\infty$ ,  $S(k) = O(\log k)$ . (Decompose the series into three sums, where respectively  $1 \leq n < x/2$ ,  $x/2 < n \leq 2x$ ,  $2x < n$ , and majorize these three sums separately.)

20. Let  $g$  be a piecewise-continuous function in  $\mathbf{R}$  with period  $2\omega$ . Put  $c = (1/2\omega) \int_0^{2\omega} g(t) dt$ .

(a) Show that

$$\int_x^{+\infty} \frac{1}{t} (c - g(t)) dt = O\left(\frac{1}{x}\right) \quad (\text{integrate by parts}).$$

(b) Suppose further that  $g$  is even. Show that for  $n$  an integer tending to  $+\infty$

$$\int_{n\omega}^{+\infty} \frac{1}{t} (c - g(t)) dt = O\left(\frac{1}{n^2}\right).$$

Prove that

$$\int_{(n-1)\omega}^{(n+1)\omega} \frac{g(t)}{t} dt = \frac{2c}{n} + O\left(\frac{1}{n^3}\right).$$

Is it then true that

$$\int_x^{+\infty} \frac{1}{t} (c - g(t)) dt = O\left(\frac{1}{x^2}\right)$$

in the neighbourhood of  $+\infty$ ?

(c) Suppose that the improper integral  $\int_0^1 g(t)/t dt$  exists. If  $\varphi(x)$  is a function tending to  $+\infty$  with  $x$  such that  $\varphi(x) = o(x)$ , show that in the neighbourhood of  $+\infty$

$$\int_0^{1/\varphi(x)} \frac{g(xt)}{t} dt = c \log \frac{x}{\varphi(x)} + \int_0^1 \frac{g(t)}{t} dt - \int_1^{+\infty} \frac{1}{t} (c - g(t)) dt + O\left(\frac{\varphi(x)}{x}\right).$$

(d) Let  $F(x, t)$  be a function whose partial derivative  $t \rightarrow \frac{\partial F}{\partial t}(x, t)$  is continuous in  $[a, b]$  for every  $x > 0$ ; suppose further that, for a function  $\varphi(x)$  increasing and  $> 0$  for  $x > 0$

$$\int_a^b \left| \frac{\partial F}{\partial t}(x, t) \right| dt = O(\varphi(x)), \quad F(x, b) = O(\varphi(x)).$$

Show that

$$\int_a^b F(x, t) (c - g(xt)) dt = O(\varphi(x)/x)$$

(integrate by parts).

In particular, if  $f(t)$  is piecewise-continuously differentiable in  $[a, b]$ , then

$$\int_a^b f(t) g(xt) dt = c \int_a^b f(t) dt + O\left(\frac{1}{x}\right).$$

# Uniform approximation

## Distance between two functions

(1.1) Just as we try to *approximate* an unknown *number* (defined in any manner) by means of decimal (or rational) numbers, so it is natural in Analysis to try to “approximate” an unknown complex *function* (which can be defined in various ways, sum of a series, integral depending on a parameter, solution of a differential equation, etc.) by means of functions considered to be *known* (polynomials, exponential functions, trigonometric functions, etc.). But we must be precise by what we mean by “approximate”, that is “measure” in some way the “distance” between two functions, just as the absolute value  $|x - y|$  measures the distance between two real or complex numbers.

The most natural idea for “approximating” a function  $f$  by means of a function  $g$  on a set  $E$ , where both are defined, is that for each  $x_0 \in E$ , the *value*  $g(x_0)$  of  $g$  *approximate* the *value*  $f(x_0)$  of  $f$  in the usual sense, i.e.  $|f(x_0) - g(x_0)|$  must be small. Since this must occur at *every* point  $x_0 \in E$ , we are led into taking for the “distance” between two complex functions  $f, g$  defined in  $E$  the number

$$(1.2) \quad d(f, g) = \sup_{x \in E} |f(x) - g(x)|.$$

When it is a question of *real* functions  $f, g$  defined in an interval  $E = [a, b]$  of  $\mathbf{R}$ , the notion of “distance” just defined can be represented graphically:  $d(f, g) \leq \varepsilon$  means that for *each*  $x \in E$  we have  $g(x) - \varepsilon \leq f(x) \leq g(x) + \varepsilon$ , i.e. that the graph of  $f$  is *wholly contained* in the “strip” of width  $2\varepsilon$  around the graph of  $g$  (Fig. 12). To distinguish this notion of “approximation” from other notions, which will be examined later (IX, 9), we say that this is a *uniform approximation* of one function by another in the set  $E$ . It is important to remark that this notion *depends essentially* on the set  $E$  under consideration: if  $f$  and  $g$  are both defined in a larger set  $E'$ , the relation  $|f(x) - g(x)| \leq \varepsilon$  for  $x \in E$  does not imply  $|f(x) - g(x)| \leq \varepsilon$  for  $x \in E'$ .

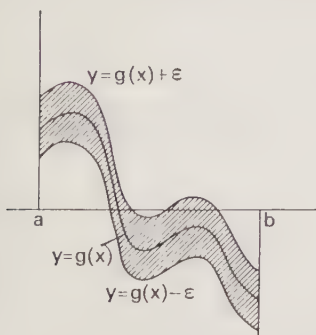


FIGURE 12

(1.3) Given two sets of functions  $\mathcal{F}$  (the “unknown” functions) and  $\mathcal{G}$  (the “known” functions) all defined

in the same set  $E$ , we say that we can *uniformly approximate* the functions of  $\mathcal{F}$  by means of the functions of  $\mathcal{G}$  in  $E$  if, for each function  $f \in \mathcal{F}$  and each number  $\varepsilon > 0$ , there exists a function  $g \in \mathcal{G}$  (depending on  $f$  and on  $\varepsilon$ ) such that the distance  $d(f, g) \leq \varepsilon$ , i.e. such that

$$(1.3.1) \quad |f(x) - g(x)| \leq \varepsilon \quad \text{for every } x \in E.$$

(1.4) The preceding notions generalize immediately to the case of functions whose values are *complex vectors* in  $\mathbf{C}^n$  (I, 1.6): if  $f, g$  are two such functions, defined in  $E$ , we set this time

$$(1.4.1) \quad d(f, g) = \sup_{x \in E} \|f(x) - g(x)\|$$

and then write the definition of (1.3) without further change.

## 2. Uniform convergence and simple convergence

(2.1) Let  $(g_n)$  be a sequence of complex functions defined in a set  $E$ . We say that the sequence  $(g_n)$  *converges uniformly* in  $E$  to a function  $f$  defined in  $E$  if

$$(2.1.1) \quad \lim_{n \rightarrow \infty} d(f, g_n) = 0.$$

A series of complex functions  $(u_n)$  defined in  $E$  is said to be *uniformly convergent* if the sequence of partial sums  $s_n = u_1 + u_2 + \dots + u_n$  converges *uniformly* in  $E$  to a function  $s$ , called the *sum* of the series of functions.

(2.2) Given two sets of complex functions  $\mathcal{F}, \mathcal{G}$  defined in  $E$ , a necessary and sufficient condition that the functions of  $\mathcal{F}$  can be uniformly approximated by means of the functions of  $\mathcal{G}$  in  $E$ , is that corresponding to each function  $f \in \mathcal{F}$ , there exists a sequence  $(g_n)$  of functions of  $\mathcal{G}$  which converges *uniformly* in  $E$  to  $f$ .

Indeed, if there is such a sequence, we have by definition the relation (2.1.1), therefore for each  $\varepsilon > 0$ , there exists an integer  $n_0$  such that for  $n \geq n_0$ ,  $d(f, g_n) \leq \varepsilon$ , which proves that the functions of  $\mathcal{F}$  can be approximated uniformly in  $E$  by those of  $\mathcal{G}$ . Conversely, if this is so, for each function  $f \in \mathcal{F}$ , we can successively determine functions  $g_1, g_2, \dots, g_n, \dots$  in  $\mathcal{G}$  so that  $d(f, g_1) \leq 1, d(f, g_2) \leq 1/2, \dots, d(f, g_n) \leq 1/n, \dots$ : by definition, we therefore have  $\lim_{n \rightarrow \infty} d(f, g_n) = 0$ , in other words the sequence  $(g_n)$  converges *uniformly* to  $f$  in  $E$ .

(2.3) A sequence  $(g_n)$  of complex functions defined in  $E$  is said to *converge simply* in  $E$  to a function  $f$ , if, for each  $x \in E$ , the sequence of complex numbers  $(g_n(x))$  has for a limit the number  $f(x)$ .

IT IS ESSENTIAL TO DISTINGUISH CAREFULLY THE NOTION OF SIMPLE CONVERGENCE FROM THAT OF UNIFORM CONVERGENCE. If the sequence  $(g_n)$  converges uniformly in  $E$  to  $f$ , then it also converges simply to  $f$ : indeed, for each  $x \in E$ , we have by definition (1.2),  $|f(x) - g_n(x)| \leq d(f, g_n)$ , and the relation (2.1.1) implies *a fortiori*

$$\lim_{n \rightarrow \infty} |f(x) - g_n(x)| = 0.$$

But it is easy to give examples of sequences  $(g_n)$  which converge *simply* to 0, but *do not converge uniformly* to 0. Define  $g_n$  in  $E = [0, 1]$  as a piecewise affine linear function (Fig. 13) such that:

$$(2.3.1) \quad \begin{cases} g_n(x) = 2nx & \text{for } 0 \leq x \leq 1/2n \\ g_n(x) = 2 - 2nx & \text{for } 1/2n \leq x \leq 1/n \\ g_n(x) = 0 & \text{for } 1/n \leq x \leq 1. \end{cases}$$

For each  $x \in E$ ,  $\lim_{n \rightarrow \infty} g_n(x) = 0$ : this is obvious for  $x = 0$ , since  $g_n(0) = 0$  for every  $n$ .

For  $x > 0$ , there exists an integer  $n_0$  (depending on  $x$ ) such that  $1/n_0 < x$ , and for  $n \geq n_0$ ,  $1/n < x$ , therefore by definition  $g_n(x) = 0$ . However, the sequence  $(g_n)$  does not converge uniformly to 0, since  $g_n(1/2n) = 1$ , therefore  $d(0, g_n) = 1$  for every  $n$ : graphically it is clear that the graph of  $g_n$  can never be contained in a "strip" of half-width  $\frac{1}{2}$  surrounding the graph of the function 0 (Fig. 13).

If we analyse in general the relationship between the two types of convergence, we see that if the sequence  $(g_n)$  converges simply to  $f$ , then for a *given*  $\varepsilon > 0$  and for *each*  $x \in E$ , there is an integer  $n_0$  such that

$$|f(x) - g_n(x)| \leq \varepsilon$$

for every  $n \geq n_0$ . This integer  $n_0$  *depends not only on  $\varepsilon$ , but also in general on  $x$*  (as we have seen in the preceding example). On the other hand, if the sequence  $(g_n)$  converges uniformly to  $f$ , once  $\varepsilon > 0$  is *given*,  $n_0$  can be determined depending on  $\varepsilon$  but *independent of  $x$*  such that  $|f(x) - g_n(x)| \leq \varepsilon$  for  $n \geq n_0$  and for every  $x \in E$ .

(2.4) When we wish to prove the uniform convergence of a sequence of functions  $(g_n)$  in a set  $E$ , the most convenient method, and the most often successful, consists in first "transforming the sequence into a series" (I, 2.6) by considering the series of functions  $u_n = g_n - g_{n-1}$  (and by agreeing to take  $g_0 = 0$ ). Stating that the sequence  $(g_n)$  converges uniformly in  $E$  to  $f$  is equivalent to stating that the series with general term  $u_n$  converges uniformly in  $E$  to the sum  $f$ , since  $g_n$  is the  $n^{\text{th}}$  partial sum of the series with general term  $u_n$ . Now, for a series of complex functions, we have a sufficient criterion for uniform convergence which makes the problem one of *majorization*:

(2.5) Suppose that there is a series with positive terms  $(\alpha_n)$  which is convergent and that, for each integer  $n$ ,  $\sup_{x \in E} |u_n(x)| \leq \alpha_n$  (i.e.  $|u_n(x)| \leq \alpha_n$  for every  $x \in E$ ). Then the series with general term  $u_n$  is uniformly convergent in  $E$ .

Indeed, for each  $x \in E$ ,  $|u_n(x)| \leq \alpha_n$  for every  $n$ , therefore the series of complex numbers  $(u_n(x))$  is *absolutely convergent* by the comparison principle (I, 2.2), and hence is

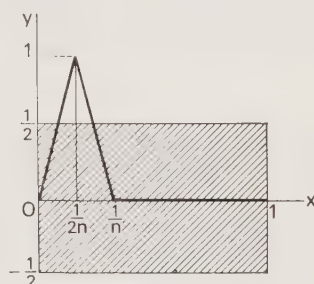


FIGURE 13

convergent; let  $s(x)$  be its sum. This defines a complex function  $s$  in  $E$ ; we show that the series  $(u_n)$  converges uniformly to  $s$ . Indeed, for a given  $n$

$$s(x) - (u_1(x) + \cdots + u_n(x)) = u_{n+1}(x) + \cdots + u_{n+p}(x) + \cdots$$

and since the series of the second member is absolutely convergent, we have by virtue of (I, 2.3.1) and (I, 2.2.1)

$$(2.5.1) \quad |s(x) - (u_1(x) + \cdots + u_n(x))| \leq |u_{n+1}(x)| + \cdots + |u_{n+p}(x)| + \cdots \\ \leq \alpha_{n+1} + \cdots + \alpha_{n+p} + \cdots$$

By hypothesis the numerical series  $(\alpha_n)$  is convergent; this means that for a given  $\varepsilon > 0$  there exists  $n_0$  depending only on  $\varepsilon$ , such that  $\alpha_{n+1} + \cdots + \alpha_{n+p} + \cdots \leq \varepsilon$  for every  $n \geq n_0$ . Thus from (2.5.1)

$$|s(x) - (u_1(x) + \cdots + u_n(x))| \leq \varepsilon$$

for every  $n \geq n_0$  and every  $x \in E$ ; since  $n_0$  does not depend on  $x$ , the uniform convergence of the series has been proved.

A series  $(u_n)$  of functions which satisfies the hypothesis of (2.5) is said to be *normally convergent*: observe, however, that a series  $(u_n)$  may be uniformly convergent without being normally convergent (problem I).

(2.6) Let  $(h_n)$  be a sequence of complex functions defined in a set  $E$ , taking all their values in a closed and bounded subset  $F$  of the complex plane  $\mathbf{C}$ ; secondly, let  $f$  be a continuous complex function in  $F$ . Then, if in  $E$  the sequence  $(h_n)$  converges uniformly to a function  $g$ , the sequence  $(f \circ h_n)$  of composed functions converges uniformly in  $E$  to  $f \circ g$ . Indeed,  $g(E) \subset F$ ; secondly, a theorem, which we shall admit (0, 5.6), states that for each  $\varepsilon > 0$ , there exists a number  $\delta > 0$  such that if  $z, z'$  are two points of  $F$  satisfying  $|z - z'| \leq \delta$ , then  $|f(z) - f(z')| \leq \varepsilon$ . But, by hypothesis, there exists an integer  $n_0$ , depending only on  $\delta$ , such that for every  $n \geq n_0$ ,  $|g(x) - h_n(x)| \leq \delta$  for every  $x \in E$ . Therefore  $|f(g(x)) - f(h_n(x))| \leq \varepsilon$  for every  $x \in E$  and every  $n \geq n_0$ , hence our assertion.

(2.7) The extension of the notions defined above to functions whose values are complex vectors (1.4) is left to the reader.

### 3. Properties of uniformly convergent sequences.

The importance of the notion of uniform convergence arises from the fact that for uniformly convergent sequences in general results can be obtained which make the sequences easy to handle, and which are *no longer necessarily valid* for simply convergent sequences.

(3.1) Let  $(g_n)$  be a sequence of complex functions defined and continuous in a set  $E \subset \mathbf{R}^p$ , and which converges uniformly in  $E$  to a function  $f$ . Then  $f$  is continuous in  $E$ .

More briefly, a uniform limit of continuous functions is continuous.

Let  $\mathbf{a} = (a_1, \dots, a_p)$  be a point of  $E$ . To show that  $f$  is continuous at the point  $\mathbf{a}$ , we must, by definition, prove:

For each  $\varepsilon > 0$  there is a number  $\delta > 0$  depending on  $\varepsilon$  such that, for every point  $\mathbf{x} \in E$  satisfying  $\|\mathbf{x} - \mathbf{a}\| \leq \varepsilon$

$$(3.1.1) \quad |f(\mathbf{x}) - f(\mathbf{a})| \leq \varepsilon.$$

The hypothesis of uniform convergence states that there exists an integer  $n_0$ , depending *only* on  $\varepsilon$ , such that, for every  $n \geq n_0$  and for every  $\mathbf{x} \in E$

$$(3.1.2) \quad |f(\mathbf{x}) - g_n(\mathbf{x})| \leq \frac{\varepsilon}{3}.$$

The number  $n_0$  being thus fixed, we use the fact that the function  $g_{n_0}$  is continuous at the point  $\mathbf{a}$ : there is therefore a number  $\delta > 0$ , depending only on  $\varepsilon$ , such that if  $\mathbf{x} \in E$  and  $\|\mathbf{x} - \mathbf{a}\| \leq \delta$ , then

$$(3.1.3) \quad |g_{n_0}(\mathbf{x}) - g_{n_0}(\mathbf{a})| \leq \frac{\varepsilon}{3}.$$

We apply (3.1.2) for  $n = n_0$ ; thus for each  $\mathbf{x} \in E$

$$(3.1.4) \quad |f(\mathbf{x}) - g_{n_0}(\mathbf{x})| \leq \frac{\varepsilon}{3}$$

and in particular for  $\mathbf{x} = \mathbf{a}$

$$(3.1.5) \quad |f(\mathbf{a}) - g_{n_0}(\mathbf{a})| \leq \frac{\varepsilon}{3}.$$

But

$$f(\mathbf{x}) - f(\mathbf{a}) = (f(\mathbf{x}) - g_{n_0}(\mathbf{x})) + (g_{n_0}(\mathbf{x}) - g_{n_0}(\mathbf{a})) + (g_{n_0}(\mathbf{a}) - f(\mathbf{a}))$$

and it follows from (3.1.3), (3.1.4) and (3.1.5) that for *every*  $\mathbf{x} \in E$  satisfying  $\|\mathbf{x} - \mathbf{a}\| \leq \delta$ , we have (3.1.1). Q.E.D.

(3.2) If we no longer suppose that the sequence  $(g_n)$  converges uniformly to  $f$ , but only simply, the functions  $g_n$  may be continuous without  $f$  also being continuous. For example if  $E = [0, 1]$  and  $g_n(x) = x^n$ ; for  $0 \leq x < 1$  we have  $\lim_{n \rightarrow \infty} x^n = 0$ , whereas if  $x = 1$ ,  $\lim_{n \rightarrow \infty} x^n = 1$  (Fig. 14).

Note, however, that the uniform convergence of the sequence  $(g_n)$  of continuous functions is not a necessary condition for the limit  $f$  to be continuous: this is shown by the example (2.3.1).

From (3.1) the following corollary is at once deduced:

(3.3) Let  $u_1 + u_2 + \cdots + u_n + \cdots$  be a series of continuous functions in  $E$ , which converges uniformly in  $E$ . Then the sum of this series is continuous in  $E$ .

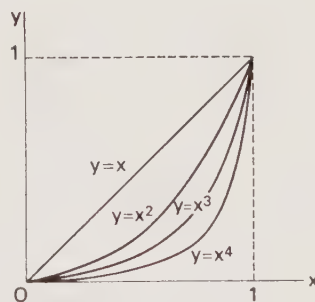


FIGURE 14

Indeed, each partial sum of the series is the sum of a finite number of continuous functions, therefore it is continuous.

(3.4) *Let  $I = [a, b]$  be a bounded, closed interval of  $\mathbf{R}$ , and let  $(g_n)$  be a sequence of piecewise-continuous complex functions in  $I$ , which converges uniformly in  $I$  to a piecewise-continuous function  $f$ . Then*

$$(3.4.1) \quad \lim_{n \rightarrow \infty} \int_a^b g_n(t) dt = \int_a^b f(t) dt.$$

We have to prove that given  $\varepsilon > 0$ , there exists an integer  $n_0$ , depending only on  $\varepsilon$ , such that for  $n \geq n_0$ ,

$$(3.4.2) \quad \left| \int_a^b f(t) dt - \int_a^b g_n(t) dt \right| \leq \varepsilon.$$

By the theorem of the mean (I, 3.3.1)

$$\left| \int_a^b f(t) dt - \int_a^b g_n(t) dt \right| \leq \int_a^b |f(t) - g_n(t)| dt.$$

The uniform convergence hypothesis implies that there exists an integer  $n_0$  depending only on  $\varepsilon$ , such that for  $n \geq n_0$  and every  $t \in I$

$$(3.4.3) \quad |f(t) - g_n(t)| \leq \varepsilon/(b-a).$$

By the theorem of the mean (I, 3.2.4), we therefore have (3.4.2) for every  $n \geq n_0$ .

Q.E.D.

Observe that if we set  $F(x) = \int_a^x f(t) dt$ ,  $G_n(x) = \int_a^x g_n(t) dt$ , then the proof shows that

$$(3.4.4) \quad |F(x) - G_n(x)| \leq \varepsilon$$

for  $n \geq n_0$  and for every  $x \in I$ . In other words, the sequence of primitives  $G_n(x) = \int_a^x g_n(t) dt$ , zero at the point  $a$ , converges uniformly to the primitive  $F(x) = \int_a^x f(t) dt$ , zero at the point  $a$ .

The following corollary is also deduced from (3.4):

(3.5) *Let  $u_1 + u_2 + \cdots + u_n + \cdots$  be a series of piecewise-continuous functions in  $I$ , which converges uniformly in  $I$  and whose sum  $f$  is piecewise-continuous. Then*

$$(3.5.1) \quad \int_a^b (u_1(t) + u_2(t) + \cdots + u_n(t) + \cdots) dt \\ = \int_a^b u_1(t) dt + \int_a^b u_2(t) dt + \cdots + \int_a^b u_n(t) dt + \cdots$$

(“term by term integration of a uniformly convergent series”).

*Remarks* (3.6.1) The conclusion of (3.4) may be false if we merely suppose that the sequence  $(g_n)$  is simply convergent to a continuous function  $f$ . For example, let  $(g_n)$  be the sequence defined in (2.3.1) and put  $h_n(x) = ng_n(x)$ ; it is at once apparent that

we again have  $\lim_{n \rightarrow \infty} h_n(x) = 0$  for every  $x \in I = [0, 1]$ . However  $\int_0^1 h_n(t) dt = \frac{1}{2}$  for every  $n$ . If  $h_n$  is replaced by  $nh_n$ , we still have  $\lim_{n \rightarrow \infty} nh_n(x) = 0$  for every  $x \in I$ , but  $\lim_{n \rightarrow \infty} \int_0^1 nh_n(t) dt = +\infty$ .

(3.6.2) However the uniform convergence of the sequence  $(g_n)$  to  $f$  is not a *necessary* condition for the validity of (3.4.1); this is also shown by the example (2.3.1), where  $\int_0^1 g_n(t) dt = 1/2n$  tends indeed to 0.

(3.6.3) Nor is the conclusion of (3.4) valid when we replace the bounded interval  $I$  by an unbounded interval, even when each of the functions  $g_n$  vanishes outside a bounded interval (depending on  $n$ ) and when the sequence  $(g_n)$  converges *uniformly* to 0. As an example of this, put  $g_n(x) = 1/n$  in the interval  $[n^2, (n+1)^2]$  and 0 outside this interval. Since  $d(0, g_n) = 1/n$ , the sequence  $(g_n)$  converges uniformly to 0 in  $[0, +\infty[$ , but

$$\int_0^{+\infty} g_n(t) dt = \frac{1}{n} ((n+1)^2 - n^2) = \frac{2n+1}{n},$$

which tends to 2.

(3.6.4) We saw in (3.4.4) that the uniform convergence of a sequence of functions defined and piecewise-continuous in  $I$  is, so to speak, “transmitted” to their primitives, zero at  $a$ . It should not be thought that this is true for *derivatives*, showing once again that differentiation is an operation much less easy to handle than integration (I, 3.7). For example, in  $I = [0, \pi]$ , the sequence of functions  $g_n(x) = (1/n) \sin nx$  converges *uniformly* to 0, but  $g'_n(x) = \cos nx$ , so  $d(0, g'_n) = 1$  for every  $n \geq 1$ .

(3.7) The conclusion of (3.4) can be obtained with weaker hypotheses. Indeed, suppose that the sequence  $(g_n)$  is *uniformly bounded* in  $I$ , i.e. that there exists a number  $M > 0$  such  $|g_n(t)| \leq M$  for every  $t \in I$  and for all  $n$ . Moreover, suppose that there is in  $I$  a finite number of points

$$a = a_0 < a_1 < a_2 < \dots < a_m = b$$

such that in every bounded, closed interval  $[\alpha, \beta]$  containing none of the  $a_k$ , the sequence of the restrictions of the  $g_n$  converges uniformly to the restriction of  $f$ . Then we have again the relation (3.4.1). Indeed, for each  $\varepsilon > 0$ , such that  $2\varepsilon < a_k - a_{k-1}$  for  $1 \leq k \leq m$ , let  $\delta$  satisfy  $0 < \delta < \varepsilon/2m$ , and decompose the interval  $I$  by the points  $a_0 + \delta, a_1 - \delta, a_1 + \delta, a_2 - \delta, \dots, a_m - \delta$ . For each  $n$ , the sum of the integrals of the function  $|f - g_n|$  over all of the intervals

$$[a_0, a_0 + \delta], \quad [a_1 - \delta, a_1 + \delta], \quad \dots, \quad [a_m - \delta, a_m]$$

is at most equal to  $2(M+N)m\delta \leq (M+N)\varepsilon$ , by the theorem of the mean, with  $N = \sup_{t \in I} |f(t)|$ . On the other hand, there exists an integer  $n_0$  such that for  $n \geq n_0$ ,  $|f(t) - g_n(t)| \leq \varepsilon/(b-a)$  in each of the remaining intervals  $[a_0 + \delta, a_1 - \delta], [a_1 + \delta, a_2 - \delta], \dots, [a_{m-1} + \delta, a_m - \delta]$ . From this the inequality

$$\int_a^b |f(t) - g_n(t)| dt \leq (M+N+1)\varepsilon$$

is deduced, hence the conclusion.

(3.8) Theorems (3.1) and (3.4) can be interpreted as theorems on *interchanges of limits* in expressions depending on *two* integral indices. To say that a function  $f$  defined in an interval  $I$  of  $\mathbf{R}$  is continuous at a point  $a \in I$  signifies in fact that for each sequence  $(r_m)$  of real numbers tending to 0,  $\lim_{m \rightarrow \infty} f(a + r_m) = f(a)$ . The result of (3.1) can be interpreted by stating that

$$(3.8.1) \quad \lim_{m \rightarrow \infty} (\lim_{n \rightarrow \infty} g_n(a + r_m)) = \lim_{n \rightarrow \infty} (\lim_{m \rightarrow \infty} g_n(a + r_m)).$$

Similarly, for a piecewise-continuous function  $f$  in  $I = [a, b]$ , let  $s_m(f)$  denote the arithmetical mean of the values of  $f$  (the “Riemann sum”)

$$s_m(f) = \frac{1}{m} \left( f\left(a + \frac{b-a}{m}\right) + f\left(a + 2\frac{b-a}{m}\right) + \cdots + f\left(a + k\frac{b-a}{m}\right) + \cdots + f(b) \right)$$

which tends to  $\int_a^b f(t) dt$  as  $m$  tends to  $+\infty$ . The result of (3.4) can be interpreted by means of the formula for the interchange of limits

$$(3.8.2) \quad \lim_{m \rightarrow \infty} (\lim_{n \rightarrow \infty} s_m(g_n)) = \lim_{n \rightarrow \infty} (\lim_{m \rightarrow \infty} s_m(g_n)).$$

Note finally that the results of Chap. IV, Sections 2 and 4, can also be interpreted as “interchanges of limits”. Indeed, if we designate by  $g_0(x)$  and  $h_0(x)$  the asymptotic developments (IV, 2.2.2) of the functions  $g$  and  $h$  in the neighbourhood of  $x = 0$ , the result of (IV, 2.3) can be expressed by stating that

$$(3.8.3) \quad \lim_{n \rightarrow \infty} \left( \lim_{N \rightarrow \infty} \frac{\int_0^{1/n} g(x) e^{Nh(x)} dx}{\int_0^{1/n} g_0(x) e^{Nh_0(x)} dx} \right) = \lim_{N \rightarrow \infty} \left( \lim_{n \rightarrow \infty} \frac{\int_0^{1/n} g(x) e^{Nh(x)} dx}{\int_0^{1/n} g_0(x) e^{Nh_0(x)} dx} \right)$$

taking into account the fact that for each  $n$

$$\lim_{N \rightarrow \infty} \frac{\int_0^{1/n} g(x) e^{Nh(x)} dx}{\int_0^{1/n} g_0(x) e^{Nh_0(x)} dx} = 0$$

and for each  $N$

$$\lim_{n \rightarrow \infty} \frac{\int_0^{1/n} g(x) e^{Nh(x)} dx}{\int_0^{1/n} g_0(x) e^{Nh_0(x)} dx} = 1$$

by virtue of (III, 10.2).

(3.9) All the results of this section can be immediately extended to the functions with values in a space  $\mathbf{C}^n$ .

## 4. Regularization

(4.1) Those continuous functions of a real variable, which, historically, have been studied from the beginnings of the Infinitesimal Calculus, and which are also those met the most frequently in applications, are very “regular” in that they are infinitely differentiable (and even *analytic* functions, a notion which will be studied and defined in Chap. VI). However, when it became important to study the most general continuous functions, it was found that they can have very surprising properties, for example no derivative at *any point* (which implies that it is impossible to draw a graph of them). Fortunately we can, as we shall see, *uniformly approximate* every continuous function in a bounded interval by means of *indefinitely differentiable* functions, which often makes their study much easier both in theory and in practice.

(4.2) The starting idea consists in replacing the function at every point by the “mean” of its values in a small interval surrounding the point. To be precise, let  $f$  be a complex function defined and piecewise-continuous in the whole of  $\mathbf{R}$ . For each  $x \in \mathbf{R}$  and each  $h > 0$ , the *mean value* of  $f$  in the interval  $[x - h, x + h]$  is by *definition* the integral

$$(4.2.1) \quad f_h(x) = \frac{1}{2h} \int_{x-h}^{x+h} f(t) dt$$

(which, with the help of the Riemann sums, can effectively be approximated by means of the *arithmetical means* of the values of  $f$  at regularly spaced points of the interval  $[x - h, x + h]$ ). When the function  $f$  is *continuous* at the point  $x$ ,  $\lim_{h \rightarrow 0} f_h(x) = f(x)$ ; indeed, for each  $\varepsilon > 0$ , there exists by hypothesis  $\delta > 0$  such that  $|f(t) - f(x)| \leq \varepsilon$  for  $x - \delta \leq t \leq x + \delta$ . From this and the theorem of the mean we deduce that as soon as  $h < \delta$

$$\left| \int_{x-h}^{x+h} (f(t) - f(x)) dt \right| \leq 2\varepsilon h$$

i.e.

$$|f_h(x) - f(x)| \leq \varepsilon$$

hence our assertion.

Thus it is natural to consider for each *fixed* “small”  $h$  the function  $x \rightarrow f_h(x)$  as an approximation of  $f$ . The approximation is interesting in that even if  $f$  is only *piecewise-continuous*, the function  $f_h$  is always *continuous* in  $\mathbf{R}$  (problem 6). We verify this for the function  $f$  equal to 0 for  $x < 0$ , to 1 for  $x \geq 0$  (“Heaviside function”);  $f_h(x) = 0$  is obtained immediately for  $x \leq -h$ ,  $f_h(x) = 1$  for  $x \geq h$  and  $f_h(x) = (x + h)/2h$  for  $-h \leq x \leq h$  (Fig. 15). When the function

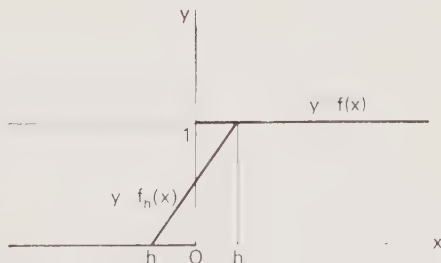


FIGURE 15

$f$  admits discontinuities, we cannot obtain a *uniform* approximation of  $f$  with the help of the  $f_h$ , by virtue of (3.1). But in any case we have constructed a sequence  $(f_{1/n})$  such

that  $(f_{1/n}(x))$  has the limit  $f(x)$  at all the points of continuity of  $f$ , and which is formed of continuous functions, therefore more “regular” ones than the function with which we started.

(4.3) The formula (4.2.1) can be written a little differently. Let us introduce the piecewise-continuous function (Fig. 16)

$$(4.3.1) \quad \varphi_h(x) = \begin{cases} 0 & \text{for } x < -h \text{ or } x > h \\ 1/2h & \text{for } -h \leq x \leq h \end{cases}$$

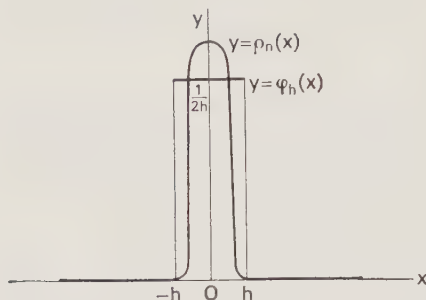


FIGURE 16

from which  $\varphi_h(x) \geq 0$  for every  $x \in \mathbf{R}$  and (cf. III, 9.7)

$$(4.3.2) \quad \int_{-\infty}^{+\infty} \varphi_h(x) dx = 1$$

Thus the formula (4.2.1) can be written as follows

$$(4.3.3) \quad f_h(x) = \int_{-\infty}^{+\infty} f(t) \varphi_h(x-t) dt$$

the integrand then vanishing outside the interval  $[x-h, x+h]$  and being equal to

$f(t)/2h$  in this interval (III, 9.7). In general, for every function  $\varphi$  piecewise-continuous in  $\mathbf{R}$  and zero outside a bounded interval  $I = [-\alpha, \alpha]$ , we call the *convolution* of  $f$  and  $\varphi$  the function

$$(4.3.4) \quad x \rightarrow \int_{-\infty}^{+\infty} f(t) \varphi(x-t) dt$$

and denote this by  $f * \varphi$ . It will be seen that the method of (4.2) can be improved by replacing the discontinuous function  $\varphi_h$  by a *continuous* and even *indefinitely differentiable* function  $\rho_n$  whose graph is in a certain sense “near” to that of  $\varphi_h$  (Fig. 16).

(4.4) To be precise, let us consider a function  $\rho$  defined in  $\mathbf{R}$  and possessing the following properties:

1.  $\rho$  is continuous and  $\geq 0$  in  $\mathbf{R}$  and zero for  $x < -\alpha$  and for  $x > \alpha$ .
2. We have

$$(4.4.1) \quad \int_{-\infty}^{+\infty} \rho(x) dx = 1.$$

(Observe that if  $\rho$  satisfies condition 1 and is not identically zero, then  $\beta = \int_{-\infty}^{+\infty} \rho(x) dx > 0$  (I, 3.2), hence replacing  $\rho$  by  $\rho/\beta$  the “normalization” condition 2 is also satisfied.)

For each integer  $n \geq 1$ , put

$$(4.4.2) \quad \rho_n(x) = n\rho(nx).$$

The function  $\rho_n$  also satisfies the property 1 but vanishes outside the interval  $[-\alpha/n, \alpha/n]$  (Fig. 17); furthermore

$$n \int_{-\infty}^{+\infty} \rho(nx) dx = \int_{-\infty}^{+\infty} \rho(t) dt = 1$$

by the change of variable  $t = nx$ ; thus the normalization condition 2 is also satisfied by  $\rho_n$ . This being so, it will be seen that *at each point  $x$  where  $f$  is continuous*

$$(4.4.3) \quad \lim_{n \rightarrow \infty} (f * \rho_n)(x) = f(x).$$

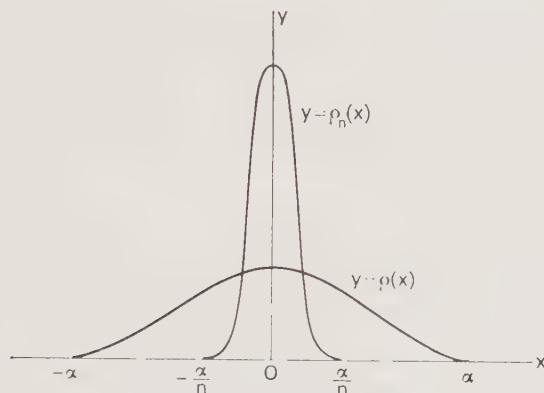


FIGURE 17

Indeed, we can write

$$(4.4.4) \quad (f * \rho_n)(x) = \int_{-\infty}^{+\infty} f(t) \rho_n(x - t) dt = \int_{-\infty}^{+\infty} f(x - u) \rho_n(u) du$$

by the change of variable  $t = x - u$ ; thus by the normalization condition

$$\begin{aligned} f(x) - (f * \rho_n)(x) &= \int_{-\infty}^{+\infty} (f(x) - f(x - u)) \rho_n(u) du \\ &= \int_{-\alpha/n}^{\alpha/n} (f(x) - f(x - u)) \rho_n(u) du \end{aligned}$$

since  $\rho_n(u) = 0$  outside the interval  $[-\alpha/n, \alpha/n]$ . By hypothesis, for each  $\varepsilon > 0$ , there exists an integer  $n_0$  such that for  $n \geq n_0$ ,  $|f(x) - f(x - u)| \leq \varepsilon$  whenever  $|u| \leq \alpha/n$ . Since  $\rho_n$  is positive, we deduce from this, the theorem of the mean and the normalization condition

$$\begin{aligned} (4.4.5) \quad \left| \int_{-\alpha/n}^{\alpha/n} (f(x) - f(x - u)) \rho_n(u) du \right| \\ \leq \int_{-\alpha/n}^{\alpha/n} |f(x) - f(x - u)| \rho_n(u) du \leq \varepsilon \int_{-\alpha/n}^{\alpha/n} \rho_n(u) du = \varepsilon \end{aligned}$$

which proves our assertion.

(4.5) We know (0, 3.4) that a complex function *continuous in a bounded closed interval*  $I = [a, b]$  of  $\mathbf{R}$  is *uniformly continuous in I*: this means that for each  $\varepsilon > 0$ , there exists a number  $\delta > 0$  *depending only on*  $\varepsilon$  such that the relation  $|x - x'| \leq \delta$  for *any* two points  $x, x'$  of  $I$  implies  $|f(x) - f(x')| \leq \varepsilon$ . With the help of this result, the conclusion of (4.4) can be improved. If the function  $f$  is continuous in a bounded open interval  $I = ]a, b[$ , then in each closed subinterval  $[a + \lambda, b - \lambda]$  (with  $\lambda > 0$ ) the sequence  $(f * \rho_n)$  tends *uniformly* to  $f$ . The reasoning is the same as before, observing that by hypothesis one can choose  $n_0$  such that  $|\alpha/n_0| \leq \lambda/2$  and such that for *each*  $x \in [a + \lambda, b - \lambda]$  and *every*  $u$  satisfying  $|u| \leq \alpha/n_0$ , we have  $|f(x) - f(x - u)| \leq \varepsilon$ . The inequality (4.4.5) then holds for *every*  $x \in [a + \lambda, b - \lambda]$  as soon as  $n \geq n_0$ .

Observe also that if the function  $f$  *vanishes outside a bounded closed interval*  $J = [c, d]$ , then for each  $\lambda > 0$  the function  $f * \rho_n$  *vanishes outside the interval*  $[c - \lambda, d + \lambda]$  provided  $n$  is *sufficiently large*, and more precisely provided  $|\alpha/n| < \lambda$ , since the function  $u \rightarrow f(x - u)\rho_n(u)$  can be  $\neq 0$  only if  $-\alpha/n \leq u \leq \alpha/n$  and  $c \leq x - u \leq d$ , and these relations imply

$$c - \frac{\alpha}{n} \leq x \leq d + \frac{\alpha}{n}.$$

(4.6) The interest of the convolutions  $f * \rho_n$  consists, so to speak, in their “inheritance” of the “regularity” properties of the function  $\rho$ , and this *whatever the continuous function*  $f$ . To be precise, if  $f$  is *continuous* (but not necessarily differentiable), and if  $\rho$  is *k times continuously differentiable*, then  $f * \rho_n$  is *k times continuously differentiable* (this is true, even if we only suppose  $f$  to be piecewise-continuous, cf. problem 7). Indeed, for each bounded interval  $]c, d[$  of  $\mathbf{R}$  and each integer  $n$ , when  $x \in ]c, d[$  we can write

$$(f * \rho_n)(x) = \int_{c-\alpha}^{d+\alpha} f(t) \rho_n(x-t) dt$$

the integrand vanishing outside the interval  $[c - \alpha, d + \alpha]$ . We can therefore apply Leibniz's rule (0, 4.6) for the existence and the calculation of the derivative of an integral depending on a parameter, since the function of two variables  $(t, x) \rightarrow f(t) \rho_n(x - t)$  and its derivatives with respect to  $x$  of order  $h \leq k$ ,  $(t, x) \rightarrow f(t) \rho_n^{(h)}(x - t)$ , are continuous in  $]c - \alpha, d + \alpha[ \times ]c, d[$ . This proves our assertion and gives

$$(4.6.1) \quad (f * \rho_n)^{(h)}(x) = \int_{-\infty}^{+\infty} f(t) \rho_n^{(h)}(x - t) dt$$

which can also be written

$$(4.6.2) \quad (f * \rho_n)^{(h)} = f * \rho_n^{(h)}.$$

(4.7) The method of “regularization by convolution”, which we have just described, has another remarkable property. The formula (4.4.4) shows that  $f$  and  $\rho_n$  can be made to play analogous roles. From this it may be concluded, by the same reasoning as in (4.6), that if the function  $f$  *itself is k times continuously differentiable in*  $\mathbf{R}$ , then also, for  $h \leq k$

$$(4.7.1) \quad (f * \rho_n)^{(h)} = f^{(h)} * \rho_n.$$

(4.5) shows that as  $n$  tends to  $+\infty$ , then in every *bounded* interval of  $\mathbf{R}$ , not only does the sequence of regularized functions  $f * \rho_n$  tend uniformly to  $f$ , but also, for each  $h \leq k$ , the sequence of the  $h^{\text{th}}$  derivatives  $(f * \rho_n)^{(h)}$  converges uniformly to  $f^{(h)}$ .

(4.8) It remains to give examples of functions  $\rho$  with the properties of (4.4) and sufficiently “regular”. It is easy to do this if we only require the function  $\rho$  to be continuously differentiable up to a certain order  $k$ : it is enough to take

$$(4.8.1) \quad \begin{cases} \rho(x) = (\alpha^2 - x^2)^{k+1} & \text{for } -\alpha \leq x \leq \alpha, \\ \rho(x) = 0 & \text{for } |x| > \alpha. \end{cases}$$

It is rather less easy to give examples of functions  $\rho$  satisfying condition 1 of (4.4) and *indefinitely differentiable* in  $\mathbf{R}$ . To obtain such a function, start from the function  $g$  defined as follows:

$$(4.8.2) \quad \begin{cases} g(x) = e^{-1/x^2} & \text{for } x > 0 \\ g(x) = 0 & \text{for } x \leq 0. \end{cases}$$

It follows immediately that this function is continuous and indefinitely differentiable for  $x \neq 0$ ; we shall see that it is also indefinitely differentiable in a *neighbourhood* of 0 and therefore everywhere. To prove this, it will be sufficient to prove, by induction on  $h$ , that  $g^{(h)}(x)/x$ , defined for  $x \neq 0$ , tends to 0 with  $x$ . Since  $e^{-1/x^2}$  is negligible compared to any power  $x^\lambda$  in the neighbourhood of 0, it will be enough to prove by induction that for  $x > 0$  and  $h \geq 1$

$$(4.8.3) \quad g^{(h)}(x) = \frac{P_h(x)}{x^{3h}} e^{-1/x^2}$$

where  $P_h(x)$  is a *polynomial*. On differentiating this formula

$$\begin{aligned} g^{(h+1)}(x) &= \left( \frac{P'_h(x)}{x^{3h}} - \frac{3hP_h(x)}{x^{3h+1}} + \frac{2P_h(x)}{x^{3h+3}} \right) e^{-1/x^2} \\ &= \frac{P_{h+1}(x)}{x^{3(h+1)}} e^{-1/x^2} \end{aligned}$$

where  $P_{h+1}$  is a polynomial, which proves our assertion. This being so, the function

$$(4.8.4) \quad \rho(x) = g(x + \alpha)g(\alpha - x)$$

satisfies condition 1 of (4.4) and is *indefinitely differentiable* in  $\mathbf{R}$  and  $> 0$  for  $-\alpha < x < \alpha$ ; multiplying by a constant, we can also make it satisfy the normalization condition (4.4.1).

## 5. Weierstrass approximation theorem

(5.1) A slightly different application of the method of convolution described in no. 4 permits us to prove the remarkable *approximation theorem of Weierstrass*:

(5.2) *Every complex function  $f$  continuous in a bounded closed interval  $[a, b]$  can be uniformly approximated in  $[a, b]$  by polynomials.*

Note first that it can be assumed that  $f(a) = f(b) = 0$  and then  $f$  may be extended to the whole of  $\mathbf{R}$  by taking  $f(x) = 0$  outside  $[a, b]$ . Indeed, if the conditions  $f(a) = f(b) = 0$  are not satisfied, we consider a larger interval  $[c, d]$  with  $c < a$ ,  $b < d$ , and extend  $f$  in  $[c, a]$  (resp. in  $[b, d]$ ) by means of an affine linear function taking the value 0 at the point  $c$  and the value  $f(a)$  at the point  $a$  (resp. the value 0 at the point  $d$  and  $f(b)$  at the point  $b$ ). By a linear change of variable it is further supposed that  $a = -\frac{1}{2}$ ,  $b = \frac{1}{2}$ .

For each integer  $n \geq 1$  put

$$(5.2.1) \quad \begin{cases} g_n(x) = (1 - x^2)^n & \text{for } -1 \leq x \leq 1 \\ g_n(x) = 0 & \text{for } |x| > 1. \end{cases}$$

Let  $a_n = \int_{-\infty}^{+\infty} g_n(t) dt = \int_{-1}^1 g_n(t) dt$  and put  $h_n = a_n^{-1} g_n$ ; then  $h_n > 0$  in  $] -1, 1[$ ,  $h_n(x) = 0$  outside this interval,  $h_n$  is continuous in  $\mathbf{R}$  and satisfies the normalization condition

$$(5.2.2) \quad \int_{-\infty}^{+\infty} h_n(t) dt = 1.$$

Furthermore, since  $1 - x^2 \geq 1 - |x|$  for  $-1 \leq x \leq 1$ ,

$$(5.2.3) \quad a_n \geq 2 \int_0^1 (1 - t)^n dt = 2/(n + 1).$$

Hence the following result:

(5.2.4) *For any number  $\delta$  satisfying  $0 < \delta \leq 1$ , the function  $h_n$  tends uniformly to 0 in each of the intervals  $[-1, -\delta]$  and  $[\delta, 1]$  as  $n$  tends to  $+\infty$ .*

Indeed, the function  $h_n$  being even it is enough to consider the interval  $[\delta, 1]$  where by virtue of (5.2.3)

$$0 \leq h_n(x) \leq (n + 1)(1 - \delta^2)^n,$$

hence the conclusion.

This being so, consider the function defined for all  $x \in \mathbf{R}$

$$(5.2.5) \quad \begin{aligned} (f * h_n)(x) &= a_n^{-1} \int_{-\infty}^{+\infty} f(t) g_n(x - t) dt \\ &= a_n^{-1} \int_{-1/2}^{1/2} f(t) g_n(x - t) dt \end{aligned}$$

since  $f$  vanishes outside the interval  $[-\frac{1}{2}, \frac{1}{2}]$ . When  $x$  belongs to the interval  $[-\frac{1}{2}, \frac{1}{2}]$ , then  $g_n(x - t) = (1 - (x - t)^2)^n$  since  $-1 \leq x - t \leq 1$ ; developing (5.2.5) we see that in  $[-\frac{1}{2}, \frac{1}{2}]$  the function  $f * h_n$  coincides with a polynomial of degree  $\leq 2n$ .

We show, on the other hand, that  $f * h_n$  converges uniformly to  $f$  in every bounded interval  $[-a, a]$  as  $n$  tends to  $+\infty$ . As in (4.4)

$$(5.2.6) \quad f(x) - (f * h_n)(x) = \int_{-\infty}^{+\infty} (f(x) - f(x - t)) h_n(t) dt.$$

Choose  $\varepsilon > 0$ ; by (0, 3.4) there exists a number  $\delta \in ]0, 1[$  depending only on  $\varepsilon$  such that for  $x, x'$  in the interval  $[-a - 1, a + 1]$  satisfying  $|x - x'| \leq \delta$ , we have  $|f(x) - f(x')| \leq \varepsilon$ . The number  $\delta$  being thus fixed, write the second member of (5.2.6) in the form

$$\begin{aligned} \int_{-\infty}^{-\delta} (f(x) - f(x - t)) h_n(t) dt + \int_{-\delta}^{\delta} (f(x) - f(x - t)) h_n(t) dt \\ + \int_{\delta}^{+\infty} (f(x) - f(x - t)) h_n(t) dt. \end{aligned}$$

The theorem of the mean, together with the fact that  $h_n$  is positive and satisfies the normalization condition (5.2.2), gives

$$(5.2.7) \quad \left| \int_{-\delta}^{\delta} (f(x) - f(x-t))h_n(t) dt \right| \leq \int_{-\delta}^{\delta} |f(x) - f(x-t)|h_n(t) dt \\ \leq \varepsilon \int_{-\delta}^{\delta} h_n(t) dt \leq \varepsilon$$

for all  $x$  satisfying  $-a \leq x \leq a$ . On the other hand the function  $f$  is bounded in  $\mathbf{R}$ , so let  $|f(x)| \leq A$ ; then, by the theorem of the mean

$$(5.2.8) \quad \left| \int_{\delta}^{+\infty} (f(x) - f(x-t))h_n(t) dt \right| \leq 2A \int_{\delta}^{+\infty} h_n(t) dt = 2A \int_{\delta}^1 h_n(t) dt$$

since  $h_n$  vanishes outside  $[-1, 1]$ . But by virtue of (5.2.4) there exists  $n_0$  (depending on  $\delta$ , therefore on  $\varepsilon$ ) such that for  $n \geq n_0$ ,  $h_n(t) \leq \varepsilon/2A$  for all  $t$  satisfying  $\delta \leq t \leq 1$ ; we conclude from this by the theorem of the mean that, for  $n \geq n_0$

$$(5.2.9) \quad \left| \int_{\delta}^1 (f(x) - f(x-t))h_n(t) dt \right| \leq \varepsilon.$$

In the same way ( $h_n$  being an even function)

$$(5.2.10) \quad \left| \int_{-1}^{-\delta} (f(x) - f(x-t))h_n(t) dt \right| \leq \varepsilon$$

for  $n \geq n_0$ ; we conclude that for  $n \geq n_0$  and for all  $x \in [-a, a]$

$$(5.2.11) \quad |f(x) - (f * h_n)(x)| \leq 3\varepsilon,$$

which proves the theorem.

Observe that when the function  $f$  is *differentiable up to the order  $k$*  in  $\mathbf{R}$ , and vanishes outside a bounded interval, the formula (4.7.1) and the previous reasoning prove that the derivatives  $(f * h_n)^{(p)}$  up to the order  $k$  *tend uniformly* in  $[-a, a]$  to  $f^{(p)}$ .

(5.3) The method of approximation described in (5.2) is not very practical and does not give an easy majorization of the distance  $d(f, f * h_n)$ ; for a better method (but of less scope) see the Appendix. Note that the most “natural” method of approximation, *interpolation*, does not necessarily give a sequence of polynomials *convergent* to  $f$  (the “Runge phenomenon” (cf. IX, Appendix)).

## APPENDIX

### Bernstein polynomials

S. Bernstein has provided an *explicit* method for approximating uniformly a continuous function  $f$  in  $[0, 1] = I$  by polynomials, whose formulation calls for no operation of the infinitesimal calculus.

The *Bernstein  $n^{\text{th}}$  degree polynomial* for  $f$  is the polynomial

$$(1) \quad B_n(f)(t) = \sum_{p=0}^n \binom{n}{p} f\left(\frac{p}{n}\right) (1-t)^{n-p} t^p.$$

We propose to show that if  $f$  is continuous in  $I$ , the sequence of Bernstein polynomials  $B_n(f)$  converges uniformly to  $f$  in  $I$ .

Start from the identity

$$(2) \quad 1 = (1 - t + t)^n = \sum_{p=0}^n \binom{n}{p} (1 - t)^{n-p} t^p.$$

It follows from this relation that for every bounded function  $f$  defined in  $I$

$$(3) \quad |B_n(f)(t)| \leq \sup_{t \in I} |f(t)| \left( \sum_{p=0}^n \binom{n}{p} (1 - t)^{n-p} t^p \right) = d(0, f),$$

since the functions  $\binom{n}{p} (1 - t)^{n-p} t^p$  are  $\geq 0$  in  $I$ . Given  $\varepsilon > 0$ , by Weierstrass's theorem there exists a polynomial  $P$  such that  $d(f, P) \leq \varepsilon$ ; thus from (3)

$$(4) \quad d(B_n(f), B_n(P)) \leq d(f, P) \leq \varepsilon$$

and hence

$$(5) \quad d(f, B_n(f)) \leq 2\varepsilon + d(P, B_n(P)).$$

Thus, if the theorem is proved when  $f$  is a polynomial, it will be true for every continuous function in  $I$ . By linearity, it is then sufficient to prove it when  $f(t) = t^m$ . In fact, we shall see, by induction on  $m$ , that putting  $f_m(t) = t^m$  we have for  $n \geq m$

$$(6) \quad B_n(f_m)(t) = t^m + \frac{1}{n} Q_{m,n}(t)$$

where  $Q_{m,n}(t)$  is a polynomial of degree  $\leq m$ , whose coefficients are majorized in absolute value by a number  $A_m$  independent of  $n$ .

Formula (6) reduces to (2) for  $m = 0$ , with  $Q_{0,n}(t) = 0$ . Suppose it is verified for an integer  $m$  and differentiate with respect to  $t$ ; by virtue of definition (1), for  $n \geq m + 1$

$$(7) \quad - \sum_{p=0}^{n-1} \binom{n}{p} \frac{t^m}{n^m} (n-p)(1-t)^{n-p-1} t^p + \sum_{p=1}^n \binom{n}{p} \frac{p^{m+1}}{n^m} (1-t)^{n-p} t^{p-1} \\ = m t^{m-1} + \frac{1}{n} Q'_{m,n}(t).$$

Since  $(n-p) \binom{n}{p} = n \binom{n-1}{p}$ , the first term of the first member of (7) is equal to

$$-n \left( \frac{n-1}{n} \right)^m B_{n-1}(f_m)(t) = -n \left( \frac{n-1}{n} \right)^m t^m - \frac{(n-1)^{m-1}}{n^m} Q_{m,n-1}(t).$$

Multiplying the two members of (7) by  $t/n$  we thus obtain, by virtue of (1)

$$B_n(f_{m+1})(t) = t^{m+1} + \frac{1}{n} Q_{m+1,n}(t)$$

and this proves (6) by induction, since

$$Q_{m+1,n}(t) = n \left( \left( \frac{n-1}{n} \right)^m - 1 \right) t^{m+1} + m t^m + \frac{t}{n} Q'_{m,n}(t) + \left( \frac{n-1}{n} \right)^{m-1} Q_{m,n-1}(t)$$

hence  $A_{m+1} \leq 2 \sup(m, mA_m)$ .

Q.E.D.

## PROBLEMS

1. If  $g_n$  is the function defined in (3.6.3), show that the series with general term  $g_n$  is uniformly convergent but not normally convergent.

2. Let  $(f_n)$  be a sequence of functions continuous and differentiable in a bounded interval  $I = [a, b]$ ; suppose that there exists a number  $M > 0$  independent of  $n$  such that  $|f'_n(t)| \leq M$  for every  $n$  and every  $t \in I$ . Show that if the sequence  $(f_n)$  converges simply to a limit  $f$  in  $I$ , it converges *uniformly* in  $I$ . (Observe that for each  $\varepsilon > 0$  we can divide  $I$  into a finite number (dependent on  $\varepsilon$ ) of intervals  $[\alpha, \beta]$  such that in each of them

$$|f_n(x) - f_n(\alpha)| \leq \varepsilon$$

for every integer  $n$  and every  $x \in [\alpha, \beta]$ .)

3. Let  $(f_n)$  be a sequence of real functions each  $p$  times differentiable in an open interval  $I$ ; suppose that in  $I$  the sequence  $(f_n)$  converges simply to a function  $f$ , which is  $p$  times differentiable. For each integer  $r$ ,  $1 \leq r \leq p$ , show that given numbers  $\delta > 0$ ,  $\varepsilon > 0$  and a point  $x_0 \in I$  there exists an integer  $N$  such that for each  $n \geq N$  there is a point  $x_n$  for which  $|x_n - x_0| \leq \delta$  and  $|f^{(r)}(x_n) - f^{(r)}(x_0)| \leq \varepsilon$ . (Prove this first for  $r = 1$  by using the mean value theorem, then use induction on  $r$ .)

4. For each integer  $n \geq 0$ , let

$$s_n = a_{n0} + a_{n1} + \cdots + a_{nm} + \cdots$$

be a series of complex numbers with the following property: there exists a convergent series of numbers  $\geq 0$

$$A_0 + A_1 + \cdots + A_m + \cdots$$

such that  $|a_{nm}| \leq A_m$  for every  $n$  (which implies that the series  $s_n$  is absolutely convergent). Suppose that for each  $m \geq 0$  the sequence  $(a_{nm})_{n \geq 0}$  has a finite limit  $a_m$ . Show that the series

$$s = a_0 + a_1 + \cdots + a_m + \cdots$$

is convergent and that  $s = \lim_{n \rightarrow \infty} s_n$ . (Observe that for each  $\varepsilon > 0$  there exists an integer  $m_0$  depending only on  $\varepsilon$  such that

$$A_{m_0+1} + \cdots + A_m + \cdots \leq \varepsilon;$$

note on the other hand that for every  $m$

$$|a_0| + |a_1| + \cdots + |a_m| \leq A_0 + A_1 + \cdots + A_m + \cdots)$$

5. Let  $F$  be a real function  $\geq 0$ , piecewise-continuous in  $\mathbf{R}$  and such that the improper integral  $\int_{-\infty}^{+\infty} F(t) dt$  is convergent. Let  $(f_n)$  be a sequence of complex functions piecewise-continuous in  $\mathbf{R}$  and having the following properties: (1)  $|f_n(t)| \leq F(t)$  for every  $n$  and every  $t \in \mathbf{R}$ ; (2) the sequence  $(f_n)$  converges uniformly in every bounded interval  $[a, b] \subset \mathbf{R}$ . If  $f(t) = \lim_{n \rightarrow \infty} f_n(t)$ , show that the improper integral  $\int_{-\infty}^{+\infty} f(t) dt$  is convergent and that

$$\lim_{n \rightarrow \infty} \int_{-\infty}^{+\infty} f_n(t) dt = \int_{-\infty}^{+\infty} f(t) dt.$$

(Note that for each  $\varepsilon > 0$ , there exists an interval  $[-\Lambda, \Lambda]$  dependent on  $\varepsilon$  and such that  $\int_{-\infty}^{-\Lambda} F(t) dt \leq \varepsilon$  and  $\int_{\Lambda}^{+\infty} F(t) dt \leq \varepsilon$ .)

6. Show that for each piecewise-continuous function  $f$  in  $\mathbf{R}$ , the “mean”  $f_h$  defined in (4.2.1) is continuous in  $\mathbf{R}$  for every  $h > 0$  (study the behaviour in the neighbourhoods of the discontinuities of  $f$ ).

7. Let  $f$  be a real function piecewise-continuous in  $\mathbf{R}$  and vanishing outside a bounded closed interval  $[a, b]$ .

(a) Show that for each  $\varepsilon > 0$  there exists a continuous function  $g$  vanishing outside a bounded closed interval, such that  $f \leq g$  and

$$\int_{-\infty}^{+\infty} g(t) dt \leq \int_{-\infty}^{+\infty} f(t) dt + \varepsilon$$

(study the behaviour in the neighbourhood of the discontinuities of  $f$ , by taking  $g$  linear in suitable intervals).

(b) With the notations of (4.4) show that, if we take  $\rho$   $k$  times continuously differentiable, then  $f * \rho_n$  is  $k$  times continuously differentiable (use (a) and the fact that the derivative of  $\rho_n$  is bounded).

(c) Show that for each  $\varepsilon > 0$ , there exist two polynomials  $P, Q$  such that in  $[a, b]$

$$P(x) \leq f(x) \leq Q(x) \quad \text{and} \quad \int_a^b Q(t) dt - \int_a^b P(t) dt \leq \varepsilon.$$

(Use (a) and Weierstrass's theorem.)

8. Let  $f$  be a piecewise-continuous function in a bounded closed interval  $[a, b]$ . If for every integer  $n \geq 0$

$$\int_a^b f(t) t^n dt = 0,$$

show that  $f(x) = 0$  except at the points of discontinuity of  $f$ . (Reduce to the case where  $f$  is real. Use problem 7(c) to prove that for each  $\varepsilon > 0$ ,  $\int_a^b f(t)^2 dt \leq \varepsilon$ .)

9. Let  $f$  be a real continuous function in a bounded closed interval  $[a, b]$ , and suppose that  $\int_a^b f(t) t^k dt = 0$  for  $0 \leq k \leq n-1$ . If  $f$  is not identically zero, show that there are at least  $n$  distinct points in  $]a, b[$  where  $f$  vanishes and changes sign (use contradiction, assuming that there are at most  $n-1$  of these points  $c_1 < c_2 < \dots < c_r$  ( $r \leq n-1$ ), and consider the function

$$f(t)(t - c_1) \dots (t - c_r)$$

which does not change sign in  $[a, b]$ ).

10. Let  $f$  be a complex continuous function in the interval  $[0, +\infty[$  such that the integral  $J(k) = \int_0^{+\infty} e^{-kt} f(t) dt$  is convergent for  $k = k_0$  real. We then know (III, problem 29) that  $J(k)$  converges for every  $k \geq k_0$ . If there exists  $\alpha > 0$  such that  $J(k_0 + n\alpha) = 0$  for every integer  $n > 1$ , show that  $f$  is identically zero. (Note that if  $F(x) = \int_0^x e^{-k_0 t} f(t) dt$ ,  $F$  is continuous and bounded for  $x \geq 0$  and that  $\int_0^{+\infty} e^{-(k-k_0)t} F(t) dt = 0$  for  $k - k_0 = n\alpha$ ; make a suitable change of variable and use problem 8.)

11. Let  $f$  be a continuous function in the interval  $[0, +\infty[$ , vanishing outside a bounded closed interval not containing 0. Show that for each  $\varepsilon > 0$ , there exists a polynomial  $Q(x)$  such that

$$\int_0^{+\infty} |f(x) - Q(x)| e^{-x} dx \leq \varepsilon.$$

(By the change of variable  $t = e^{-x}$ , show first that there exists a polynomial  $P(t)$  such that

$$\int_0^{+\infty} |f(x) - P(e^{-x})| e^{-x} dx \leq \varepsilon$$

then use problem 18 of Chap. IV.) Show that we have the same conclusion if  $f$  is any bounded continuous function in  $[0, +\infty[$ .

12. Deduce from problem 11 that for each function  $f$  continuous and bounded in  $\mathbf{R}$ , there exists a polynomial  $R(x)$  such that

$$\int_{-\infty}^{+\infty} |f(x) - R(x)| e^{-x^2} dx \leq \varepsilon.$$

(Consider separately the case where  $f$  is odd and the case where  $f$  is even and zero in an open interval containing 0.)

13. (a) Show that for every  $x > 0$ , we have

$$(*) \quad \Gamma'(x) = \int_0^{+\infty} t^{x-1} e^{-t} \log t \, dt$$

where the integral is absolutely convergent. (For each  $n$  consider the integral

$$g_n(x) = \int_{1/n}^n t^{x-1} e^{-t} \log t \, dt$$

and show that it converges *uniformly* to the second member of (\*) when  $x$  remains in a bounded closed interval  $[a, b]$  with  $0 < a < b$ ; then use (3.4.4).) In particular, using the relation  $\Gamma'(1) = -\gamma$  (Chap. IX),

$$-\gamma = \int_0^{+\infty} e^{-t} \log t \, dt.$$

(b) Deduce from (a) the asymptotic development

$$\int_y^{+\infty} \frac{e^{-t}}{t} dt = \log \frac{1}{y} - \gamma + o(1)$$

as  $y$  tends to 0 through positive values. If  $f(t)$  is continuous in  $]0, +\infty[$  and the integral  $\int_y^{+\infty} f(t) dt$  is convergent for every  $y > 0$  with

$$\int_y^{+\infty} f(t) dt = \log \frac{1}{y} + o(1),$$

deduce that

$$\gamma = \int_0^{+\infty} \left( f(t) - \frac{e^{-t}}{t} \right) dt$$

where the integral of the second member is convergent. In particular

$$\gamma = \int_0^{+\infty} e^{-t} \left( \frac{1}{1-e^{-t}} - \frac{1}{t} \right) dt.$$

# Analytic functions

## 1. Taylor series

Consider a complex function  $f$  defined in a neighbourhood

$$I = [x_0 - \alpha, x_0 + \alpha]$$

of a point  $x_0 \in \mathbf{R}$  and *indefinitely differentiable* in this interval. Taylor's formula gives, for every integer  $n$  and every  $x \in I$

$$(1.1) \quad f(x) = f(x_0) + \frac{f'(x_0)}{1!} (x - x_0) + \cdots + \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n + R_n(x)$$

with a majorization of the remainder

$$(1.2) \quad |R_n(x)| \leq \frac{M_{n+1}}{(n+1)!} |x - x_0|^{n+1}$$

where  $M_{n+1}$  is the least upper bound of  $|f^{(n+1)}(x)|$  in  $I$ . Naturally one hopes that the polynomials

$$(1.3) \quad P_n(x) = f(x_0) + \frac{f'(x_0)}{1!} (x - x_0) + \cdots + \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n$$

converge uniformly to  $f$  in  $I$ . In general, *this is not the case*, as is shown by the example of the function  $g$  defined in (V, 4.8.2) for  $x_0 = 0$ : *all the derivatives of  $g$  vanish at the point 0*, so the polynomials  $P_n$  are all *identically zero* and thus cannot converge to  $g$  in any interval (cf. problem 2).

The indefinitely differentiable functions  $f$  for which the polynomials (1.3) do converge to  $f$  in a neighbourhood of  $x_0$ , or which are sums of their *Taylor series*

$$f(x) = \frac{f'(x_0)}{1!} (x - x_0) + \cdots + \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n + \cdots$$

thus form only a *subset* of the set of indefinitely differentiable functions. These functions have quite remarkable properties; this and the two following chapters are devoted to their study.

## 2. Power series

We begin by studying the question of the *convergence* of a series of the form

$$(2.1) \quad c_0 + c_1 z + \cdots + c_n z^n + \cdots$$

where the coefficients  $c_n$  are any complex numbers. The terms of such a series make sense, not only when  $z$  is a real number, but when  $z$  itself is any *complex number* and we shall see that the study of these series for complex  $z$  explains phenomena which are surprising when we confine ourselves to the real domain. Therefore, from now on, we assume that the  $c_n$  and  $z$  are *complex numbers* and for the moment arbitrary. When the series (2.1) converges, its sum  $f(z)$  is thus a *complex function of the complex number  $z$* . A series of the type (2.1) is called a *power series in  $z$* .

(2.2) (Abel's Lemma) *Suppose that the series (2.1) has uniformly bounded terms for a value  $z = z_0 \neq 0$ :*

$$(2.2.1) \quad |c_n z_0^n| \leq M$$

where  $M$  is independent of  $n$ . Then :

- (i) *for each complex number  $z$  satisfying  $|z| < |z_0|$ , the series (2.1) is absolutely convergent;*
- (ii) *for each  $r$  such that  $0 < r < |z_0|$ , the series (2.1) is normally convergent in the closed disc  $|z| \leq r$ .*

(i) It follows from (2.2.1) that for  $|z| < |z_0|$

$$|c_n z^n| = |c_n z_0^n| \left| \frac{z}{z_0} \right|^n \leq M \left| \frac{z}{z_0} \right|^n$$

hence the conclusion, by virtue of the comparison principle (I, 2.2).

(ii) In the same way, we have for  $|z| \leq r$

$$|c_n z^n| \leq M \left( \frac{r}{|z_0|} \right)^n$$

and the series of the second member is convergent, so the series of  $c_n z^n$  is normally convergent (V, 2.5).

(2.3) Let  $B$  be the set of numbers  $r \geq 0$  such that the sequence  $(|c_n| r^n)$  is majorized; it is clear that if  $r \in B$ , then  $r' \in B$  for every  $r' < r$ . The *supremum*  $R$  of the set  $B$  (0, 2.2) is a number  $\geq 0$  or  $+\infty$ ; we shall call it the *radius of convergence* of the series with general term  $c_n z^n$ . It follows from Abel's Lemma that for  $0 \leq r < R$ , the series with general term  $c_n z^n$  is *normally convergent* in the *closed disc*  $|z| \leq r$ . On the other hand, if  $|z| > R$ , the sequence  $(|c_n z^n|)$  is *not bounded*, so *a fortiori* the series with general term  $c_n z^n$  is not convergent. The *open disc*  $|z| < R$  is called the *disc of convergence* of the series; the plane  $\mathbf{C}$  is therefore divided into three mutually disjoint subsets:

- (i) The disc of convergence  $|z| < R$ , where the series converges.
- (ii) The exterior of the disc of convergence:  $|z| > R$ , where *not only* does the series not converge, but its terms are *not bounded*.

- (iii) The circle of convergence  $|z| = R$ , at whose points the series *may or may not* converge.

We can have  $R = 0$  or  $R = +\infty$ , and in these cases one or two of the three preceding subsets is empty.

*Examples (2.4)* Suppose that the sequence  $(|c_n|^{1/n})$  has a limit  $\rho$  finite or not. If  $0 < \rho < +\infty$ , for each pair of numbers  $\rho', \rho''$  satisfying  $0 < \rho' < \rho < \rho''$ , we have  $(\rho'r)^n \leq |c_n|r^n \leq (\rho''r)^n$  as soon as  $n$  is sufficiently large, by definition of limit. Therefore the sequence  $(|c_n|r^n)$  is bounded if  $r < 1/\rho$ , unbounded if  $r > 1/\rho$ , and the radius of convergence is  $R = 1/\rho$  ("Cauchy's rule"). If  $\rho = 0$  (resp.  $\rho = +\infty$ ), the same reasoning shows that the sequence  $(|c_n|r^n)$  is unbounded for *every*  $r > 0$  (resp. bounded for *every*  $r > 0$ ); therefore  $R = +\infty$  (resp.  $R = 0$ ). One may write in every case  $R = 1/\rho$  (agreeing that  $1/0 = +\infty$ ,  $1/+\infty = 0$ ).

If the sequence  $(|c_n|)$  has a *principal part*  $|c_n| \sim a \cdot g(n)$  with  $g \in \mathcal{E}$  (III, 2.1) and  $a > 0$ , we are then led to seek the limit of the sequence  $((1/n) \log g(n))$ .

It is at once apparent that this limit depends only on the factor  $e^{P(x)}$  in (III, 2.1), and is equal to  $c_1$  if  $\gamma_1 = 1$ , to 0 if  $P = 0$  or if  $\gamma_1 < 1$ , to  $+\infty$  if  $\gamma_1 > 1$  and  $c_1 > 0$ , to  $-\infty$  if  $\gamma_1 > 1$  and  $c_1 < 0$ . Hence  $R = e^{-c_1}$  if  $\gamma_1 = 1$ ,  $R = 1$  if  $\gamma_1 < 1$  or if  $P = 0$ ,  $R = 0$  if  $\gamma_1 > 1$  and  $c_1 > 0$ ,  $R = +\infty$  if  $\gamma_1 > 1$  and  $c_1 < 0$ .

(2.4.1) All the series  $(n^\alpha z^n)$  ( $\alpha$  any real number) have radius of convergence 1. Observe that if  $\alpha < -1$ , the series *converges normally* in the *closed* disc  $|z| \leq 1$ ; if  $\alpha \geq 0$ , the series does not converge *at any point* of the circle  $|z| = 1$ , its general term not tending to 0. Finally if  $-1 \leq \alpha < 0$ , the series is divergent at the point  $z = 1$ , but it can be shown (problem 4) that it converges at every other point  $z = e^{i\theta}$  ( $0 < \theta < 2\pi$ ) of the circle  $|z| = 1$ .

(2.4.2) The series with general term  $z^n/n!$  has radius of convergence  $+\infty$ , since by Stirling's formula (IV, 3.8.2),  $(1/n) \log(n!) \sim \log n$  which tends to  $+\infty$ .

Note that this series *converges uniformly* in *every* open disc  $|z| < r$ , but nevertheless *does not converge uniformly in the whole plane*  $\mathbf{C}$ .

(2.4.3) The same calculations show that the series with general term  $n! z^n$  has radius of convergence 0.

(2.4.4) Cauchy's rule does not apply to the series

$$1 + z + z^2 + z^4 + z^8 + \cdots + z^{2^n} + \cdots$$

since the coefficients  $c_n$  take the value 0 infinitely often and the value 1 infinitely often. However, clearly  $|c_n|r^n \leq r^n$  for all  $n$ , and for  $|z| > 1$  the general term of the series is not bounded; therefore we again have the radius of convergence  $R = 1$ .

*Remark (2.5)* If  $f(z) = c_0 + c_1 z + \cdots + c_n z^n + \cdots$  is a power series convergent in an open disc  $|z| < r$ , then for every  $z$  in this disc we have at once, by passage to the limit

$$\overline{f(z)} = \bar{c}_0 + \bar{c}_1 \bar{z} + \cdots + \bar{c}_n \bar{z}^n + \cdots$$

and therefore

$$\overline{f(\bar{z})} = \bar{c}_0 + \bar{c}_1 z + \cdots + \bar{c}_n z^n + \cdots$$

When the coefficients  $c_n$  are all *real*, we thus have  $\overline{f(\bar{z})} = f(z)$ ; this is naturally not the case if the  $c_n$  are not all real.

### 3. Principle of isolated zeros

(3.1) Consider a power series

$$f(z) = c_0 + c_1z + \cdots + c_nz^n + \cdots$$

with radius of convergence  $R > 0$ . Then the function  $f$  is continuous in the open disc  $|z| < R$ .

Indeed, for  $0 < r < R$ , the series is normally convergent in the closed disc  $|z| \leq r$ ; the conclusion follows from (V, 3.1), each point  $z_0$  such that  $|z_0| < R$  lying inside a disc  $|z| < r$  with  $0 < r < R$ .

(3.2) (Principle of isolated zeros) With the hypotheses of (3.1), suppose further that the  $c_n$  are not all zero. Then there exists a number  $r_0$  such that  $0 < r_0 < R$  and such that  $f(z) \neq 0$  for  $0 < |z| < r_0$ .

By hypothesis, there exists a smallest integer  $k \geq 0$  such that  $c_k \neq 0$ . Then

$$(3.2.1) \quad f(z) = z^k g(z)$$

with

$$(3.2.2) \quad g(z) = c_k + c_{k+1}z + \cdots + c_{k+n}z^n + \cdots.$$

For each  $z \neq 0$  such that  $0 < |z| < R$ , the power series  $(c_{k+n}z^n)$  converges, since its terms are obtained by dividing by  $z^k \neq 0$  the terms of the convergent series  $(c_{k+n}z^{k+n})$ . Therefore  $R$  is also the radius of convergence of the series  $g(z)$ , and the function  $g$  is thus continuous for  $|z| < R$  (3.1). But  $g(0) = c_k \neq 0$ ; since there exists  $r_0 > 0$  such that  $|g(z) - g(0)| \leq |c_k|/2$  for  $|z| < r_0$ , for  $|z| < r_0$

$$|g(z)| \geq |g(0)| - |c_k|/2 = |c_k|/2,$$

and *a fortiori*  $g(z) \neq 0$ . Because of (3.2.1) the result follows with this number  $r_0$ . The result may also be expressed in the following form:

(3.3) If a power series

$$f(z) = c_0 + c_1z + \cdots + c_nz^n + \cdots$$

is convergent for  $|z| < r$  ( $r > 0$ ) and such that  $f(z_p) = 0$  for a sequence  $(z_p)$  of distinct points of this disc tending to 0, then all the coefficients  $c_n$  are zero (and  $f$  is therefore identically zero).

For example, there is no convergent power series in an open disc of centre 0 and equal to the function  $g$  defined in (V, 4.8.2) in an interval of centre 0 in  $\mathbf{R}$ .

(3.4) Let

$$\begin{aligned} f(z) &= a_0 + a_1z + \cdots + a_nz^n + \cdots \\ g(z) &= b_0 + b_1z + \cdots + b_nz^n + \cdots \end{aligned}$$

be two power series convergent in the same open disc  $|z| < r$  ( $r > 0$ ). If there is a sequence of distinct points  $(z_p)$  of this disc tending to 0 and such that

$$f(z_p) = g(z_p)$$

for every  $p$ , then  $a_n = b_n$  for every  $n$  (and hence  $f(z) = g(z)$  for every  $z$  such that  $|z| < r$ ).

It suffices to apply (3.3) to the series  $f(z) - g(z)$ .

This can be expressed more briefly by stating that a function can be the sum of at most one power series convergent in an open disc of centre 0.

#### 4. Substitution of a power series in another power series

(4.1) First consider two *polynomials* of degree  $\leq N$ , with complex coefficients

$$f(z) = \sum_{n \leq N} a_n z^n, \quad g(z) = \sum_{n \leq N} b_n z^n.$$

The composed function  $f(g(z))$  is also a polynomial (of degree  $\leq N^2$ ). To calculate it we first calculate each of the products

$$(4.1.1) \quad (g(z))^m = \sum_{n_1 \leq N, \dots, n_m \leq N} b_{n_1} b_{n_2} \dots b_{n_m} z^{n_1 + n_2 + \dots + n_m}$$

where, for each  $m \leq N$ , the sum is taken over all the systems  $(n_j)_{1 \leq j \leq m}$  of integers  $n_j \leq N$ . We then have

$$(4.1.2) \quad f(g(z)) = \sum_{m \leq N} a_m (g(z))^m$$

and by substituting for each term  $(g(z))^m$  its expansion (4.1.1) we obtain the required polynomial

$$(4.1.3) \quad f(g(z)) = \sum_{p \leq N^2} c_p z^p$$

where the coefficients  $c_p$  are given by

$$(4.1.4) \quad c_p = \sum_{n_1 + n_2 + \dots + n_m = p} a_m b_{n_1} b_{n_2} \dots b_{n_m}$$

the sum being taken over *all* the values of  $m \leq N$ , and for each  $m$ , over *all* the systems  $(n_j)_{1 \leq j \leq m}$  such that  $n_1 + n_2 + \dots + n_m = p$ .

(4.2) It will be seen that with sufficiently restrictive hypotheses formulae (4.1.3) and (4.1.4) can be generalized to convergent *power series*. The prime difficulty is that we must now consider powers  $(g(z))^m$ , where  $m$  is an arbitrarily large exponent, and thus we must make sense (where possible) of sums such as occur in the second member of (4.1.4) when the number  $m$  and the numbers  $n_j$  can take *all* integer values  $\geq 0$ .

We begin by arranging in a fixed order the terms which we can thus obtain. For each integer  $N$  designate by  $A_N$  the set of all finite sequences of integers  $(n_j)_{1 \leq j \leq m}$  such that the *number of terms*  $m$  is  $\leq N$  and such that *each* term  $n_j \leq N$ . Clearly  $A_N \subset A_{N+1}$  for every  $N$ ; the sets  $A_N$  are finite and so also are the sets  $B_N = A_{N+1} - A_N$  (put  $A_0 = \emptyset$ ). We *enumerate* all the sequences  $(n_j)_{1 \leq j \leq m}$  by choosing in each finite set  $B_N$  an arbitrary order fixed once and for all; then first enumerate the sequences of  $B_1$  in the chosen order, *then* the sequences of  $B_2$  in the chosen order, *then* those of  $B_3$ , and so on. For example, for the first terms

$$B_1: (0), (1)$$

$$B_2: (2), (0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2), (2, 0), (2, 1), (2, 2)$$

$$B_3: (3), (0, 3), (1, 3), (2, 3), (3, 3), (3, 0), (3, 1), (3, 2), (0, 0, 0), (0, 0, 1), (0, 0, 2), (0, 0, 3), (1, 0, 0), \dots$$

If, for each integer  $m$  and each sequence of  $m$  integer terms  $(n_j)_{1 \leq j \leq m}$ , we choose a complex number  $\varphi(m, n_1, \dots, n_m)$ , to say that the sum

$$(4.2.1) \quad \sum_{(n_j)} \varphi(m, n_1, \dots, n_m)$$

exists means *by definition* that the series obtained by arranging the sequences  $(n_j)$  in the chosen order is *convergent*, and the sum of this series will be by definition the number (4.2.1).

If furthermore this series is *absolutely convergent*, we know (I, 2.4) that any series obtained by *changing the order* of these terms in an arbitrary way is also absolutely convergent and has the same sum, which shows that in this case the above choice of order has no importance.

(4.3) Consider now two *power series*

$$(4.3.1) \quad f(z) = a_0 + a_1 z + \dots + a_n z^n + \dots$$

$$(4.3.2) \quad g(z) = b_0 + b_1 z + \dots + b_n z^n + \dots$$

supposed respectively *convergent* in the open discs  $|z| < r$ ,  $|z| < r'$  ( $r > 0$ ,  $r' > 0$ ). If it is desired to substitute  $g(z)$  for  $z$  in (4.3.1), i.e. form the series  $f(g(z)) = \sum_{m=0}^{\infty} a_m (g(z))^m$ , clearly this will not make sense unless  $|g(z)| < r$ . In fact to obtain  $f(g(z))$  as the sum of a power series a more restrictive hypothesis is necessary. Consider the power series

$$(4.3.3) \quad G(z) = |b_0| + |b_1|z + \dots + |b_n|z^n + \dots$$

It follows from Abel's lemma (2.2) that the power series  $g(z)$  and  $G(z)$  have the *same disc of convergence*.

(4.4) Suppose there exists a number  $r''$ ,  $0 < r'' < r'$ , such that for  $|z| < r''$  we have  $G(|z|) < r$ . Then for  $|z| < r''$  we have  $|g(z)| < r$  and the number  $f(g(z))$  is the sum of the convergent series

$$(4.4.1) \quad f(g(z)) = c_0 + c_1 z + \dots + c_p z^p + \dots$$

where each  $c_p$  is the sum of the absolutely convergent series

$$(4.4.2) \quad c_p = \sum_{n_1 + n_2 + \dots + n_m = p} a_m b_{n_1} b_{n_2} \dots b_{n_m}$$

(the terms of the second member being arranged in the order described in (4.2)).

Thus, from a practical point of view, the powers  $(g(z))^m$  are treated as if they were polynomials, *collecting together* the terms of the same degree in  $z$  (observe that for a *fixed*  $m$  there are only a *finite* number of terms of a given degree  $p$ ). Then "term by term" the power series  $a_m (g(z))^m$  so obtained are added (this time there is an infinity of terms of a given degree.)

Prove first that the series (in the sense of (4.2))

$$(4.4.3) \quad h(z) = \sum_{(n_j)} a_m b_{n_1} b_{n_2} \dots b_{n_m} z^{n_1 + n_2 + \dots + n_m}$$

is *absolutely convergent* for  $|z| < r''$ . It is enough to show that there exists a fixed number  $C > 0$  such that for every integer  $N$

$$(4.4.4) \quad \sum_{(n_j) \in A_N} |a_m b_{n_1} b_{n_2} \dots b_{n_m} z^{n_1 + \dots + n_m}| \leq C.$$

Now, by definition of  $A_N$ , in the finite sum of the first member of (4.4.4),  $0 \leq m \leq N$  and  $0 \leq n_j \leq N$  for  $j \leq N$ ; therefore this sum is just

$$\sum_{m=0}^N |a_m| \left( \sum_{n=0}^N |b_n z^n| \right)^m \leq \sum_{m=0}^N |a_m| G(|z|)^m \leq \sum_{m=0}^{\infty} |a_m| G(|z|)^m = C$$

since the last series written is convergent by Abel's lemma. The relation  $|g(z)| < r$  for  $|z| < r'$  follows immediately from  $|g(z)| \leq G(|z|)$  for  $|z| < r'$  (I, 2.3.1). The series  $f(g(z)) = \sum_{m=0}^{\infty} a_m (g(z))^m$  is thus convergent for  $|z| < r''$ ; we show that its sum is equal to (4.4.3).

To do this we show that the difference of these two sums is *arbitrarily small*. Fix  $z$  such that  $|z| < r''$  and choose  $\varepsilon > 0$ . Since the series (4.4.3) is absolutely convergent, there exists an integer  $N_1$  such that for  $N \geq N_1$

$$(4.4.5) \quad \sum_{(n_j) \in A_N} |a_m b_{n_1} \dots b_{n_m} z^{n_1 + \dots + n_m}| - \sum_{(n_j) \in A_{N_1}} |a_m b_{n_1} \dots b_{n_m} z^{n_1 + \dots + n_m}| \leq \varepsilon.$$

For each  $N$ , put  $g_N(z) = \sum_{n=0}^N b_n z^n$ . From (4.4.5)

$$(4.4.6) \quad \left| \sum_{m=0}^N a_m (g_N(z))^m - \sum_{m=0}^{N_1} a_m (g_{N_1}(z))^m \right| \leq \varepsilon$$

since when the first member is developed, we obtain the absolute value of a sum of a finite number of terms, whose absolute values have for sum the first member of (4.4.5). The relation (4.4.6) can also be written

$$(4.4.7) \quad \left| \sum_{(n_j) \in A_N} a_m b_{n_1} \dots b_{n_m} z^{n_1 + \dots + n_m} - \sum_{m=0}^{N_1} a_m (g_{N_1}(z))^m \right| \leq \varepsilon$$

and thus letting  $N$  tend to  $+\infty$

$$(4.4.8) \quad \left| h(z) - \sum_{m=0}^{N_1} a_m (g_{N_1}(z))^m \right| \leq \varepsilon.$$

Now  $|g(z)| < r$ ; choose  $\delta$  satisfying  $0 < \delta < r - |g(z)|$ ; since the series of general term  $a_m u^m$  is uniformly convergent for  $|u| \leq |g(z)| + \delta < r$  (2.2), there exists an integer  $N_2 \geq N_1$  such that for  $N \geq N_2$  and  $|u| \leq |g(z)| + \delta$

$$(4.4.9) \quad \left| f(u) - \sum_{m=0}^N a_m u^m \right| \leq \varepsilon.$$

On the other hand, since the series of general term  $b_n z^n$  has the sum  $g(z)$ , there exists an integer  $N_3 \geq N_2$  such that for  $N \geq N_3$ ,  $|g(z) - g_N(z)| \leq \delta$  and hence by virtue of (4.4.9)

$$(4.4.10) \quad \left| f(g_N(z)) - \sum_{m=0}^N a_m (g_N(z))^m \right| \leq \varepsilon.$$

Finally, because of the continuity of  $f$  (3.1),  $\delta$  may be supposed taken so small that

$$(4.4.11) \quad |f(g(z)) - f(g_N(z))| \leq \varepsilon$$

for  $N \geq N_3$ . We then deduce from (4.4.6), (4.4.8), (4.4.10) and (4.4.11) that

$$|h(z) - f(g(z))| \leq 4\epsilon$$

which proves our assertion.

We show that each of the series (4.4.2) converges absolutely; since  $r'' > 0$ , there exists  $z \neq 0$  such that  $|z| < r''$ , and it is enough to show that the series obtained by multiplying all the terms of (4.4.2) by  $z^p$  is absolutely convergent; but the series thus obtained is a *partial* series of  $h(z)$ , hence our assertion (I, 2.5). Similarly, for each integer  $N$ , by virtue of (4.4.4)

$$(4.4.12) \quad |c_0| + |c_1 z| + \cdots + |c_N z^N| \leq C,$$

i.e. the series of general term  $c_p z^p$  is absolutely convergent. Lastly, for  $p \geq N^2$ , all the terms of the sum  $\sum_{(n_j) \in A_N} a_n b_{n_1} \dots b_{n_m} z^{n_1 + \dots + n_m}$  occur in the partial sum

$$c_0 + c_1 z + \cdots + c_p z^p$$

of the series (4.4.3). Therefore from (4.4.5) and (I, 2.5.3)

$$|h(z) - (c_0 + c_1 z + \cdots + c_p z^p)| \leq \epsilon \quad \text{for } p \geq N_1^2. \quad \text{Q.E.D.}$$

From theorem (4.4) the following corollary is deduced:

(4.5) *With the notations of (4.3) suppose that*

$$(4.5.1) \quad |b_0| = |g(0)| < r.$$

*Then there exists a number  $r''$ ,  $0 < r'' < r'$ , such that  $G(|z|) < r$  for  $|z| < r''$ ; hence the conclusions of (4.4) are valid for  $|z| < r''$ .*

Indeed,  $G(0) = |b_0|$  and the conclusion follows from the continuity of the function  $G$  at the point 0 (3.1).

Note, however, that although the relation (4.5.1) proves the existence of a number  $r''$  satisfying the hypothesis of (4.4), it does not on its own enable one to *minorize* this number.

## 5. Analytic functions

(5.1) Let  $D$  be an *open* set in the plane  $\mathbf{C}$ . A complex function  $f: D \rightarrow \mathbf{C}$  defined in  $D$  is said to be *analytic* (or *holomorphic*) in  $D$  if, for *each* point  $z_0 \in D$ , there exists an *open disc*  $\Delta: |z - z_0| < r$  contained in  $D$  such that in this disc

$$(5.1) \quad f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$$

where the right hand side is a powerseries in  $z - z_0$ , *convergent* in  $\Delta$  (Fig. 18). More briefly,  *$f$  can be developed in a power series in  $z - z_0$  in the neighbourhood of each point  $z_0 \in D$* . Taking into account (3.4), this power series (if it exists) is necessarily *unique*; we shall characterize the coefficients below (6.3). A complex function analytic in the whole of  $\mathbf{C}$  is called an *entire function*.



FIGURE 18

Note that it is not *a priori* evident that a power series in  $z$

$$(5.1.2) \quad f(z) = \sum_{n=0}^{\infty} a_n z^n$$

convergent in a disc  $|z| < r$ , is *analytic* in this disc, since the hypothesis made on  $f$  shows that the condition (5.1.1) is satisfied for  $z_0 = 0$ , but does not immediately prove that this condition is also satisfied at the *other* points  $z_0$  of the disc. However, this is the case, and more precisely:

(5.2) *If the power series (5.1.2) is convergent in the disc  $D: |z| < r$ , then for each  $z_0 \in D$ , there exists*

*one, and only one, power series convergent for*

$$|z - z_0| < r - |z_0|$$

*and satisfying (5.1.1) (Fig. 19).*

Apply the substitution theorem (4.4) with  $g(t) = z_0 + t$ ; then  $G(t) = |z_0| + t$  and the condition  $G(|t|) < r$  is satisfied for  $|t| < r - |z_0|$ . Since by the binomial theorem

$$(5.2.1) \quad (z_0 + t)^n = \sum_{p=0}^n \binom{n}{p} z_0^{n-p} t^p$$

the substitution theorem shows that each of the series

$$(5.2.2) \quad c_p = a_p + \binom{p+1}{1} a_{p+1} z_0 + \cdots \\ + \binom{p+m}{m} a_{p+m} z_0^m + \cdots$$

is absolutely convergent. For  $|t| < r - |z_0|$

$$(5.2.3) \quad f(z_0 + t) = \sum_{p=0}^{\infty} c_p t^p$$

the power series of the second member being convergent for  $|t| < r - |z_0|$  (it is actually possible for the radius of convergence of (5.2.3) to be *strictly greater* than  $r - |z_0|$  (VIII, 7.3)). More briefly, *a power series in  $z$  is an analytic function in its disc of convergence.*

Note also that it is not at all evident *a priori* that conversely if a function  $f$  is analytic in an open disc  $|z| < r$ , then there is a power series in  $z$  convergent *in this disc* and with sum equal to  $f$ . The definition says only that there is such a series convergent in a disc  $|z| < r'$  and equal to  $f$  in this disc, for a *certain*  $r' < r$ , but not that  $r' = r$ . It will be seen later (VIII, 7.3) that we *can* effectively take  $r' = r$ .

(5.3) *Let  $D, D'$  be two open sets in  $\mathbf{C}$ ,  $f$  a complex function analytic in  $D$ ,  $g$  a complex function*

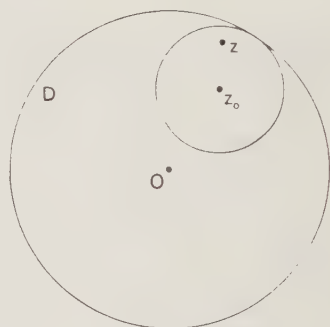


FIGURE 19

analytic in  $D'$  and suppose that  $g(D') \subset D$ . Then the composed function  $f \circ g$  is defined and analytic in  $D'$  (Fig. 20).†

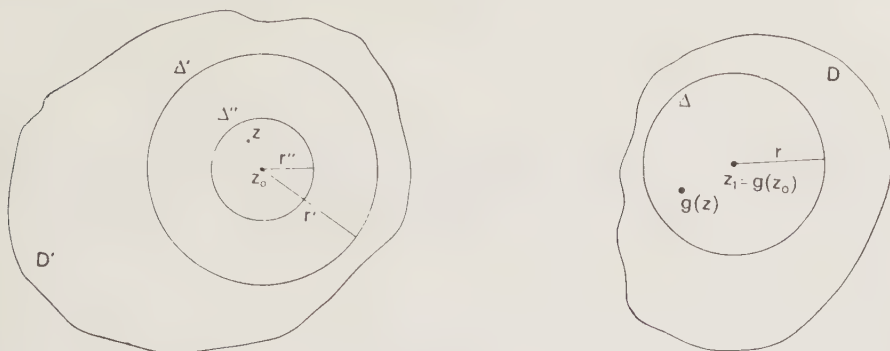


FIGURE 20

Let  $z_0 \in D'$ . By hypothesis there exists an open disc  $\Delta'$ :  $|z - z_0| < r'$  such that in  $\Delta'$

$$(5.3.1) \quad g(z) = \sum_{n=0}^{\infty} b_n(z - z_0)^n$$

the series converging in  $\Delta'$ . On the other hand the point  $z_1 = g(z_0)$  belongs to  $D$  by hypothesis, therefore there exists an open disc  $\Delta$ :  $|z - z_1| < r$  such that in  $\Delta$

$$(5.3.2) \quad f(z) = \sum_{n=0}^{\infty} a_n(z - z_1)^n$$

the series converging in  $\Delta$ . Since the power series

$$g(z) - g(z_0) = \sum_{n=1}^{\infty} b_n(z - z_0)^n$$

has its constant term zero, (4.5) can be applied and thus there exists a number  $r''$ ,  $0 < r'' < r'$ , such that in the disc  $\Delta''$ :  $|z - z_0| < r''$ ,  $g(z) \in \Delta$  and

$$f(g(z)) = \sum_{n=0}^{\infty} c_n(z - z_0)^n$$

the series converging in this disc.

Q.E.D.

(5.4) A vector function  $\mathbf{f}$ , defined in an open set  $D \subset \mathbf{C}$  and with values in  $\mathbf{C}^n$ , is said to be analytic in  $D$  if each of the components  $f_j$  ( $1 \leq j \leq n$ ) is analytic in  $D$ . It is evidently the same (I, 2.7) to say that for each  $z_0 \in D$ , there exists an open disc  $\Delta$ :  $|z - z_0| < r$  contained in  $D$  such that in this disc

$$(5.4.1) \quad \mathbf{f}(z) = \sum_{m=0}^{\infty} \mathbf{a}_m(z - z_0)^m,$$

† For clearer figures it is convenient to represent the values of  $z$  and those of  $g(z)$  in two distinct planes.

where the  $a_m$  are vectors of  $\mathbf{C}^n$  and the series of the second member is convergent in  $\Delta$ . It follows from this definition and Abel's lemma that for each  $r'$ ,  $0 < r' < r$ , the series of the second member of (5.4.1) is *normally convergent* (V, 2.7) in the disc  $|z - z_0| \leq r'$ .

## 6. Derivatives and primitives of an analytic function

(6.1) Let  $D$  be an open set in  $\mathbf{C}$ ,  $f$  a continuous complex function defined in  $D$ . We say that the function  $f$  is *differentiable with respect to the complex variable  $z$*  at a point  $z_0 \in D$  if, as  $u = s + it$  tends to 0 in  $\mathbf{C}$  while remaining  $\neq 0$  (i.e.  $(s, t)$  tends to  $(0, 0)$  in  $\mathbf{R}^2$  (0, 5.5) while remaining  $\neq (0, 0)$ ) the limit of the expression

$$(6.1.1) \quad \frac{f(z_0 + u) - f(z_0)}{u}$$

exists. (Observe that this expression is defined as soon as  $u$  is small enough, since by hypothesis there is a disc  $|z - z_0| < r$  with  $r > 0$ , which is contained in  $D$ .) The limit of (6.1.1) is called the *derivative of  $f$  at the point  $z_0$*  and denoted by  $f'(z_0)$  or  $Df(z_0)$ .

It should be observed that a function of  $z = s + it$  may possess *partial derivatives* of all orders *with respect to  $s$  and  $t$*  without possessing a derivative *with respect to  $z$* . The simplest example is the function  $f: z \rightarrow \bar{z} = s - it$ : at the point  $z_0 = 0$ , the expression (6.1.1) is  $\bar{u}/u$ ; if  $u = re^{i\theta}$ ,  $\bar{u}/u = e^{-2i\theta}$ . On each half line  $\theta = \theta_0$  this function is constant, and therefore tends to a limit when  $u$  tends to 0 *along the half line*; but the limit *depends on  $\theta_0$* , so there is *no limit in  $\mathbf{R}^2$*  at the point  $(0, 0)$ . We shall consider later conditions under which the derivative  $f'(z_0)$  exists (VIII, 9.4)

(6.2) A complex analytic function  $f$  in an open set  $D \subset \mathbf{C}$  possesses a derivative  $f'(z)$  at every point  $z \in D$ , and the function  $f'$  (also written  $df/dz$ ) is analytic in  $D$ .

By translation, we may confine ourselves to the case where  $D$  is a disc  $|z| < r$ , in which

$$(6.2.1) \quad f(z) = \sum_{n=0}^{\infty} a_n z^n$$

the power series being convergent in  $D$ . We have then seen (5.2) that for each  $z_0 \in D$  and every  $u \neq 0$  such that  $|u| < r - |z_0|$

$$(6.2.2) \quad \frac{f(z_0 + u) - f(z_0)}{u} = c_1 + c_2 u + \dots + c_p u^{p-1} + \dots$$

where the series of the second member is convergent and the  $c_p$  are given by the convergent series (5.2.2). Because of the continuity of a power series in its disc of convergence (3.1), it is therefore seen that the limit of (6.1.1) exists and is given by the formula

$$(6.2.3) \quad f'(z_0) = a_1 + 2a_2 z_0 + \dots + n a_n z_0^{n-1} + \dots$$

i.e. is simply obtained by *differentiating term by term* the series (6.2.1).

An immediate consequence of this result is:

(6.3) A function  $f$  analytic in an open set  $D \subset \mathbf{C}$  is indefinitely differentiable in  $\mathbf{C}$  and all its

derivatives are analytic in  $D$ ; furthermore, for each  $z_0 \in D$ , there exists a disc  $|z - z_0| < \rho$  in which the function is equal to its Taylor series

$$(6.3.1) \quad f(z) = f(z_0) + \frac{1}{1!}f'(z_0)(z - z_0) + \cdots + \frac{1}{p!}f^{(p)}(z_0)(z - z_0)^p + \cdots$$

which converges in this disc.

The last assertion follows from the expression (5.2.2) for the coefficients of the power series (5.2.3) and from the formula (6.2.3) applied by induction. We also write  $d^p f(z)/dz^p$  instead of  $f^{(p)}(z)$  and agree that  $f^{(0)} = f$ . Later (VII, 9.1) it will be shown that conversely, a complex function  $f$  possessing a continuous derivative in an open set  $D \subset \mathbf{C}$  is necessarily analytic in  $D$ .

(6.4) Given an analytic function  $f$  in an open set  $D \subset \mathbf{C}$ , we say that a function  $F$  analytic in  $D$  is a *primitive* of  $f$  if  $F'(z) = f(z)$  for every  $z \in D$ . Contrary to what one might expect, because of the preceding results, an analytic function in an open set  $D \subset \mathbf{C}$  does not necessarily admit a primitive in  $D$  (VII, 3.2). This problem is studied in more detail in Chap. VII and VIII; here we merely prove the existence of a primitive in a special case:

(6.5) *If the power series*

$$(6.5.1) \quad f(z) = a_0 + a_1(z - z_0) + \cdots + a_n(z - z_0)^n + \cdots$$

*is convergent in the disc  $|z - z_0| < r$ , then the series*

$$(6.5.2) \quad F(z) = a_0(z - z_0) + \frac{a_1}{2}(z - z_0)^2 + \cdots + \frac{a_n}{n+1}(z - z_0)^{n+1} + \cdots$$

*is convergent in this disc and its sum is a primitive of  $f$ .*

By virtue of (6.2.3) it need only be shown that the series (6.5.2) converges for  $|z - z_0| < r$ . For each  $\rho$ ,  $0 < \rho < r$ , the sequence of complex numbers  $(a_n \rho^n)$  is bounded; the same is true *a fortiori* of the sequence  $\left(\frac{1}{n+1} a_n \rho^{n+1}\right)$ , and the conclusion follows from Abel's lemma (2.2).

*Remark (6.6)* The formal calculus of derivatives is valid without modifications for analytic functions: for two analytic functions  $f, g$ ,

$$(6.6.1) \quad \begin{aligned} (f + g)'(z) &= f'(z) + g'(z), & (fg)'(z) &= f'(z)g(z) + f(z)g'(z), \\ (f^n)'(z) &= n(f(z))^{n-1}f'(z) & (n \in \mathbf{Z}) \end{aligned}$$

$$(6.6.2) \quad (f \circ g)'(z) = f'(g(z))g'(z)$$

whenever these formulae have a meaning. Let us prove, for example (6.6.2): for each  $\varepsilon \in ]0, 1[$ , there exists  $r > 0$  such that for  $|u| \leq r$ ,  $|v| \leq r$

$$(6.6.3) \quad |f(g(z) + u) - f(g(z)) - f'(g(z))u| \leq \varepsilon|u|$$

$$(6.6.4) \quad |g(z + v) - g(z) - g'(z)v| \leq \varepsilon|v|.$$

Since  $g$  is continuous at the point  $z$ , there exists  $r' < r$  such that for  $|v| \leq r'$  we have  $|g(z+v) - g(z)| \leq r$ . Replacing  $u$  by  $g(z+v) - g(z)$  in (6.6.3), and taking into account (6.6.4)

$$|f(g(z+v)) - f(g(z)) - f'(g(z))g'(z)v| \leq A\varepsilon|v|$$

where  $A = |g'(z)| + |f'(g(z))| + 1$  is independent of  $\varepsilon$ , hence our assertion. The same reasoning shows that if  $f$  is analytic in an open set  $D \subset \mathbf{C}$  and  $\gamma$  is a complex function of the real variable  $t$ , continuous and differentiable in an interval  $I \subset \mathbf{R}$  and such that  $\gamma(I) \subset D$ , then the composed function  $t \rightarrow f(\gamma(t))$  of the real variable  $t$  is differentiable in  $I$  and

$$(6.6.5) \quad (f \circ \gamma)'(t) = f'(\gamma(t))\gamma'(t).$$

Suppose that  $f$  is analytic in an open set  $D \subset \mathbf{C}$  containing a segment with end-points  $a, b$  (set of points of the form  $a + t(b-a)$  for  $0 \leq t \leq 1$ ). The function  $t \rightarrow f(a + t(b-a))$  has then for each  $n$  an  $n^{\text{th}}$  derivative equal to

$$(b-a)^n f^{(n)}(a + t(b-a))$$

in the interval  $[0, 1]$ . If  $|f^{(n)}(z)| \leq M$  in  $D$ , the *Taylor formula*

$$(6.6.6) \quad f(b) = f(a) + \frac{b-a}{1!}f'(a) + \cdots + \frac{(b-a)^{n-1}}{(n-1)!}f^{(n-1)}(a) + R_n$$

holds with the majorization

$$(6.6.7) \quad |R_n| \leq M \frac{(b-a)^n}{n!}$$

by simple application of the method of (I, 3.6). The formula (I, 3.6.2) is extended in the same way.

(6.7) For functions continuous for  $0 < |z - z_0| < r$ , it is convenient to generalize the notations of Landau (III, 3.2). If  $g$  is continuous for  $0 < |z - z_0| < r$ , we agree that  $O(g)$  (resp.  $o(g)$ ) designates a function  $f$  defined for  $0 < |z - z_0| < r$ , such that there exists  $M > 0$  and a neighbourhood  $|z - z_0| \leq c$  of  $z_0$  in which  $|f(z)| \leq M|g(z)|$  (resp. such that for each  $\varepsilon > 0$ , there exists  $\delta > 0$  such that the relation  $0 < |z - z_0| \leq \delta$  implies  $|f(z)| \leq \varepsilon|g(z)|$ ). The Taylor formula (6.6.6) for an analytic function  $f$  at the point  $z_0$  is therefore written in the neighbourhood of  $z_0$

$$(6.7.1) \quad f(z) = f(z_0) + \frac{f'(z_0)}{1!}(z - z_0) + \cdots + \frac{f^{(n-1)}(z_0)}{(n-1)!}(z - z_0)^{n-1} + O((z - z_0)^n).$$

(6.8) All the results of this section can be extended immediately to analytic functions with values in  $\mathbf{C}^n$  (5.4); this extension is left to the reader.

## 7. Principle of analytic continuation

(7.1) An indefinitely differentiable function  $f$  of a real variable, defined in an open interval  $I$ , can be modified in an arbitrarily small interval  $J \subset I$  keeping the same values in  $I - J$ ,

without ceasing to be *indefinitely differentiable*. Indeed, we have seen (V, 4.8.4) that there are indefinitely differentiable functions  $h$  zero in  $I - J$  and  $\neq 0$  in  $J$  and the indefinitely differentiable function  $f + \alpha h$  is therefore equal to  $f$  in  $I - J$  and can take *arbitrarily large* values (for  $\alpha$  sufficiently large) in  $J$  (Fig. 21).

(7.2) It will be seen on the contrary, that the values of an *analytic* function at various points of an open set  $D$  in which the function is defined, are in certain ways dependent on one another; we cannot modify the function in the neighbourhood of a point (if it is to remain analytic in  $D$ ) without it being *also* modified at points *very far* from the considered point. This is already shown by the principle of isolated zeros (3.4), which is a particular case of what is called the principle of “analytic continuation”.

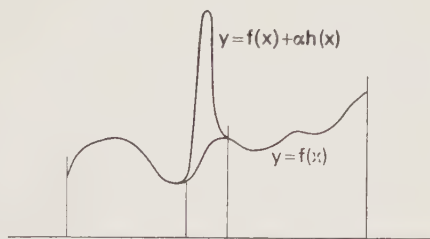


FIGURE 21

To enunciate this principle, the open set  $D$  under consideration must be restricted. If, for example,  $D$  is the union of two *disjoint* open discs  $D'$ ,  $D''$ , then a function equal to a complex constant  $a'$  in  $D'$  and to *another* complex constant  $a''$  in  $D''$  is evidently analytic, and it is clear that two such functions can be equal in  $D'$  and different in  $D''$ . The restriction which should be imposed on  $D$  is that this open set be *connected* (0, 5.8):

(7.3) (Principle of Analytic Continuation) *Let  $f, g$  be two functions analytic in an open connected set  $D \subset \mathbf{C}$ . If there exists a non-empty open set (arbitrarily small)  $U \subset D$  such that  $f|_U = g|_U$ , then  $f = g$ .*

Let  $a$  be a point of  $U$ ,  $b$  any point of  $D$ . By hypothesis, there is a *polygonal line*  $L: t \rightarrow \lambda(t)$ , contained in  $D$ , the image of  $[0, 1]$  under a piecewise affine linear function

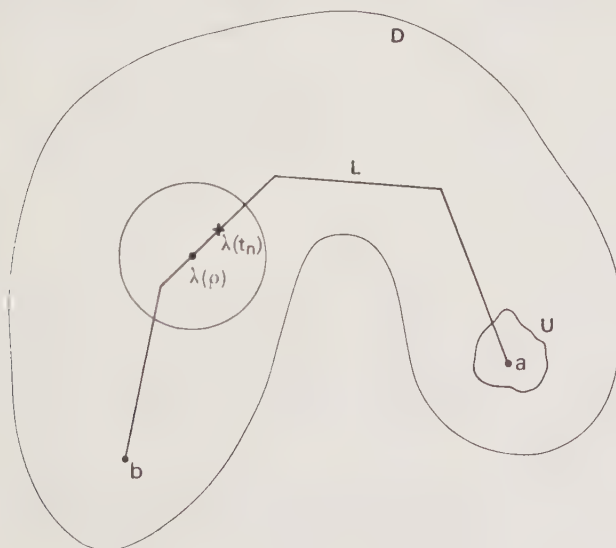


FIGURE 22

$\lambda$  such that  $\lambda(0) = a$  and  $\lambda(1) = b$  (Fig. 22). Consider the set  $A$  of those values of  $t$  such that for  $0 \leq s \leq t$ , we have  $f(\lambda(s)) = g(\lambda(s))$ . Since  $\lambda$  is continuous and  $a \in U$ , there is by hypothesis an *interval*  $0 \leq t \leq \alpha$  with  $\alpha > 0$ , which is *contained in*  $A$ . We have to show that  $1 \in A$ ; to do this, consider the *least upper bound*  $\rho \geq \alpha > 0$  of  $A$  in  $[0, 1]$ . Note first that  $\rho \in A$ : indeed, there is an increasing sequence  $(t_n)$  of points of  $A$  tending to  $\rho$ , and since  $f(\lambda(t_n)) = g(\lambda(t_n))$  for every  $n$ , we have also by continuity,  $f(\lambda(\rho)) = g(\lambda(\rho))$ . We must therefore show that  $\rho = 1$ . Suppose on the contrary that  $\rho < 1$ . Put  $z_0 = \lambda(\rho) \in D$ . By hypothesis, there exists an open disc  $\Delta: |z - z_0| < r$  contained in  $D$  and in which  $f$  and  $g$  are equal to convergent power series in  $z - z_0$ . Now, since  $\rho > 0$ , there is an *increasing* sequence  $(t_n)$  of *distinct* real numbers tending to  $\rho$  such that the  $\lambda(t_n)$  are distinct and tend to  $z_0$ . Thus  $\lambda(t_n) \in \Delta$  for large enough values of  $n$ ; since, by hypothesis,  $f(\lambda(t_n)) = g(\lambda(t_n))$  for all  $n$ , it follows from the principle of isolated zeros (3.4) that the restrictions of  $f$  and  $g$  to  $\Delta$  are *identical*. But there is an interval  $[\rho, \rho + h]$  with  $h > 0$  such that  $\rho + h < 1$  and such that  $\lambda(t) \in \Delta$  for  $\rho \leq t \leq \rho + h$ , by continuity of  $\lambda$ . Thus  $f(\lambda(t)) = g(\lambda(t))$  for every  $t$  such that  $0 \leq t \leq \rho + h$ , so  $\rho + h \in A$ , which is contrary to the definition of the *least upper bound*  $\rho$  (0, 2.2); the theorem is thus proved.

Note that it is clear that (7.3) applies to vector analytic functions.

(7.4) The conclusion of (7.3) is still valid when instead of supposing that the restrictions of  $f$  and  $g$  to an *open* non-empty set  $U \subset D$  are equal, one supposes only that there is a point  $a \in D$  and a *sequence of distinct points*  $z_n$  of  $D$  with limit  $a$ , such that  $f(z_n) = g(z_n)$  for every  $n$ . Indeed, since  $f$  and  $g$  are equal to convergent power series in an open disc  $U: |z - a| < \alpha$  contained in  $D$ , the principle of isolated zeros (3.4) applied to  $U$  shows that  $f(z) = g(z)$  in  $U$ , and we return to (7.3). This remark will apply, for example, when  $f(z) = g(z)$  at all the points of a *line segment* not reducing to a point (but as small as we please) and contained in  $D$ .

*Remark (7.5)* Suppose that  $f$  is a function analytic in an open *connected* set  $D$  and not identically zero in  $D$ . Then for every *closed and bounded* subset  $K \subset D$ , there is only a *finite* number of roots of the equation  $f(z) = 0$  in  $K$ . Otherwise, by a theorem which will be admitted, an infinite sequence  $(z_n)$  of distinct roots of  $f(z) = 0$  can be formed possessing a limit  $a \in K \subset D$ . From (7.4) applied to  $f$  and to the constant function 0, this would give  $f(z) = 0$  for every  $z \in D$ , contrary to the hypothesis.

## 8. Examples of analytic functions

(8.1) A *polynomial*  $P(z) = a_0 + a_1z + \cdots + a_nz^n$  is evidently a power series which converges for *every*  $z \in C$ , and is therefore an *entire function* (5.1). It follows immediately from (5.3) that for each function  $f$  analytic in an open set  $D \subset C$ , every power  $f^n$  ( $n$  an integer  $\geq 0$ ) is analytic in  $D$ ; since the sum  $f + g$  of two functions analytic in  $D$  is clearly also analytic in  $D$ , the *product*  $fg$  is analytic in  $D$ , since  $fg = \frac{1}{4}((f + g)^2 - (f - g)^2)$ .

It is seen at once from (4.4) that given the developments of  $f$  and  $g$  in power series in  $z - z_0$  in the neighbourhood of a point  $z_0 \in D$ , the development in a power series of  $fg$  can be obtained by “multiplying term by term” the two series and collecting together the terms of the same degree in  $z - z_0$ .

(8.2) The function  $1/z$  is analytic in  $\mathbf{C} - \{0\}$ . It suffices to show that, for all  $z_0 \neq 0$ , the power series in  $z - z_0$

$$(8.2.1) \quad \frac{1}{z_0} - \frac{(z - z_0)}{z_0^2} + \frac{(z - z_0)^2}{z_0^3} + \cdots + (-1)^n \frac{(z - z_0)^n}{z_0^{n+1}} + \cdots$$

is convergent for  $|z - z_0| < |z_0|$  and has the sum  $1/z$ . Indeed, for each complex number  $u$  and every integer  $n \geq 1$

$$\frac{1}{1+u} = 1 - u + u^2 - \cdots + (-1)^n u^n + \frac{1 - (-1)^{n+1} u^{n+1}}{1+u}.$$

When  $|u| < 1$ , we have  $\lim_{n \rightarrow \infty} u^{n+1} = 0$  and hence

$$\frac{1}{1+u} = 1 - u + u^2 - \cdots + (-1)^n u^n + \cdots$$

the series converging for  $|u| < 1$ ; replacing  $u$  by  $(z - z_0)/z_0$ , (8.2.1) converges and has the sum  $1/z$  for  $|z - z_0| < |z_0|$ . We conclude from this result and from (5.3) that if  $f$  is analytic in an open set  $D$ , then  $1/f$  is analytic in the open set  $D' \subset D$  of those points  $z$  where  $f(z) \neq 0$ . If  $D$  is connected and if  $f$  is not identically zero in  $D$ , the set  $D - D'$  consists of *isolated* points in  $D$ , by virtue of the principle of analytic continuation (7.4).

From this together with (8.1) it is seen that if  $f$  and  $g$  are analytic in  $D$ , then  $f/g$  is analytic in the open set  $D' \subset D$  of those points where  $g(z) \neq 0$ . In particular, a *rational fraction*  $P(z)/Q(z)$ , where  $P$  and  $Q$  are polynomials ( $Q$  not identically zero) is an analytic function in the complement of the *finite* set of roots of  $Q(z) = 0$ .

(8.3) We suppose known the definitions of  $e^x$ ,  $\cos x$ ,  $\sin x$  for  $x$  *real*, the expressions for the derivatives of these functions and the fundamental relation  $e^{x+x'} = e^x e^{x'}$  for  $x, x'$  *real*.

Taylor's formula gives for every real  $x$  and every integer  $n > 0$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + R_n(x)$$

with  $R_n(x) = x^{n+1}e^{\theta x}/(n+1)!$  for a  $\theta$  (depending on  $x$ ) such that  $0 \leq \theta \leq 1$ . Since  $|e^{\theta x}| \leq e^{|x|}$ , it follows immediately from Stirling's formula (IV, 3.9.2) that  $R_n(x)$  tends to 0 (for  $x$  fixed) as  $n$  tends to  $+\infty$ , i.e. for every *real*  $x$

$$(8.3.1) \quad e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots$$

the power series converging. It follows from Abel's lemma (2.2) that the series with general term  $z^n/n!$  converges for *every complex*  $z$ . We therefore *define* an entire function (5.1) by the formula

$$(8.3.2) \quad e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \cdots + \frac{z^n}{n!} + \cdots$$

for every *complex*  $z$  and call  $e^z$  the *exponential function*; it is also denoted by  $\exp(z)$ . The

justification of this notation and terminology comes on the one hand from the formula (8.3.1) for  $x$  real, and on the other from the *fundamental* property of the function  $e^z$ :

$$(8.3.3) \quad e^{z+z'} = e^z e^{z'}$$

for any two *complex* numbers  $z$  and  $z'$ . To prove this formula without calculation, proceed in two steps. First, if  $x$  is *real*,  $e^{z+x} = e^z e^x$  for every *complex*  $z$ ; indeed, the two members are entire functions of  $z$  (5.3) which coincide for every *real*  $z$ ; they are therefore identical by virtue of the principle of analytic continuation (7.4). Considering then the functions  $z \rightarrow e^{z+z'}$  and  $z \rightarrow e^z e^{z'}$  for *any* fixed complex  $z'$ , it follows from the preceding and from (5.3) that they are entire functions of  $z$  which coincide for every *real*  $z$ , hence again by (7.4) the formula (8.3.3) for any complex  $z, z'$ .

Writing  $z' = -z$  in (8.3.3)

$$(8.3.4) \quad e^{-z} = 1/e^z$$

for every  $z \in \mathbf{C}$ . On the other hand, reasoning by induction on the integer  $n > 0$ , it is deduced from (8.3.3) that  $e^{nz} = (e^z)^n$ , and taking into account (8.3.4)

$$(8.3.5) \quad (e^z)^n = e^{nz}$$

for every *integer*  $n$  (positive or negative). Note, however, that the formula  $(e^z)^{z'} = e^{zz'}$  (which is true for  $z$  and  $z'$  *real*) cannot be written at the moment, because up to now no meaning has been given to the expression  $a^z$  when  $a$  and  $z$  are *both* complex (cf. (VIII, 9.6)).

(8.4) The same reasoning as in (8.3) gives for  $\cos x$  and  $\sin x$  the developments in convergent power series

$$\begin{aligned} \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots \\ \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots \end{aligned}$$

for every *real*  $x$ . On the other hand, substituting  $ix$  for  $z$  in (8.3.1), yields for every *real*  $x$  the Euler formula

$$(8.4.1) \quad e^{ix} = \cos x + i \sin x$$

which justifies the notation  $e^{ix}$  already adopted by convention for the second member. Replacing  $x$  by  $-x$  in (8.4.1), the formulae

$$(8.4.2) \quad \cos x = \frac{1}{2}(e^{ix} + e^{-ix}), \quad \sin x = \frac{1}{2i}(e^{ix} - e^{-ix})$$

are immediately obtained for  $x$  *real*.

(8.5) Consider now any complex number  $z = x + iy$ ,  $x$  and  $y$  being the real and imaginary parts of  $z$ ,  $x = \Re z$ ,  $y = \Im z$ . It is deduced from (8.3.2) and (8.4.1) that

$$(8.5.1) \quad e^z = e^x(\cos y + i \sin y);$$

in other words

$$(8.5.2) \quad \mathcal{R}(e^z) = e^{\mathcal{R}z} \cos(\mathcal{I}z), \quad \mathcal{I}(e^z) = e^{\mathcal{R}z} \sin(\mathcal{I}z).$$

Hence

$$(8.5.3) \quad |e^z| = e^x = e^{\mathcal{R}z} \neq 0 \quad \text{for all } z \in \mathbf{C},$$

$$(8.5.4) \quad \overline{e^z} = \overline{e^x} = e^x(\cos y - i \sin y)$$

and in particular

$$(8.5.5) \quad |e^{iy}| = 1$$

$$(8.5.6) \quad \overline{e^{iy}} = e^{-iy} = 1/e^{iy}$$

for every *real*  $y$ . For  $z = x + iy$ , the *amplitude* (or *argument*) of the complex number  $e^z$  is an angle of which the real number  $y$  is a *measure* in radians (Fig. 23). In particular

$$(8.5.7) \quad e^{i\pi/2} = i, \quad e^{i\pi} = -1, \quad e^{3i\pi/2} = -i, \quad e^{2i\pi} = 1$$

hence, by (8.3.3)

$$(8.5.8) \quad e^{z+i\pi} = -e^z, \quad e^{z+2i\pi} = e^z.$$

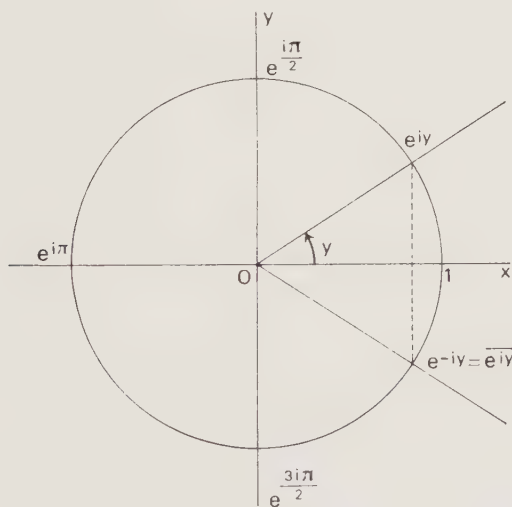


FIGURE 23

The exponential function is thus *periodic* with imaginary period  $2i\pi$ . We shall study later (VIII, 9.2) the equation  $e^z = a$ , for any complex  $a$ . Here we confine ourselves to determining the roots of  $e^z = 1$ ; if  $z = x + iy$  we deduce, by virtue of (8.5.1) and (8.5.3), that  $e^x = 1$ , so  $x = 0$ , and then that  $\cos y = 1$ ,  $\sin y = 0$  which gives  $z = 2ni\pi$  ( $n$  positive or negative integer).

(8.6) The function  $z \rightarrow e^{iz}$  of the *complex* variable  $z$  is clearly an entire function equal in the whole of  $\mathbf{C}$  to the convergent power series

$$(8.6.1) \quad e^{iz} = 1 + i \frac{z}{1!} - \frac{z^2}{2!} - i \frac{z^3}{3!} + \cdots + \frac{i^n z^n}{n!} + \cdots$$

We can thus *continue* the cosine and sine functions to the whole of  $\mathbf{C}$  by putting, for every *complex*  $z$ , by *definition*

$$(8.6.2) \quad \cos z = \frac{1}{2}(e^{iz} + e^{-iz}), \quad \sin z = \frac{1}{2i}(e^{iz} - e^{-iz})$$

which clearly gives

$$(8.6.3) \quad e^{iz} = \cos z + i \sin z.$$

However one should take care not to think that  $\cos z$  and  $\sin z$  are the real and imaginary parts of  $e^{iz}$  for  $z$  complex! It is easy in fact to decompose into their real and imaginary parts the trigonometric functions thus continued. Indeed, for  $z = x + iy$ ,  $x, y$  real

$$\cos z = \frac{1}{2}(e^{ix-y} + e^{-ix+y}) = \frac{1}{2}e^{-y}(\cos x + i \sin x) + \frac{1}{2}e^y(\cos x - i \sin x)$$

which can also be written

$$(8.6.4) \quad \cos z = \cos x \cosh y - i \sin x \sinh y$$

and similarly

$$(8.6.5) \quad \sin z = \sin x \cosh y + i \cos x \sinh y$$

hence the absolute values

$$(8.6.6) \quad \begin{cases} |\cos z| = \sqrt{\cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y} \\ |\sin z| = \sqrt{\sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y}. \end{cases}$$

In particular, for  $x = 0$

$$(8.6.7) \quad \cos(iy) = \cosh y, \quad \sin(iy) = i \sinh y \quad (y \text{ real})$$

which shows that on the imaginary axis the cosine and sine functions *increase exponentially* in absolute value. Note finally that the functions  $\cos z$  and  $\sin z$  are entire functions equal in the whole of  $\mathbf{C}$  to the convergent series

$$(8.6.8) \quad \begin{aligned} \cos z &= 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots + (-1)^n \frac{z^{2n}}{(2n)!} + \cdots \\ \sin z &= z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots + (-1)^n \frac{z^{2n+1}}{(2n+1)!} + \cdots \end{aligned}$$

and satisfy the classical relations

$$(8.6.9) \quad \cos^2 z + \sin^2 z = 1$$

$$(8.6.10) \quad \begin{cases} \cos \left( z + \frac{\pi}{2} \right) = -\sin z, & \sin \left( z + \frac{\pi}{2} \right) = \cos z \\ \cos (z + \pi) = -\cos z, & \sin (z + \pi) = -\sin z \\ \cos (z + 2\pi) = \cos z, & \sin (z + 2\pi) = \sin z \end{cases}$$

$$(8.6.11) \quad \cos (-z) = \cos z, \quad \sin (-z) = -\sin z$$

$$(8.6.12) \quad \begin{cases} \cos (z + z') = \cos z \cos z' - \sin z \sin z' \\ \sin (z + z') = \sin z \cos z' + \cos z \sin z' \end{cases}$$

as we see directly from the definitions or by analytic continuation as in (8.3).

The roots of the equation  $\sin z = 0$  are the numbers satisfying the relation  $e^{iz} = e^{-iz}$ , i.e.  $e^{2iz} = 1$ ; they are therefore the real numbers

$$z = n\pi \quad (n \in \mathbf{Z}).$$

From this result and from (8.6.10) we deduce that the roots of the equation  $\cos z = 0$  are the numbers  $(n + \frac{1}{2})\pi$  ( $n$  integer positive or negative). We write

$$(8.6.13) \quad \begin{aligned} \tan z &= \frac{\sin z}{\cos z} \quad \text{for } z \neq (n + \tfrac{1}{2})\pi \quad (n \in \mathbf{Z}) \\ \cot z &= \frac{\cos z}{\sin z} \quad \text{for } z \neq n\pi \quad (n \in \mathbf{Z}) \end{aligned}$$

functions which thus continue the known functions for  $x$  real to the non-real values of  $z$ ; each of these functions is analytic in the open set where it is defined. From (8.6.2), (8.6.10) and (8.6.11)

$$(8.6.14) \quad \tan z = \frac{1}{i} \frac{e^{2iz} - 1}{e^{2iz} + 1}, \quad \cot z = i \frac{e^{2iz} + 1}{e^{2iz} - 1}$$

$$(8.6.15) \quad \tan \left( z + \frac{\pi}{2} \right) = -\cot z, \quad \tan (z + \pi) = \tan z$$

$$(8.6.16) \quad \tan (-z) = -\tan z.$$

(8.7) Let  $A$  be a square matrix of order  $n$  with complex elements (which can therefore be considered as a vector of  $\mathbf{C}^{n^2}$ ). The power series

$$(8.7.1) \quad \sum_{k=0}^{\infty} \frac{A^k z^k}{k!}$$

is *absolutely convergent* for every  $z \in \mathbf{C}$ : indeed, (I, 1.6.5)

$$\|A^k z^k\| = |z^k| \|A^k\| \leq (n|z|)^k \|A\|^k$$

and the assertion follows, because of the convergence of the exponential series. Thus if  $B$  is a second square matrix of order  $n$ ,

$$(8.7.2) \quad \left( \sum_{k=0}^{\infty} \frac{A^k z^k}{k!} \right) \left( \sum_{k=0}^{\infty} \frac{B^k z^k}{k!} \right) = \sum_{j,k} \frac{A^j B^k z^{j+k}}{j!k!}$$

the series of the second member (with any ordering of the terms) being absolutely convergent.

Suppose now that  $A$  and  $B$  commute; we have then seen in Algebra that if  $P(u, v)$ ,  $Q(u, v)$  are two polynomials with complex coefficients and  $R(u, v) = P(u, v)Q(u, v)$  is their product

$$P(A, B)Q(A, B) = R(A, B).$$

Now, it follows from the identity  $e^{uz+vs} = e^{uz} e^{vs}$  that

$$\sum_{j+k=m} \frac{u^j v^k}{j!k!} = \frac{(u+v)^m}{m!}.$$

and we deduce from (8.7.2) that for two matrices  $A, B$  which commute

$$(8.7.3) \quad \left( \sum_{k=0}^{\infty} \frac{A^k z^k}{k!} \right) \left( \sum_{k=0}^{\infty} \frac{B^k z^k}{k!} \right) = \sum_{k=0}^{\infty} \frac{(A+B)^k z^k}{k!}.$$

For this reason, we put

$$(8.7.4) \quad e^{Az} = \sum_{k=0}^{\infty} \frac{A^k z^k}{k!}$$

which is also denoted by  $\exp(Az)$ , and which is an *entire* function with values in the space of complex matrices of order  $n$ ; the formula (8.7.3) is thus written

$$(8.7.5) \quad e^{(A+B)z} = e^{Az} e^{Bz}$$

for two matrices  $A, B$  which commute (cf. problem 10). In particular, for any complex numbers  $z, z'$

$$(8.7.6) \quad e^{A(z+z')} = e^{Az} e^{Az'} = e^{Az'} e^{Az}, \quad e^{Az} e^{-Az} = e^{-Az} e^{Az} = I$$

since clearly  $eO = I$  ( $O$  being the zero matrix,  $I$  the identity matrix). This shows that every matrix  $e^{Az}$  is *invertible* (cf. VIII, 9.9). On the other hand, it follows from (8.7.4) that  $A$  commutes with  $e^{Az}$ .

For  $A = I$  we have  $A^k = I$  for every integer  $k > 1$ , hence

$$(8.7.7) \quad e^{Iz} = I e^z.$$

If  $P$  is an *invertible* square matrix, then  $(PAP^{-1})^k = PA^kP^{-1}$  for every integer  $k > 0$ , hence

$$(8.7.8) \quad \exp(PAP^{-1}z) = P \exp(Az)P^{-1}.$$

If  $n = r + s$  and if the matrix  $A$  has the form

$$(8.7.9) \quad A = \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix}$$

where  $B$  is a square matrix of order  $r$  and  $C$  a square matrix of order  $s$ , then for every integer  $k > 0$

$$A^k = \begin{pmatrix} B^k & 0 \\ 0 & C^k \end{pmatrix}$$

and hence

$$(8.7.10) \quad e^{Az} = \begin{pmatrix} e^{Bz} & 0 \\ 0 & e^{Cz} \end{pmatrix}.$$

Note finally that, since the series of the second member of (8.7.4) is the Taylor series at the point 0 of  $e^{Az}$ , it follows from (8.7.6) that the Taylor series of this function at the point  $z_0$  is

$$(8.7.11) \quad e^{Az} = \sum_{k=0}^{\infty} e^{Az_0} A^k \frac{(z - z_0)^k}{k!}$$

and therefore

$$(8.7.12) \quad \frac{d^k}{dz^k} (e^{Az}) = A^k e^{Az} = e^{Az} A^k.$$

## 9. Maximum principle

An indefinitely differentiable real function defined in an interval of  $\mathbf{R}$  can attain a *relative maximum* at *any* point of this interval. *Analytic* functions of a *complex* variable behave very differently:

(9.1) (The maximum Principle) *Let D be an open connected set in  $\mathbf{C}$ , and let  $f$  be a complex function analytic in D. If at a point  $z_0 \in D$  the function  $z \rightarrow |f(z)|$  attains a relative maximum (i.e. if there exists an open disc  $|z - z_0| < r$  contained in D such that  $|f(z)| \leq |f(z_0)|$  in this disc), then  $f$  is constant in D.*

The reason for this behaviour, which at first sight seems very surprising, is easily seen in the particular case of a polynomial  $f(z) = c_0 + c_k(z - z_0)^k$  with two terms  $c_0 \neq 0, c_k \neq 0$  ( $k$  integer  $\geq 1$ ). If we put  $z - z_0 = \rho e^{i\theta}$  ( $\rho > 0, 0 \leq \theta \leq 2\pi$ ), then

$$f(z) = c_0 + c_k \rho^k e^{ik\theta} = c_0 \left( 1 + \left| \frac{c_k}{c_0} \right| \rho^k e^{i(k\theta + \alpha)} \right)$$

where  $\alpha$  is the amplitude of  $c_k/c_0$ . As  $\theta$  varies between 0 and  $2\pi$ ,  $k\theta$  varies between 0 and  $2k\pi$ ; in particular, we can find  $\theta$  such that  $k\theta + \alpha$  has the form  $2h\pi$  ( $h$  an integer) and hence

$$1 + \left| \frac{c_k}{c_0} \right| \rho^k e^{i(k\theta + \alpha)} = 1 + \left| \frac{c_k}{c_0} \right| \rho^k$$

is *real* and  $> 1$ . For the corresponding value of  $z$ , we therefore have

$$|f(z)| > |c_0| = |f(z_0)|,$$

and this holds *however small*  $\rho = |z - z_0|$ . Thinking in geometric terms, as  $z$  “rotates” about  $z_0$ ,  $f(z)$  “rotates” about  $f(z_0)$  and its absolute value therefore attains values *strictly greater* than  $|f(z_0)|$ .

The proof of (9.1) is a “formalization” of this idea. By virtue of the principle of analytic continuation (7.3), it suffices to show that if  $|f(z)|$  attains a relative maximum

at the point  $z_0$ ,  $f$  is constant in an *open disc*  $|z - z_0| < r$  contained in  $D$ . Now there is, by definition, such a disc where

$$f(z) = c_0 + c_1(z - z_0) + \cdots + c_n(z - z_0)^n + \cdots$$

the series converging in the disc, with  $c_0 = f(z_0)$ . It will be sufficient to show that the hypothesis implies  $c_n = 0$  for every  $n \geq 1$ .

We need only consider the case where  $c_0 \neq 0$ , otherwise the hypothesis implies  $|f(z)| \leq 0$ , so  $f(z) = 0$  in a neighbourhood of  $z_0$  and the theorem is then proved. Suppose then that  $c_0 \neq 0$  and assume that there exists a smallest integer  $k \geq 1$  such that  $c_k \neq 0$ . For  $|z - z_0| < r$

$$(9.1.1) \quad f(z) = c_0(1 + b_k(z - z_0)^k(1 + (z - z_0)g(z)))$$

where  $b_k = c_k/c_0$  and  $g(z)$  is analytic for  $|z - z_0| < r$ ; thus we can suppose that  $|g(z)| \leq M$  for  $|z - z_0| \leq r/2$  ( $M$  independent of  $z$ ).

Choose  $r' \leq r/2$  satisfying  $Mr' \leq 1/2$ ; put  $b_k = \rho e^{i\alpha}$  ( $\rho > 0$ ,  $0 \leq \alpha \leq 2\pi$ ),  $z - z_0 = |z - z_0| e^{i\theta}$  ( $0 \leq \theta \leq 2\pi$ ) and choose  $\theta$  such that  $k\theta + \alpha$  is an integral multiple of  $2\pi$ , which is possible since  $k \geq 1$ . Then

$$b_k(z - z_0)^k = \rho|z - z_0|^k,$$

and hence, if  $|z - z_0| \leq r'$ ,  $1 + b_k(z - z_0)^k = 1 + \rho|z - z_0|^k > 0$ , and

$$\begin{aligned} |1 + b_k(z - z_0)^k + b_k(z - z_0)^{k+1}g(z)| &\geq |1 + b_k(z - z_0)^k| - |b_k(z - z_0)^{k+1}g(z)| \\ &\geq 1 + \rho|z - z_0|^k - \frac{1}{2}\rho|z - z_0|^k \geq 1 + \frac{1}{2}\rho|z - z_0|^k > 1. \end{aligned}$$

In other words, for every  $z$  such that  $|z - z_0| \leq r'$  and for which  $\theta$  takes the value determined above, we have from (9.1.1),  $|f(z)| > |c_0| = |f(z_0)|$ , contrary to the hypothesis. Q.E.D.

(9.2) A theorem, which we shall admit, says that a real function  $g$  continuous in a bounded closed set  $A \subset \mathbf{C}$  is bounded in  $A$  and that there exists at least one point  $a' \in A$  (resp.  $a'' \in A$ ) such that  $g(a') = \sup_{z \in A} g(z)$  (resp.  $g(a'') = \inf_{z \in A} g(z)$ ) (0, 5.6). There may be infinitely many points having this property (for example, when  $g$  is constant); we say that at such a point  $a'$  (resp.  $a''$ )  $g$  attains its least upper bound (resp. its greatest lower bound) in  $A$ . This being so, the maximum principle implies the following consequence:

(9.3) Let  $D$  be an open connected set in  $\mathbf{C}$ ,  $f$  a non-constant analytic function in  $D$ . For every closed and bounded set  $A$  contained in  $D$ , the points of  $A$  at which  $|f(z)|$  attains its least upper bound in  $A$  are points of the boundary (0, 5.5) of  $A$ .

Since  $|f(z)|$  is continuous in  $D$ , these points always exist; if such a point  $z_0$  was an interior point (0, 5.5) of  $A$ , there would exist, by definition, an open disc  $|z - z_0| < r$  ( $r > 0$ ) contained in  $A$  (therefore in  $D$ ), in which  $|f(z)| \leq |f(z_0)|$ , contradicting (9.1) and the hypothesis that  $f$  is not constant.

Note that  $|f(z)|$  may be constant on the boundary of  $A$ , as is shown by the example where  $D = \mathbf{C}$ ,  $A$  is the unit disc  $|z| \leq 1$  and  $f(z) = z$ .

As a consequence of (9.3), the “fundamental theorem of algebra” (or “d’Alembert-Gauss theorem”) is easily proved:

(9.4) (Fundamental theorem of algebra) *Let*

$$(9.4.1) \quad P(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n$$

*be a non-constant polynomial with complex coefficients ( $n \geq 1$ ,  $a_0 \neq 0$ ). Then there exists a finite sequence  $(z_j)_{1 \leq j \leq n}$  of  $n$  complex numbers (not necessarily distinct) such that*

$$(9.4.2) \quad P(z) = a_0(z - z_1)(z - z_2) \cdots (z - z_n).$$

It is sufficient to show that there exists  $z_1 \in \mathbf{C}$  such that  $P(z_1) = 0$ , since algebra then shows that we can write  $P(z) = (z - z_1)Q(z)$ , where  $Q$  is a polynomial of degree  $n - 1$ , and it is enough to proceed by induction on  $n$ . To prove the existence of  $z_1$ , we assume on the contrary that  $P(z) \neq 0$  for all  $z \in \mathbf{C}$ . It then follows that  $1/P(z)$  is *analytic in the whole of  $\mathbf{C}$* ; it will be seen that this contradicts the maximum principle.

By hypothesis  $P(0) = a_n \neq 0$ ; we show that there exists a number  $R > 0$  such that

$$(9.4.3) \quad |P(z)| \geq 2|a_n| \text{ for every } z \text{ such that } |z| \geq R.$$

Indeed, for  $z \neq 0$

$$P(z) = a_0 z^n \left( 1 + \frac{a_1}{a_0 z} + \cdots + \frac{a_n}{a_0 z^n} \right)$$

and hence

$$(9.4.4) \quad |P(z)| \geq |a_0| |z|^n \left( 1 - \left| \frac{a_1}{a_0 z} + \cdots + \frac{a_n}{a_0 z^n} \right| \right).$$

Now, for  $|z| \geq 1$

$$(9.4.5) \quad \left| \frac{a_1}{a_0 z} + \cdots + \frac{a_n}{a_0 z^n} \right| \leq \left| \frac{a_1}{a_0} \right| \frac{1}{|z|} + \cdots + \left| \frac{a_n}{a_0} \right| \frac{1}{|z|^n} \\ \leq \frac{1}{|z|} \left( \left| \frac{a_1}{a_0} \right| + \cdots + \left| \frac{a_n}{a_0} \right| \right).$$

Let us then take the number  $R$  so that

$$R \geq 1, \quad \frac{1}{R} \left( \left| \frac{a_1}{a_0} \right| + \cdots + \left| \frac{a_n}{a_0} \right| \right) \leq \frac{1}{2}, \quad \frac{1}{2}|a_0|R^n \geq 2|a_n|$$

which is clearly possible. For  $|z| \geq R$ , from (9.4.5)

$$\left| \frac{a_1}{a_0 z} + \cdots + \frac{a_n}{a_0 z^n} \right| \leq \frac{1}{2}$$

therefore, because of (9.4.4),

$$|P(z)| \geq \frac{1}{2}|a_0|R^n \geq 2|a_n|.$$

The result of (9.3) can now be applied to the analytic function  $1/P(z)$  with  $D = \mathbf{C}$ ,  $A$  being the closed disc  $|z| \leq R$ . Since  $1/P$  is not constant, its absolute value  $1/|P(z)|$  attains its least upper bound in  $A$  at a boundary point of  $A$ , i.e. at a point  $z$  such that  $|z| = R$ . But we have seen that at such a point  $1/|P(z)| \leq 1/2|a_n|$ , and also  $1/|P(0)| = 1/|a_n|$ . Since  $a_n \neq 0$ , this is contradictory. Q.E.D.

Note that this argument not only proves the existence of a root of the equation  $P(z) = 0$ , but also gives a *majorization* for the absolute values of these roots:

$$(9.4.6) \quad |z_j| \leq R_0 = \sup \left( 1, 2 \left( \left| \frac{a_1}{a_0} \right| + \cdots + \left| \frac{a_n}{a_0} \right| \right) \right)$$

since for  $|z| \geq R_0$ ,  $|P(z)| \geq \frac{1}{2}|a_0| \cdot |z|^n$ .

Note also that the theorem *does not extend to entire functions* since  $e^z \neq 0$  for all  $z \in \mathbf{C}$ .

## PROBLEMS

1. Let  $(z_n)_{n \geq 1}$  be a sequence of complex numbers such that  $\Re z_n \geq 0$  for every  $n$ . Show that if the two series

$$z_1 + z_2 + \cdots + z_n + \cdots, \quad z_1^2 + z_2^2 + \cdots + z_n^2 + \cdots$$

are convergent, so is the series

$$|z_1|^2 + |z_2|^2 + \cdots + |z_n|^2 + \cdots.$$

Is it true in this case that the series of general term  $|z_n|$  is convergent?

2. Consider the function of a *real* variable

$$f(x) = \sum_{n=0}^{\infty} e^{-n} e^{n^2 i x}$$

(a) Show that this series, and each of the series obtained by differentiating term by term any number of times is normally convergent in  $\mathbf{R}$ , and hence that  $f$  is indefinitely differentiable in  $\mathbf{R}$ .

b) Show that the Taylor series of  $f$  at the point  $x = 0$

$$f(0) + x f'(0) + \cdots + f^{(k)}(0) \frac{x^k}{k!} + \cdots$$

is not convergent for any  $x \neq 0$ . (Use a *reductio ad absurdum* argument using the Abel lemma: if the series is *absolutely* convergent for an  $x > 0$ , observe that, for every integer  $N$ , we would have

$$\sum_{k=0}^{\infty} |f^{(k)}(0)| \frac{x^k}{k!} \geq \sum_{n=0}^N e^{-n} e^{n^2 x}.)$$

3. Let  $(a_n)$ ,  $(b_n)$  be two sequences of complex numbers.

(a) Put  $\sigma_n = a_1 + a_2 + \cdots + a_n$ ,  $\sigma_0 = 0$ , so that  $a_n = \sigma_n - \sigma_{n-1}$  for  $n \geq 1$ . Deduce from this

$$a_m b_m + \cdots + a_n b_n = \sum_{k=m}^{n-1} \sigma_k (b_k - b_{k+1}) - \sigma_{m-1} b_m + \sigma_n b_n$$

("Abel's partial summation").

(b) Deduce from (a) that if the sequence  $(a_n)$  is such that the sequence  $(\sigma_n)$  is *bounded* and if the sequence  $(b_n)$  consists of real numbers  $> 0$ , is decreasing and tends to 0, then the series

$$(*) \quad s = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n + \cdots$$

is convergent and satisfies  $|s| \leq A b_1$ , where  $A = \sup_n |\sigma_n|$ .

(c) Suppose that: 1. the sequence  $(\sigma_n/\sqrt{n})$  is bounded; 2. the series of general term  $|b_n - b_{n+1}|\sqrt{n}$  is convergent; 3.  $\lim_{n \rightarrow \infty} b_n \sqrt{n} = 0$ .

Show then that the series  $(*)$  is convergent.

4. (a) Show that if  $(a_n)$  is a decreasing sequence of real numbers tending to 0 and such that the series of general term  $a_n$  is divergent, then the power series  $a_0 + a_1 z + \cdots + a_n z^n + \cdots$  has radius of convergence 1 and is convergent at every point of the circle  $|z| = 1$  except at the point  $z = 1$  (use problem 3(b)).

(b) Show that the power series of general term  $(-1)^{[\sqrt{n}]} z^n/n$  is convergent at *every* point of the circle of convergence  $|z| = 1$ , but is absolutely convergent at no point of this circle. (Consider separately the case where  $z = 1$  and the case where  $z \neq 1$ ; in the latter case, use problem 3(c).)

5. Consider the power series  $c_0 + c_1 z + \cdots + c_n z^n + \cdots$ , where the coefficients  $c_n$  are defined in the following way:  $c_n = p_n/n$ , where  $p_n$  is the number of the integers  $k \geq 1$  such that  $k!$  divides  $n$ . Show that the radius of convergence of the series is 1 and that the series is divergent at all points  $z = \exp(2r i \pi)$ , where  $r$  is *rational*. Show on the other hand that for  $z = \exp(2e i \pi)$  the series is convergent. (Reduce to studying the finite sums

$$\sum_{k=h}^K \frac{1}{k!} \sum_{n=m_k}^{N_k} \frac{1}{n} z^{k!n}$$

and note that for every integer  $k \geq 1$ , we have

$$\frac{1}{k+1} \leq k! e - [k! e] \leq \frac{e}{k+1}$$

6. Do there exist analytic functions  $f$  in a neighbourhood of 0 and having the property that

$$f\left(\frac{1}{2n}\right) = f\left(\frac{1}{2n+1}\right) = \frac{1}{n}$$

when  $n$  tends to  $+\infty$  or the property that

$$f\left(\frac{1}{n}\right) = f\left(-\frac{1}{n}\right) = \frac{1}{n^3}$$

when  $n$  tends to  $+\infty$ ? (Consider the function  $f(z) = 2z$  or the function  $f(z) = z^3$ .)

7. (a) Let  $f$  be an analytic function in a neighbourhood of  $z = 0$  and taking real values on a sequence of distinct points  $(a_n)$  of  $\mathbf{R}$  which tends to 0. Show that  $f(\bar{z}) = \overline{f(z)}$  in the neighbourhood of 0.

(b) Deduce from (a) that if  $a_n > 0$  for every  $n$ , if  $f$  takes real values at the points  $a_n$ , and if furthermore  $f(a_{2n}) = f(a_{2n+1})$  for every  $n$ ,  $f$  is constant in a neighbourhood of 0.

8. Let  $\gamma$  be a continuous mapping of  $[0, +\infty[$  in  $\mathbf{C}$  such that  $|\gamma(t)|$  tends to  $+\infty$  with  $t$ . Show that the function  $e^{\gamma(t)}$  can tend to a limit when  $t$  tends to  $+\infty$  only if  $\Re\gamma(t)$  tends to  $-\infty$ .

9. Prove that

$$\int_0^1 t^{-t} dt = \sum_{n=0}^{\infty} n^{-n}.$$

10. Given the two matrices

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

show that  $e^{A+B} \neq e^A e^B$ .

11. (a) Let  $f$  be an analytic function in a disc  $\Delta: |z| < R$ . For every  $r$  such that  $0 < r < R$ , let  $M(r)$  be the least upper bound of  $|f(z)|$  for  $|z| = r$ . Show that if  $f$  is not constant,  $M(r)$  is a strictly increasing function of  $r$ .

(b) Show that if for a value of  $r$  ( $0 < r < R$ ), the function  $\theta \rightarrow |f(re^{i\theta})|$  is constant and if  $f(z) \neq 0$  for  $|z| < r$ ,  $f$  is constant.

12. (a) Let  $f$  be an analytic function in an open set  $D \subset \mathbf{C}$ , and let  $F$  be a closed and bounded subset of  $D$ . Show that  $\Re f(z)$  cannot reach its maximum or its minimum in  $F$  at an interior point of  $F$  (consider  $e^{f(z)}$ ).

(b) Show that if  $D$  is connected and if  $f(z)$  is *real-valued* on a circle  $|z - z_0| = r$  contained in  $D$ , then  $f$  is constant (use (a)).

13. Let  $f$  be a bounded analytic function in the exterior of a disc  $E: |z| > R$ . For every  $r > R$ , let  $M(r)$  be the least upper bound of  $|f(z)|$  for  $|z| = r$ . Show that  $|f(z)| \leq M(r)$  for  $|z| > r$ , and that if the function  $f$  is not constant,  $M(r)$  is a strictly decreasing function of  $r$ .

14. Let  $f$  be a polynomial of degree  $n$ ; with the notation of problem 11, show that if  $0 < r_1 < r_2$

$$\frac{M(r_1)}{r_1^n} \geq \frac{M(r_2)}{r_2^n}.$$

15. Let  $\varphi$  be a real function  $> 0$  defined for  $x \geq 0$ , increasing and tending to  $+\infty$  with  $x$ . Show that there exists a sequence  $(k_n)$  of integers such that

$$k_1 = 1, \quad k_n > k_{n-1}, \quad \left(\frac{n}{n-1}\right)^{k_n} > \varphi(n+1) \quad \text{for } n \geq 2;$$

the function

$$f(z) = \varphi(2) + 1 + \sum_{n=2}^{\infty} \left(\frac{z}{n-1}\right)^{k_n}$$

is then an entire function such that  $f(x) \geq \varphi(x)$  for every  $x \geq 0$ .

16. Let  $f(z)$  be the entire function

$$z - \frac{z^{2m+1}}{3!(2m+1)} + \cdots + (-1)^n \frac{z^{2nm+1}}{(2n+1)!(2nm+1)} + \cdots.$$

Show that when  $t$  tends to  $+\infty$ ,  $|f(te^{(2k-1)\pi i/2m})|$  tends to  $+\infty$ , but  $f(te^{k\pi i/m})$  tends to a finite limit

$$e^{k\pi i/m} \int_0^{+\infty} \frac{\sin x^m}{x^m} dx \quad \text{for } 1 \leq k \leq 2m.$$

17. Let  $f(z) = a_0 + a_1z + \dots + a_nz^n + \dots$  be a power series with radius of convergence 1.

(a) Suppose that the series  $s = a_0 + a_1 + \dots + a_n + \dots$  is convergent. Show that when  $x$  is real and tends to 1 while remaining  $< 1$ ,  $f(x)$  tends to  $s$  ("Abels' theorem"). (Majorize in absolute value the sum  $\sum_{n=m}^N a_n(1-x^n)$  with the help of problem 3(a).)

(b) Suppose that all  $a_n$  are  $\geq 0$  and that the series of general term  $a_n$  is divergent. Show that when  $x$  is real and tends to 1 while remaining  $< 1$ ,  $f(x)$  tends to  $+\infty$  (observe that for every  $N$ , we have  $f(x) \geq \sum_{n=0}^N a_n x^n$ ).

(c) If  $f(z) = (1+z)^{-2}$ , all  $a_n$  are real and the sum  $\left| \sum_0^{2N} a_n \right|$  tends to  $+\infty$  when  $N$  tends to  $+\infty$ , but  $f(x)$  tends to a finite limit when  $x$  tends to 1 while remaining  $< 1$ .

18. Show that when  $x$  tends to 1 while remaining real and  $< 1$

$$1 + x + x^4 + \dots + x^{n^2} + \dots \sim \frac{\sqrt{\pi}}{2} \frac{1}{\sqrt{1-x}}$$

$$1 + 1^\alpha x + 2^\alpha x^2 + \dots + n^\alpha x^n + \dots \sim \Gamma(\alpha + 1) \frac{1}{(1-x)^\alpha} \quad (\alpha > 0)$$

(IV, problems 15 and 16).

Show in the same way that

$$\sum_{n=1}^{\infty} \frac{x^n}{1-x^n} \sim \frac{1}{1-x} \log \frac{1}{1-x}.$$

19. Show that when  $x$  is real and tends to  $+\infty$

$$\sum_{n=1}^{\infty} \frac{(\log n)^x}{n^2} \sim \int_1^{+\infty} \frac{(\log t)^x}{t^2} dt \sim \Gamma(x+1)$$

$$\sum_{n=1}^{\infty} \frac{x^n}{(n!)^p} \sim \frac{1}{\sqrt[p]{p}} (2\pi)^{(1-p)/2} x^{(1-p)/2p} e^{px^{1/p}} \quad (p > 0).$$

# The Cauchy theory

The mutual dependence between the various values of an analytic function in an open connected set, which has already been made manifest by the principle of analytic continuation (VI, 7.3), is expressed in an even more striking way by introducing, following Cauchy, a new and powerful tool, the *curvilinear integral* of complex functions along “paths” contained in  $\mathbf{C}$ . We shall see that this enables us, both in theory and in practice, to calculate the values of an analytic function in a disc (for example) when we are given the values on the *boundary* circle of the disc.

## 1. Paths and loops

(1.1) A *path* is a *continuous* mapping

$$(1.1.1) \quad \gamma: I = [a, b] \rightarrow \mathbf{C}$$

of a closed interval  $I$  of  $\mathbf{R}$  (not reduced to a point) into  $\mathbf{C}$ , which is assumed to be *piecewise-continuously differentiable*, i.e. which is a *primitive*  $\gamma(t) = c + \int_a^t \gamma'(s) ds$  of a function  $\gamma'$  *piecewise-continuous* in  $I$  (0, 4.3). As  $t$  varies from  $a$  to  $b$ , the point  $\gamma(t)$  describes in the plane  $\mathbf{C}$  a “trajectory”  $\gamma(I)$  (the image of  $\gamma$ ), which, at the points  $\gamma(t)$  such that  $\gamma'$  is continuous and  $\neq 0$  at the point  $t$ , possesses a *tangent* whose direction is that of the vector  $\gamma'(t) \in \mathbf{C}$  (Fig. 24). Corresponding to the points  $t$  where  $\gamma'$  is discontinuous and has

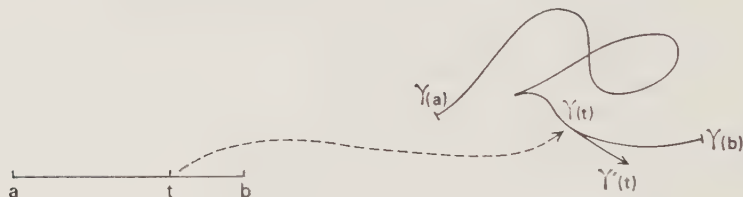


FIGURE 24

non-zero limits on the right and on the left, we have “angular points” at  $\gamma(t)$ . The “trajectory”  $\gamma(I)$  may have “multiple points”, corresponding to distinct values of  $t$  at which  $\gamma$  takes the same value; it may even happen that *all* the points of  $\gamma(I)$  are “multiple”. By a theorem which we admit (0, 5.6) the set  $\gamma(I)$  is *closed*.

*Examples (1.2.1)* If  $\gamma$  is *constant* in  $I$ ,  $\gamma(I)$  reduces to just one point (“constant path”).

(1.2.2) Let  $\alpha$  be a real number  $\neq 0$ ; the mapping  $\varepsilon_\alpha: t \mapsto e^{2\pi i \alpha t}$  defined in  $I = [0, 1]$  is a path such that  $\gamma(I)$  is a subset of the *unit circle*  $|z| = 1$ . If  $\alpha$  is an *integer*  $n \neq 0$  (positive or negative),  $\gamma(I)$  is the whole unit circle, but every point of the circle is attained at  $|n|$  distinct values of  $t$ ; we also say that  $\varepsilon_n$  is the “unit circle described  $n$  times”.

(1.2.3) Let  $c, d$  be two points of  $\mathbf{C}$ ; for  $0 \leq t \leq 2$ , let  $\gamma(t)$  be defined as follows:

$$\begin{cases} \gamma(t) = c(1-t) + dt & \text{for } 0 \leq t \leq 1; \\ \gamma(t) = d(2-t) + c(t-1) & \text{for } 1 \leq t \leq 2. \end{cases}$$

The image of  $\gamma$  is therefore the *segment* with endpoints  $c, d$  but the path  $\gamma$  may be said to be the “segment described in both senses”. The point  $t = 1$  is a point of discontinuity of  $\gamma'$ .

These examples show that it is essential NOT TO CONFUSE THE PATH WITH THE “CURVE”  $\gamma(I)$ . We may say that a path is a “parametrized curve”, and the “parametrization” will be seen to have as much importance as the curve.

(1.3) A path (1.1.1) is said to be *contained* in an open set  $D \subset \mathbf{C}$  if  $\gamma(I) \subset D$ . The point  $\gamma(a)$  is called the *initial point* of the path  $\gamma$ , the point  $\gamma(b)$  its *terminal point*; we also say that  $\gamma(a)$  and  $\gamma(b)$  are the “endpoints” of  $\gamma$ , and that  $\gamma$  “joins the points  $\gamma(a)$  and  $\gamma(b)$ ”. A sufficient condition that any two points  $c, d$  of an open set  $D \subset \mathbf{C}$  can be joined by a path in  $D$ , is that  $D$  be *connected* (0, 5.8); by a theorem, which we admit, this condition is also necessary. We say that a path (1.1.1) is a *loop* if  $\gamma(a) = \gamma(b)$ ; also that  $\gamma$  is a “loop with initial point  $\gamma(a)$ ”. Examples (1.2.1) and (1.2.3) are loops; so is  $\varepsilon_\alpha$  when  $\alpha$  is an *integer*, but *not* in the other cases (although the path  $\varepsilon_\alpha$  may “repass” several times through its initial point.)

(1.4) Given a path (1.1.1), the path

$$\gamma^0: t \mapsto \gamma(a + b - t)$$

defined in  $I$  is called the path *opposite* to  $\gamma$ ; the initial point of  $\gamma^0$  is the terminal point  $\gamma(b)$  of  $\gamma$  and the terminal point of  $\gamma^0$  is the initial point  $\gamma(a)$  of  $\gamma$ ; it may be said that  $\gamma^0$  is “the path  $\gamma$  described in the opposite sense”.

(1.5) Given two paths

$$\begin{aligned} \gamma_1: I_1 = [a, b] &\rightarrow \mathbf{C} \\ \gamma_2: I_2 = [c, d] &\rightarrow \mathbf{C} \end{aligned}$$

such that the *initial point*  $\gamma_2(c)$  of  $\gamma_2$  is the *terminal point*  $\gamma_1(b)$  of  $\gamma_1$ , the *juxtaposition* of  $\gamma_1$  and  $\gamma_2$ , denoted by  $\gamma = \gamma_1 \vee \gamma_2$ , is the path

$$\gamma: [a, d + b - c] \rightarrow \mathbf{C}$$

defined as follows

$$\begin{cases} \gamma(t) = \gamma_1(t) & \text{for } a \leq t \leq b \\ \gamma(t) = \gamma_2(t - b + c) & \text{for } b \leq t \leq d + b - c \end{cases}$$

(Fig. 25). The initial point of  $\gamma_1 \vee \gamma_2$  is the initial point of  $\gamma_1$  and the terminal point of

$\gamma_1 \vee \gamma_2$  is the terminal point of  $\gamma_2$ . The example (1.2.3) can be written  $\gamma_1 \vee \gamma_1^0$ , where  $\gamma_1$  is the path  $t \rightarrow c(1-t) + dt$  defined in  $[0, 1]$ . In general, for every path  $\gamma$ , the juxtaposition  $\gamma \vee \gamma^0$  is a *loop* whose initial point is that of  $\gamma$ .

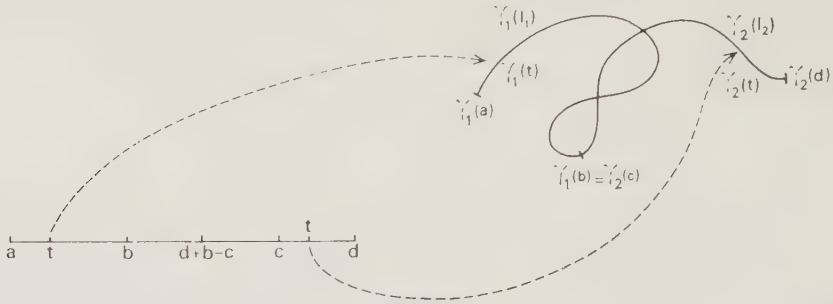


FIGURE 25

For every path  $\gamma: [a, b] \rightarrow \mathbf{C}$  and every point  $c$  satisfying  $a < c < b$ ,  $\gamma$  is therefore a juxtaposition of its *restrictions*  $\gamma_1, \gamma_2$  to  $[a, c]$  and to  $[c, b]$  respectively. In particular, a loop is always the juxtaposition of two paths (in infinitely many ways); moreover, if, with the same notations,  $\gamma$  is a loop with initial point  $\gamma(a)$ , the path  $\gamma' = \gamma_2 \vee \gamma_1$  is defined (since the terminal point of  $\gamma_2$  is the initial point of  $\gamma_1$ ) and is also a loop with initial point  $\gamma(c)$ . We say that it is the loop obtained from  $\gamma$  by “taking the point  $\gamma(c)$  for initial point”.

(1.6) Let  $\gamma_1: I_1 = [a, b] \rightarrow \mathbf{C}$ ,  $\gamma_2: I_2 = [c, d] \rightarrow \mathbf{C}$  be two paths. We say that  $\gamma_1$  and  $\gamma_2$  are *equivalent* if there exists an increasing *bijection*  $\varphi: I_2 \rightarrow I_1$ , together with its inverse  $\varphi^{-1}$  *continuous and piecewise-continuously differentiable*, such that  $\gamma_2(t) = \gamma_1(\varphi(t))$  in  $I_2$ . The images  $\gamma_1(I_1)$  and  $\gamma_2(I_2)$  are then identical, and the initial and terminal points of  $\gamma_1$  and  $\gamma_2$  are the same; but the example (1.2.2) shows that these conditions are not sufficient for two paths to be equivalent. Note that for every path (1.1.1), the path  $t \rightarrow \gamma(\lambda t + \mu)$ , where  $\lambda > 0$  and  $\mu$  is any real number, is equivalent to  $\gamma$ . Up to equivalence, we can therefore, if we wish, consider only those paths defined in a *fixed* interval of  $\mathbf{R}$ , for example  $[0, 1]$ . When there can be no confusion, we shall define a path by the figure describing its image in the correct sense.

## 2. Integration along a path

(2.1) Let  $\gamma: [a, b] \rightarrow \mathbf{C}$  be a path,  $f$  a *continuous* complex function in  $\gamma(I)$  (we do not necessarily suppose  $f$  defined outside  $\gamma(I)$  and *a fortiori* we do not assume  $f$  analytic). Then the composed function  $t \rightarrow f(\gamma(t))\gamma'(t)$  is *piecewise-continuous* in  $[a, b]$  and hence its integral in this interval is defined (I, 3.1). By *definition*, the *integral of  $f$  along the path  $\gamma$*  is the complex number

$$(2.1.1) \quad \int_{\gamma} f(z) dz = \int_a^b f(\gamma(t))\gamma'(t) dt.$$

In this notation,  $z$  can of course be replaced by any other letter.

This definition shows that the integral along a path  $\gamma$  depends not only on the *set*  $\gamma(\mathbf{I})$ , but also on its “parametrization” (see below (2.2.1)). However, if  $\gamma_1: [a, b] \rightarrow \mathbf{C}$  and  $\gamma_2: [c, d] \rightarrow \mathbf{C}$  are two *equivalent* paths (1.6), then

$$(2.1.2) \quad \int_{\gamma_1} f(z) dz = \int_{\gamma_2} f(z) dz.$$

Indeed, if  $\varphi: [c, d] \rightarrow [a, b]$  is an increasing bijection, continuous and piecewise-continuously differentiable, such that  $\gamma_2 = \gamma_1 \circ \varphi$ , then  $\gamma'_2(t) = \gamma'_1(\varphi(t))\varphi'(t)$  in  $\mathbf{I}_2$  except at a finite number of points, and hence, by the formula for change of variables (0, 4.5.5)

$$\begin{aligned} \int_{\gamma_2} f(z) dz &= \int_c^d f(\gamma_2(t))\gamma'_2(t) dt = \int_c^d f(\gamma_1(\varphi(t)))\gamma'_1(\varphi(t))\varphi'(t) dt \\ &= \int_{\varphi(c)}^{\varphi(d)} f(\gamma_1(u))\gamma'_1(u) du = \int_a^b f(\gamma_1(u))\gamma'_1(u) du \\ &= \int_{\gamma_1} f(z) dz. \end{aligned}$$

If the function  $f$  satisfies  $|f(z)| \leq M$  for  $z \in \gamma(\mathbf{I})$ , we deduce from the definition (2.1.1) and from the formula of the mean (I, 3.3.2)

$$(2.1.3) \quad \left| \int_{\gamma} f(z) dz \right| \leq M \int_a^b |\gamma'(t)| dt = ML$$

where  $L$  is none other than the *length* of the “curve”  $t \rightarrow \gamma(t)$ .

(2.2) The definition (2.1.1) and the rules of the integral calculus (0, 4.5) show at once that for two *opposite* paths (1.4)

$$(2.2.1) \quad \int_{\gamma_0} f(z) dz = - \int_{\gamma} f(z) dz,$$

and that for the *juxtaposition*  $\gamma = \gamma_1 \vee \gamma_2$  of two paths (1.5)

$$(2.2.2) \quad \int_{\gamma} f(z) dz = \int_{\gamma_1} f(z) dz + \int_{\gamma_2} f(z) dz.$$

Since the second member of (2.2.2) remains the same when the two terms are interchanged, we deduce from (2.2.2) and (1.5) that if  $\gamma$  is a *loop*, the integral  $\int_{\gamma} f(z) dz$  is *independent of the initial point* of the closed path.

If  $\gamma$  is a *constant* path (1.2.1), then

$$\int_{\gamma} f(z) dz = 0$$

for every function  $f$ .

Finally, suppose that the path  $\gamma$  is contained in an open set  $D \subset \mathbf{C}$ , and let  $u$  be an *analytic* function in  $D$ ; then, if  $\Gamma$  is the composed path  $t \rightarrow u(\gamma(t))$ , by (VI, 6.6.5)

$$(2.2.3) \quad \int_{\gamma} f(u(z))u'(z) dz = \int_{\Gamma} f(w) dw.$$

### 3. Problem of the primitives of analytic functions

(3.1) Let  $D \subset \mathbf{C}$  be an open *connected* set. It has already been pointed out (VI, 6.4) that it may happen that some functions analytic in  $D$  do not however possess a primitive in  $D$ . The notion of integral along a path enables us to give a necessary and sufficient condition for such a primitive to exist.

(3.2) A function  $f$  analytic in  $D$  possesses a primitive in  $D$ , if, and only if, for each loop  $\gamma$  contained in  $D$ ,  $\int_{\gamma} f(z) dz = 0$ . In this case every primitive  $F$  of  $f$  in  $D$  is obtained in the following way:

$$(3.2.1) \quad F(z) = C + \int_{\alpha(z)} f(u) du$$

where  $\alpha(z)$  is any path contained in  $D$  with initial point  $z_0$  fixed (and arbitrary) and terminal point  $z$  and  $C$  is a constant. The difference of two primitives of  $f$  in  $D$  is constant.

The necessity of the condition is immediate: if  $F$  is a primitive of  $f$  in  $D$ , then for every path  $\gamma: [a, b] \rightarrow \mathbf{C}$  contained in  $D$

$$f(\gamma(t))\gamma'(t) = \frac{d}{dt} F(\gamma(t))$$

in  $[a, b]$  (VI, 6.6.5), hence  $\int_{\gamma} f(u) du = F(\gamma(b)) - F(\gamma(a))$  and in particular, if  $\gamma$  is a loop,  $\int_{\gamma} f(u) du = 0$ . This also shows that if a primitive  $F$  of  $f$  exists in  $D$ , then  $F(z) - F(z_0) = \int_{\alpha(z)} f(u) du$  (with the notations of (3.2)) for any path  $\alpha(z)$  with initial point  $z_0$  and terminal point  $z$  (there are always some such paths, since  $D$  is connected). This proves the last assertion of the theorem. It remains to show that the condition is *sufficient* for the existence of a primitive. When this condition is satisfied, the integral in the second member of (3.2.1) depends only on  $z$  and  $z_0$ , and not on the path  $\alpha(z)$  with initial point  $z_0$  and terminal point  $z$ . Indeed, if  $\alpha(z)$  and  $\beta(z)$  are two such paths, then  $\gamma = \alpha(z) \vee \beta(z)^0$  is a loop with initial point  $z_0$ , so our assertion follows from the fact that  $\int_{\alpha(z)} f(u) du - \int_{\beta(z)} f(u) du = \int_{\gamma} f(u) du = 0$  ((2.2.1) and (2.2.2)). The formula (3.2.1) thus defines a function  $F$  in  $D$  (once  $C$  and  $z_0$  are chosen). It remains to show that this function is analytic and has the derivative  $f$ . Now, for each point  $z_1 \in D$ , there is a disc  $\Delta: |z - z_1| < r$  contained in  $D$ , in which  $f$  can be developed in a power series in  $z - z_1$ , therefore there exists a primitive  $F_1$  (analytic) of  $f$  in  $\Delta$  (VI, 6.5), which may be supposed to vanish at the point  $z_1$ . For every path  $\lambda(z)$  in  $\Delta$  with initial point  $z_1$  and terminal point  $z$ , we therefore have  $F_1(z) = \int_{\lambda(z)} f(u) du$ . Now, if  $\alpha(z_1)$  is a path in  $D$  with initial point  $z_0$  and terminal point  $z_1$ ,  $\alpha(z_1) \vee \lambda(z)$  is a path in  $D$  with initial point  $z_0$  and terminal point  $z$ , hence from (2.2.1)

$$F(z) = F(z_1) + F_1(z)$$

for all  $z \in \Delta$ , which proves the proposition.

Because the integral  $\int_{\alpha(z)} f(u) du$  is independent of the chosen path  $\alpha(z)$  joining  $z_0$  to

$z$ , it may also be written when no confusion may occur,  $\int_{z_0}^z f(u) du$ ; note that this symbol has a meaning only if  $\int_{\gamma} f(z) dz = 0$  for every loop  $\gamma$  contained in  $D$ .

(3.3) The typical example where the condition of (3.2) is not satisfied is the case where  $D = \mathbf{C} - \{0\}$  (which is connected (0, 5.8)), and  $f(z) = 1/z$  (which is analytic in  $D$  (VI, 8.2)). If we consider the loop  $\varepsilon_1: t \rightarrow e^{it}$  defined in  $[0, 2\pi]$ , which is clearly contained in  $D$

$$\int_{\varepsilon_1} \frac{dz}{z} = \int_0^{2\pi} i \frac{e^{it}}{e^{it}} dt = 2i\pi \neq 0.$$

In general, for a given open connected set  $D \subset \mathbf{C}$ , there are some analytic functions in  $D$  which possess a primitive in  $D$  (for example the polynomials), but there may be functions analytic in  $D$  which do not possess a primitive in  $D$ . It will be seen that if  $D$  satisfies a certain *geometrical* condition, then *all* the analytic functions in  $D$  possess a primitive in  $D$ .

#### 4. Homotopies of paths. Simply connected domains

(4.1) The intuitive idea of the *homotopy* of two paths is that of a “continuous deformation” sending one path onto the other. This is made precise by the following mathematical definition:

(4.2) Let  $D$  be an open set in  $\mathbf{C}$ ,  $\gamma_1: I \rightarrow \mathbf{C}$ ,  $\gamma_2: I \rightarrow \mathbf{C}$  be two paths contained in  $D$ , defined in the same interval  $I = [a, b]$  (cf. (1.6)). A homotopy from  $\gamma_1$  to  $\gamma_2$  in  $D$  is a continuous mapping  $\varphi: I \times J \rightarrow D$ , where  $J = [c, d]$  is an interval of  $\mathbf{R}$ , such that  $\varphi(t, c) = \gamma_1(t)$  and  $\varphi(t, d) = \gamma_2(t)$  for every  $t \in I$ .

Note that the *continuity* of the function of two variables is required

$$(t, s) \rightarrow \varphi(t, s),$$

and not just the continuity in each of the two variables with the other variable fixed; on the other hand no differentiability conditions are imposed on  $\varphi$ , except those imposed on  $\gamma_1$  and  $\gamma_2$  by the definition of a path.

We say that  $\gamma_2$  is *homotopic* to  $\gamma_1$  in  $D$  when there exists a homotopy from  $\gamma_1$  to  $\gamma_2$  in  $D$ .

When  $\gamma_1$  and  $\gamma_2$  are *loops* in  $D$ , we say that  $\gamma_1$  and  $\gamma_2$  are *homotopic as loops* in  $D$  if there exists a homotopy  $\varphi: I \times J \rightarrow D$  from  $\gamma_1$  to  $\gamma_2$  in  $D$  with the additional property that  $\varphi(a, s) = \varphi(b, s)$  for every  $s \in J$ . Intuitively speaking the homotopy  $\varphi$  “does not untie” the loop on the way; such a homotopy is called a *homotopy of loops*.

Note carefully that two loops contained in  $D$  may be homotopic (as loops) in an open set  $D'$  containing  $D$ , without being homotopic in  $D$  (cf. (4.5)).

The homotopy of paths in  $D$  is an *equivalence relation*. The reflexivity of this relation is clear (take  $\varphi(t, s) = \gamma(t)$  for every  $s$ ). To see that if there exists a homotopy  $\varphi: I \times J \rightarrow D$  from  $\gamma_1$  to  $\gamma_2$ , there also exists a homotopy from  $\gamma_2$  to  $\gamma_1$  in  $D$ , it is sufficient to consider the mapping  $\varphi^0: I \times J \rightarrow D$  defined by  $\varphi^0(t, s) = \varphi(t, c + d - s)$ . Lastly, the transitivity of the relation is proved as follows: consider a homotopy  $\varphi_1: I \times J_1 \rightarrow D$  from  $\gamma_1$  to  $\gamma_2$  and a homotopy  $\varphi_2: I \times J_2 \rightarrow D$  from  $\gamma_2$  to  $\gamma_3$ , with  $J_1 = [c_1, d_1]$  and

$J_2 = [c_2, d_2]$ . Put  $J_3 = [c_1, d_1 + d_2 - c_2]$  and define  $\varphi_3: I \times J_3 \rightarrow D$  by the conditions  $\varphi_3(t, s) = \varphi_1(t, s)$  in  $I \times J_1$  and  $\varphi_3(t, s) = \varphi_2(t, s + c_2 - d_1)$  for  $(t, s) \in I \times [d_1, d_1 + d_2 - c_2]$ . It is immediately verified that  $\varphi_3$  is a homotopy from  $\gamma_1$  to  $\gamma_3$  in  $D$ . Furthermore, the same reasoning shows that the homotopy of *loops* is also an equivalence relation, the homotopies  $\varphi^0$  and  $\varphi_3$  constructed above being homotopies of loops if the same is true of  $\varphi$ ,  $\varphi_1$  and  $\varphi_2$ .

(4.3) We say that a loop  $\gamma$  contained in  $D$  is *homotopic to a point* in  $D$  if it is homotopic as a loop in  $D$  to a *constant* path (1.2.1). An open connected set  $D$  is called *simply connected* (or a *simply connected domain*) if every loop in  $D$  is homotopic to a point in  $D$ .

(4.4) *Examples of simply connected domains.* We say that an open set  $D \subset \mathbf{C}$  is *starlike with respect to a point*  $a \in D$  if, for each  $z \in D$ , the segment  $t \rightarrow (1 - t)a + tz$  ( $0 \leq t \leq 1$ ) joining  $a$  and  $z$  is contained in  $D$ . Such an open set is evidently connected (0, 5.8); furthermore it is *simply connected*. Indeed, let  $\gamma: I \rightarrow D$  be a loop in  $D$  with  $I = [\alpha, \beta]$ . We define a homotopy  $\varphi: I \times [0, 1] \rightarrow D$  by the formula  $\varphi(t, s) = sa + (1 - s)\gamma(t)$ . It is clear that  $\varphi(t, 0) = \gamma(t)$  and  $\varphi(t, 1) = a$ ;  $\varphi$  is evidently continuous and the hypothesis implies that the values of  $\varphi$  belong to  $D$ , hence our assertion.

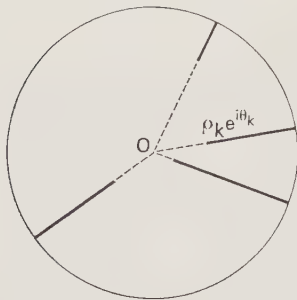


FIGURE 26

Examples of open starlike sets with respect to a point, are the whole plane  $\mathbf{C}$ , a half-plane, an open disc, the inside of a rectangle, an ellipse, etc.; in all these cases the set is starlike with respect to *each* of its points (the set is then said to be *convex*). Another example is the disc  $|z| < 1$  from which a finite number of radial segments have been deleted:

$$z = t e^{i\theta_k} \quad \text{with } \rho_k \leq t \leq 1,$$

$\rho_k$  being a number satisfying  $0 < \rho_k < 1$  (Fig. 26). It is immediately verified that this open set is starlike with respect to 0. Every *intersection* of a finite number of open sets starlike with respect to a point  $a$  (resp.

convex) is starlike with respect to  $a$  (resp. convex).

An example of a simply connected domain which is not starlike, is the open half-plane  $\mathcal{I}z > 0$  from which has been deleted a certain number of closed half-lines  $z = t + i\beta_k$ , with  $-\infty < t \leq \alpha_k$  (Fig. 27) (problem 1).

(4.5) It will be seen below, as a consequence of Cauchy's theorem and of (3.3), that the connected set  $\mathbf{C} - \{0\}$ , the complement of a point, is not simply connected, although  $\mathbf{C}$  is. A loop in  $\mathbf{C} - \{0\}$  is homotopic to a point *in*  $\mathbf{C}$ , but not necessarily *in*  $\mathbf{C} - \{0\}$ .

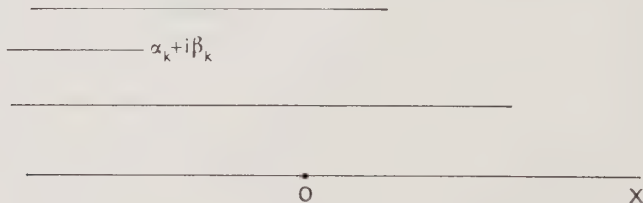


FIGURE 27

## 5. Cauchy's theorem

(5.1) (Cauchy's theorem) *Let  $D \subset \mathbf{C}$  be an open connected set, and let  $f$  be a function analytic in  $D$ . If  $\gamma_1$  and  $\gamma_2$  are two loops contained in  $D$  and homotopic as loops in  $D$ , then*

$$(5.1.1) \quad \int_{\gamma_1} f(z) dz = \int_{\gamma_2} f(z) dz.$$

*In particular, if  $D$  is simply connected (4.3), then*

$$(5.1.2) \quad \int_{\gamma} f(z) dz = 0$$

*for every loop contained in  $D$ .*

We merely give an idea of the proof without attempting to demonstrate rigorously every step (cf. [FA], (9.6.3)).

It can be supposed that the loops  $\gamma_1$  and  $\gamma_2$  are defined in the same interval  $I = [a, b]$  of  $\mathbf{R}$ ; let  $\varphi: I \times J \rightarrow D$  be a homotopy from  $\gamma_1$  to  $\gamma_2$  in  $D$ , where  $J = [c, d]$ . We begin by dividing  $I$  (resp.  $J$ ) into a finite number of subintervals  $[a_h, a_{h+1}]$  ( $0 \leq h \leq n-1$ ) (resp.  $[c_k, c_{k+1}]$  ( $0 \leq k \leq m-1$ )) in such a way that the image  $\varphi(R_{hk})$  of the rectangle  $[a_h, a_{h+1}] \times [c_k, c_{k+1}]$  is entirely contained in a disc  $\Delta_{hk} \subset D$ , of centre  $z_{hk}$ , in which the function  $f$  can be developed in a power series in  $z - z_{hk}$ , and therefore (VI, 6.5) possesses a primitive in  $\Delta_{hk}$  (Fig. 28).

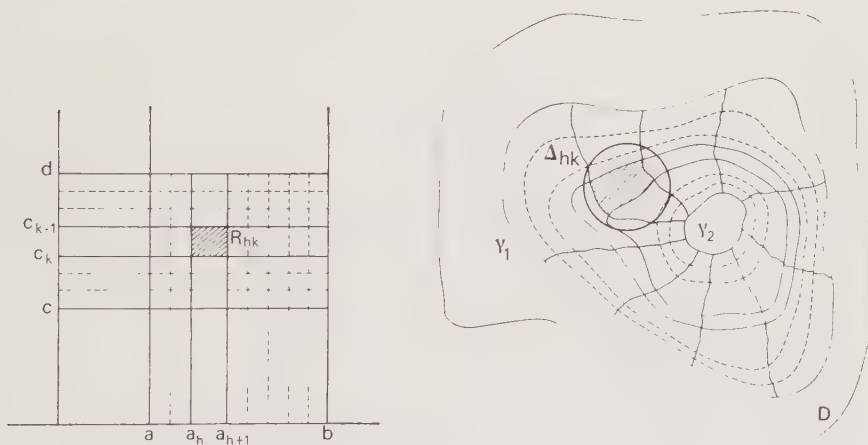


FIGURE 28

Consider then in  $\Delta_{hk}$  the four segments joining the images under  $\varphi$  of the four vertices of the rectangle  $R_{hk}$  (Fig. 29):

- $\beta_{hk}$  with initial point  $u_{hk} = \varphi(a_h, c_k)$  and terminal point  $u_{h,k+1} = \varphi(a_h, c_{k+1})$ .
- $\beta_{h+1,k}$  with initial point  $u_{h+1,k} = \varphi(a_{h+1}, c_k)$  and terminal point  $u_{h+1,k+1} = \varphi(a_{h+1}, c_{k+1})$ .
- $\alpha_{h,k}$  with initial point  $u_{hk}$  and terminal point  $u_{h+1,k}$ .
- $\alpha_{h,k+1}$  with initial point  $u_{h,k+1}$  and terminal point  $u_{h+1,k+1}$ .

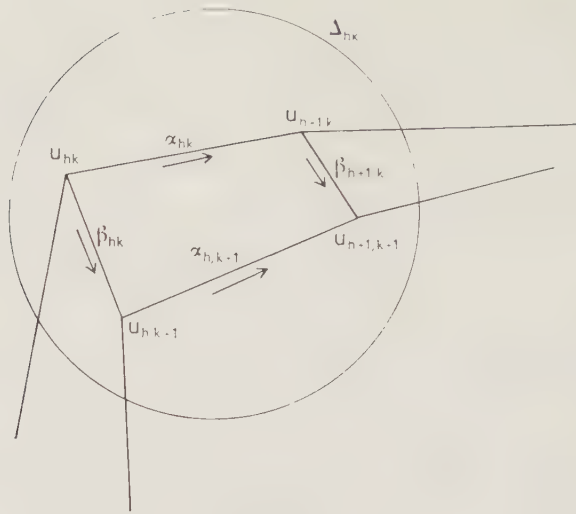


FIGURE 29

It follows from (3.2) applied to the loop

$$\alpha_{hk} \vee \beta_{h+1,k} \vee \alpha_{h,k+1}^0 \vee \beta_{h,k}^0$$

that

$$(5.1.3) \quad \int_{\beta_{hk}} f(z) dz + \int_{\alpha_{h,k+1}} f(z) dz = \int_{\alpha_{hk}} f(z) dz + \int_{\beta_{h+1,k}} f(z) dz.$$

Observe now that the juxtaposition  $\alpha_k = \alpha_{0,k} \vee \alpha_{1,k} \vee \cdots \vee \alpha_{n-1,k}$  is a loop in  $D$  (since  $\varphi$  is a homotopy of loops); for the same reason  $\beta_{0,k} = \beta_{n,k}$ . Summing over the  $n$  relations (5.1.3)

$$(5.1.4) \quad \int_{\alpha_k} f(z) dz = \int_{\alpha_{k+1}} f(z) dz,$$

hence in particular

$$(5.1.5) \quad \int_{\alpha_0} f(z) dz = \int_{\alpha_m} f(z) dz.$$

But the same reason shows that

$$\int_{\alpha_0} f(z) dz = \int_{\gamma_1} f(z) dz \quad \text{and} \quad \int_{\alpha_m} f(z) dz = \int_{\gamma_2} f(z) dz. \quad \text{Q.E.D.}$$

(5.2) If  $D$  is a simply connected domain, then every analytic function in  $D$  possesses a primitive in  $D$ .

This follows from (5.1) and (3.2).

## 5. Index of a point with respect to a loop

(6.1) Let  $\gamma: I = [c, d] \rightarrow \mathbf{C}$  be a loop in  $\mathbf{C}$  and let  $a$  be a point of  $\mathbf{C}$  not belonging to  $\gamma(I)$ . Then the number

$$(6.1.1) \quad j(a; \gamma) = \frac{1}{2\pi i} \int_{\gamma} \frac{dz}{z - a}$$

is an integer (positive or negative).

For every  $t \in I$ , put  $h(t) = \int_c^t \gamma'(s) ds / (\gamma(s) - a)$ , so that  $2\pi i j(a; \gamma) = h(d)$ . Since  $h$  is a primitive of a piecewise-continuous function,  $h'(t) = \gamma'(t) / (\gamma(t) - a)$  except at a finite number of points of  $I$ . Put

$$g(t) = e^{-h(t)}(\gamma(t) - a);$$

then, except at a finite number of points of  $I$ ,

$$g'(t) = -h'(t)e^{-h(t)}(\gamma(t) - a) + \gamma'(t)e^{-h(t)} = 0.$$

Hence the continuous function  $g$  is constant in  $I$ , and in particular  $g(c) = g(d)$ . But  $h(c) = 0$ , therefore

$$g(c) = \gamma(c) - a;$$

thus

$$e^{-h(d)}(\gamma(d) - a) = \gamma(c) - a.$$

But by hypothesis  $\gamma$  is a loop so  $\gamma(d) = \gamma(c)$ , and thus for the complex number  $h(d)$  we have the relation  $e^{-h(d)} = 1$ , and therefore by (VI, 8.5)  $h(d) = 2n\pi i$  for some  $n \in \mathbf{Z}$ .

The integer  $j(a; \gamma)$  thus defined is called the *index of  $a$  with respect to the loop  $\gamma$* ; we recall that it is defined *only if  $a \notin \gamma(I)$* . It will be seen that this number is the precise mathematical notion which corresponds to the intuitive idea of the “number of times the loop  $\gamma$  winds around  $a$ ”.

By virtue of the definition (6.1.1) and the formulae (2.2.1) and (2.2.2)

$$(6.1.2) \quad j(a; \gamma^0) = -j(a; \gamma)$$

$$(6.1.3) \quad j(a; \gamma_1 \vee \gamma_2) = j(a; \gamma_1) + j(a; \gamma_2)$$

where  $\gamma, \gamma_1, \gamma_2$  are loops whose images do not contain  $a$ , the loops  $\gamma_1$  and  $\gamma_2$  being assumed to have the *same initial point* (in order that  $\gamma_1 \vee \gamma_2$  be also a loop with the same initial point).

(6.2) Let  $a$  be a point of  $\mathbf{C}$ ,  $\gamma_1, \gamma_2$  two loops in  $\mathbf{C} - \{a\}$  homotopic as loops in  $\mathbf{C} - \{a\}$ ; then  $j(a; \gamma_1) = j(a; \gamma_2)$ .

This is an immediate consequence of Cauchy's theorem (5.1) applied to the function  $1/(z - a)$  analytic in  $\mathbf{C} - \{a\}$  (VI, 8.2).

(6.3) Let  $\gamma$  be a loop in  $\mathbf{C}$ ; for every open connected set  $D$  contained in  $\mathbf{C} - \gamma(I)$ , the function  $z \rightarrow j(z; \gamma)$  is constant in  $D$ .

It is sufficient to prove that for each  $z_0 \in D$  and each disc  $\Delta: |z - z_0| < r$  ( $r > 0$ ) contained in  $D$ , we have  $j(z; \gamma) = j(z_0; \gamma)$  for  $z \in \Delta$ . Indeed, this will prove that the function  $z \rightarrow j(z; \gamma)$  is *locally constant* in  $D$ , therefore *constant* since  $D$  is connected (0, 5.9). Thus let  $\Delta$  be an open disc of centre  $z_0$  contained in  $D$ . Note that

$$j(z; \gamma) = \frac{1}{2\pi i} \int_{\gamma} \frac{du}{u - z} = \frac{1}{2\pi i} \int_{\gamma_1} \frac{du}{u - z_0} = j(z_0; \gamma_1)$$

where  $\gamma_1$  is the loop  $t \rightarrow \gamma(t) - (z - z_0)$ . Now  $\gamma_1$  is *homotopic* as a loop to  $\gamma$  in  $\mathbf{C} - \{z_0\}$ , by the homotopy

$$\varphi(t, s) = \gamma(t) - s(z - z_0) \quad (0 \leq s \leq 1).$$

Indeed, if we had  $\varphi(t, s) = z_0$ , this would imply that  $\gamma(t) = sz + (1 - s)z_0$  for  $t \in I$  and an  $s$  satisfying  $0 \leq s \leq 1$ . But the point  $sz + (1 - s)z_0$  belongs to  $\Delta$  and by hypothesis  $\gamma(I)$  does not meet  $\Delta$ . It is now sufficient to apply (6.2).

(6.4) Let  $\varepsilon_n: t \rightarrow e^{n i t}$  ( $0 \leq t \leq 2\pi$ ) be the “unit circle described  $n$  times” (1.2.1) ( $n$  integer positive or negative). For each  $z$  satisfying  $|z| < 1$ , we have  $j(z; \varepsilon_n) = n$ , and for each  $z$  satisfying  $|z| > 1$ , we have  $j(z; \varepsilon_n) = 0$ .

The unit disc  $|z| < 1$  and the exterior of the disc  $|z| > 1$  are open *connected* sets not meeting the unit circle  $|z| = 1$  (0, 5.8). It thus suffices to prove that  $j(z_0; \varepsilon_n) = n$  for one point  $z_0$  in the unit disc  $|z| < 1$  and  $j(z_1; \varepsilon_n) = 0$  for one point  $z_1$  exterior to this disc. Now, if we put  $z_0 = 0$

$$j(0; \varepsilon_n) = \frac{1}{2\pi i} \int_{\varepsilon_n} \frac{du}{u} = \frac{1}{2\pi i} \int_0^{2\pi} \frac{n i e^{n i t} dt}{e^{n i t}} = n.$$

On the other hand, for  $|z_1| > 1$ ,  $z_1$  is not contained in the open disc  $|z| < |z_1|$ , and the unit circle is contained in this disc. Since we have seen that an open disc is simply connected (4.4), the assertion concerning  $z_1$  is a consequence of the following more general result:

(6.5) Let  $D \subset \mathbf{C}$  be a simply connected domain, and let  $\gamma$  be a loop contained in  $D$ . Then for every point  $a \notin D$ ,  $j(a; \gamma) = 0$ .

Indeed, the function  $z \rightarrow 1/(z - a)$  is analytic in  $D$  and the conclusion follows from Cauchy's theorem (5.1) and the definition of the index (6.1).

(6.6) *Method for calculating the index.* In numerous cases the following practical method is available for calculating the index  $j(a; \gamma)$ . We can confine ourselves, by means of a translation, to the case where  $a = 0$  (and of course  $0 \notin \gamma(I)$ ). We assume that the real axis  $\mathcal{R}z = 0$  meets  $\gamma(I)$  at only a *finite* number of points. Put  $\gamma(t) = u(t) + iv(t)$ , where  $u(t)$  and  $v(t)$  are real; there is thus in  $I$  a finite number of points

$$t_1 < t_2 < \dots < t_N$$

where the function  $v$  changes sign. Furthermore, we may suppose (by changing the initial point of  $\gamma$  (2.2)) that  $I = [t_1, t_N]$ , so that  $\gamma(t_1) = \gamma(t_N)$  and  $y(t) \neq 0$  in  $I$ . This being so, we extend  $\gamma$  to the whole of  $\mathbf{R}$  by periodicity with period  $t_N - t_1$ ; the points  $t_k$  ( $1 \leq k \leq N$ ) are distributed among four sets:

$M_1: u(t_k) > 0$  and  $v(t)$  changes from the sign  $-$  to the sign  $+$  as  $t$  passes through  $t_k$  in the increasing sense.

$M_2: u(t_k) > 0$  and  $v(t)$  changes from the sign  $+$  to the sign  $-$  as  $t$  passes through  $t_k$  in the increasing sense.

$M_3: u(t_k) < 0$  and  $v(t)$  changes from the sign  $+$  to the sign  $-$  as  $t$  passes through  $t_k$  in the increasing sense.

$M_4: u(t_k) < 0$  and  $v(t)$  changes from the sign  $-$  to the sign  $+$  as  $t$  passes through  $t_k$  in the increasing sense (Fig. 30).

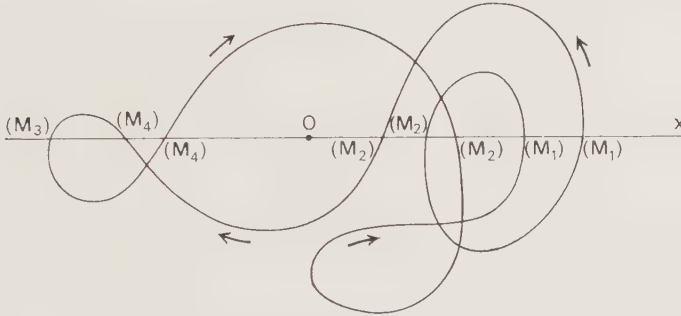


FIGURE 30

Note that  $t_1$  and  $t_N$  belong to the same set  $M_k$  and that  $N$  is *odd*, since for  $t = t_1 - h$  and  $t = t_N + h$  with  $h$  sufficiently small,  $v(t)$  has different signs and has changed its sign  $N$  times. Now define for every integer  $k$  satisfying  $1 \leq k \leq N$  a number

$$(6.6.1) \quad \begin{cases} \delta_k = +1 & \text{if } t_k \in M_1 \text{ or } t_k \in M_3 \\ \delta_k = -1 & \text{if } t_k \in M_2 \text{ or } t_k \in M_4 \end{cases}$$

(intuitively,  $\delta_k$  is positive if in the neighbourhood of  $t_k$ ,  $\gamma(t)$  “turns in the positive sense” around 0, negative in the contrary case). Then

$$(6.6.2) \quad j(0; \gamma) = \frac{1}{2} \sum_{k=1}^{N-1} \delta_k.$$

The proof is easy: in the integral

$$(6.6.3) \quad j(0; \gamma) = \frac{1}{2i\pi} \int_{t_1}^{t_N} \frac{\gamma'(t) dt}{\gamma(t)}$$

the interval of integration is partitioned at the points  $t_k$  ( $2 \leq k \leq N-1$ ). Suppose for example that  $t_k \in M_2$ ; by definition,  $v(t) < 0$  for

$$t_k < t < t_{k+1}$$

and so  $t_{k+1} \in M_1$  or  $t_{k+1} \in M_4$ . Thus, for  $t_k \leq t \leq t_{k+1}$ , we can write  $u(t) + iv(t) = \sqrt{u^2(t) + v^2(t)} e^{i\theta(t)}$ , where  $\theta(t)$  is a *well-defined* differentiable function *with values in*  $[-\pi, 0]$ , taking the value 0 at  $t = t_k$ , the value 0 at  $t = t_{k+1}$  if  $t_{k+1} \in M_1$ , and the value

$-\pi$  if  $t_{k+1} \in M_4$ . On the other hand, we need only calculate the real part of the integral

$$\frac{1}{2i\pi} \int_{t_k}^{t_{k+1}} \frac{\gamma'(t) dt}{\gamma(t)}$$

i.e.

$$(6) \quad \frac{1}{2\pi} \int_{t_k}^{t_{k+1}} \frac{u(t)v'(t) - v(t)u'(t)}{u^2(t) + v^2(t)} dt.$$

But by elementary differential calculus

$$u' + iv' = e^{i\theta} \left( i\theta' \sqrt{u^2 + v^2} + \frac{uu' + vv'}{\sqrt{u^2 + v^2}} \right)$$

and since

$$(6.6.5) \quad \begin{aligned} e^{-i\theta} &= \frac{u - iv}{\sqrt{u^2 + v^2}} \\ \theta' &= \frac{uv' - vu'}{u^2 + v^2}. \end{aligned}$$

Hence, substituting into (6.6.4)

$$\Re \left( \frac{1}{2i\pi} \int_{t_k}^{t_{k+1}} \frac{\gamma'(t) dt}{\gamma(t)} \right) = \frac{1}{2\pi} (\theta(t_{k+1}) - \theta(t_k))$$

and finally we obtain the expression  $\frac{1}{2}(\delta_k + \delta_{k+1})$ . The other cases are treated similarly and yield finally the formula (6.6.2).

By means of a rotation round the origin, we pass at once from the above case to the case when *any one line*  $L$  passing through 0 meets  $\gamma(I)$  at only a finite number of points.

An interesting case occurs when the *real positive half-axis*  $\mathbf{R}_+ : \mathcal{I}z = 0, \mathcal{R}z \geq 0$ , meets  $\gamma(I)$  in *exactly one point* (or the cases deduced from this by rotation); it can be supposed that this point corresponds to  $t = t_1$  and to  $t = t_N$ . It then follows from the definitions that there is an *odd* number of the other  $t_j$  ( $2 \leq j \leq N - 1$ ). If, for example,  $t_1 \in M_1$ , then necessarily  $t_2 \in M_3$  and the  $t_j$  such that  $2 \leq j \leq N - 1$  are in  $M_3$  if  $j$  is even, in  $M_4$  if  $j$  odd; the contribution of these points in (6.6.2) is therefore  $\frac{1}{2}$ , and therefore

$$(6.6.6) \quad i(0; \gamma) = 1.$$

Similarly, if  $t_1 \in M_2$ , we deduce that  $j(0; \gamma) = -1$ .

## 7. Cauchy's formula

(7.1) (Cauchy's formula) *Let  $D \subset \mathbf{C}$  be a simply connected domain, and let  $\gamma: I \rightarrow D$  be a loop in  $D$  (Fig. 31). Then, for every function  $f$  analytic in  $D$  and every point  $x \in D - \gamma(I)$ ,*

$$(7.1.1) \quad j(x; \gamma) f(x) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z) dz}{z - x}.$$

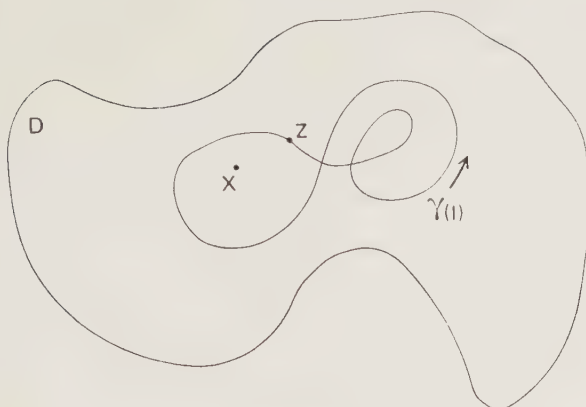


FIGURE 31

We define the function  $g(z)$  in  $D$  by the conditions

$$(7.1.2) \quad \begin{cases} g(z) = \frac{f(z) - f(x)}{z - x} & \text{for } z \neq x \\ g(x) = f'(x) \end{cases}$$

and show that this function is *analytic* in  $D$ . This is clear for  $z \neq x$  (VI, 8.2); it remains to examine the situation in a neighbourhood of the point  $x$ . Let  $\Delta: |z - x| < r$  be a disc contained in  $D$ , in which the Taylor series

$$f(z) = f(x) + \frac{f'(x)}{1!} (z - x) + \cdots + \frac{f^{(n)}(x)}{n!} (z - x)^n + \cdots$$

is convergent (VI, 6.3). The definition (7.1.2) shows that for every  $z \in \Delta$ ,  $g(z)$  is the sum of the convergent power series in  $z - x$

$$f'(x) + \frac{f''(x)}{2!} (z - x) + \cdots + \frac{f^{(n)}(x)}{n!} (z - x)^{n-1} + \cdots$$

therefore  $g$  is analytic in  $D$ . Since  $D$  is simply connected,  $\int_{\gamma} g(z) dz = 0$  by Cauchy's theorem (5.1), which gives, since  $x \notin \gamma(I)$ ,

$$\int_{\gamma} \frac{f(z) - f(x)}{z - x} dz = 0,$$

or again

$$\int_{\gamma} \frac{f(z) dz}{z - x} = f(x) \int_{\gamma} \frac{dz}{z - x} = 2\pi i j(x; \gamma) f(x).$$

The Cauchy formula is particularly interesting when  $j(x; \gamma) \neq 0$ . For example, if  $D$  contains a closed disc  $|z - a| \leq r$ , then for every point  $x$  of the open disc  $|x - a| < r$  of the same centre and radius

$$(7.1.3) \quad f(x) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z) dz}{z - x}$$

where  $\gamma$  is the closed path  $t \rightarrow a + re^{it}$  defined in  $[0, 2\pi]$  (6.4). Thus we have an explicit formula giving the values of  $f$  in the disc  $|x - a| < r$  when these are known on the circle  $|x - a| = r$ .

Conversely, integrals of the type occurring in the second member of (7.1.1) are analytic functions of the parameter  $x$ ; more generally:

(7.2) *Let  $\gamma: I = [b, c] \rightarrow \mathbf{C}$  be a path in  $\mathbf{C}$  and let  $g: \gamma(I) \rightarrow \mathbf{C}$  be defined and continuous on  $\gamma(I)$  (we do not assume  $g$  defined outside the set  $\gamma(I)$ , and a fortiori  $g$  is not assumed analytic). Then the function*

$$(7.2.1) \quad f(z) = \int_{\gamma} \frac{g(u) du}{u - z}$$

*is defined and analytic in the open set  $\mathbf{C} - \gamma(I)$ . To be precise, for every point  $a \in \mathbf{C} - \gamma(I)$ , if we put*

$$(7.2.2) \quad c_n = \int_{\gamma} \frac{g(u) du}{(u - a)^{n+1}}$$

*for each integer  $n \geq 0$ , then we have the development in a convergent power series*

$$(7.2.3) \quad f(z) = \sum_{n=0}^{\infty} c_n (z - a)^n$$

*in the open disc of centre  $a$  and radius  $d(a, \gamma(I))$ , the distance from  $a$  to the closed set  $\gamma(I)$  (0, 5.6). Moreover*

$$(7.2.4) \quad f^{(n)}(a) = n! c_n = n! \int_{\gamma} \frac{g(u) du}{(u - a)^{n+1}}.$$

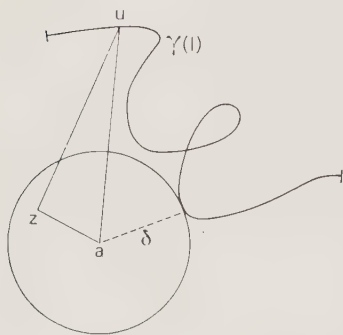


FIGURE 32

Put  $\delta = d(a, \gamma(I)) > 0$  and suppose that  $|z - a| = q\delta$  with  $0 \leq q < 1$ . Since by definition

$$\delta = \inf_{u \in \gamma(I)} |u - a|,$$

$|u - a| \geq \delta$  for every  $u \in \gamma(I)$ , and hence  $|(z - a)/(u - a)| \leq q < 1$  for every  $u \in \gamma(I)$  (Fig. 32).

Thus for every  $u \in \gamma(I)$ ,

$$(7.2.5) \quad \frac{1}{u - z} = \frac{1}{(u - a) \left(1 - \frac{z - a}{u - a}\right)} = \sum_{n=0}^{\infty} \frac{(z - a)^n}{(u - a)^{n+1}}$$

where the series is convergent, with the majorizations

$$(7.2.6) \quad \left| \frac{(z - a)^n}{(u - a)^{n+1}} \right| \leq \frac{q^n}{\delta} \quad \text{for } n \geq 0.$$

Thus, by definition

$$f(z) = \int_b^c \frac{g(\gamma(t)) \gamma'(t) dt}{\gamma(t) - z} = \int_b^c g(\gamma(t)) \gamma'(t) \left( \sum_{n=0}^{\infty} \frac{(z - a)^n}{(\gamma(t) - a)^{n+1}} \right) dt$$

and the inequalities (7.2.6) show that for  $t \in I$ , the integrated series is *normally convergent* (V, 2.5). Indeed, since  $g$  is continuous and  $\gamma'$  piecewise-continuous in  $[b, c]$ , there exists a number  $M$  such that

$$|g(\gamma(t))\gamma'(t)| \leq M$$

for all  $t \in I$ , and hence

$$\left| g(\gamma(t))\gamma'(t) \frac{(z-a)^n}{(\gamma(t)-a)^{n+1}} \right| \leq M \frac{q^n}{\delta}$$

for all  $t \in I$ , which proves our assertion. Applying (V, 3.5), the development in a convergent series is obtained:

$$f(z) = \sum_{n=0}^{\infty} (z-a)^n \int_b^c \frac{g(\gamma(t))\gamma'(t) dt}{(\gamma(t)-a)^{n+1}}$$

i.e. (7.2.3).

If  $|g(u)| \leq M$  in  $\gamma(I)$ , we deduce from (7.2.2) a *majorization* of the coefficients  $c_n$ :

$$(7.2.7) \quad |c_n| \leq \frac{ML}{\delta^{n+1}}$$

where  $L$  is the length of the "curve"  $\gamma(I)$  (cf. (2.1.3)).

By combining (7.2) and Cauchy's formula (7.1), we can answer a question which occurred naturally in (VI, 5.2):

(7.3) *If a function  $f$  is analytic in an open set  $D$ , then, for each point  $a \in D$ , the Taylor series of  $f$  at the point  $a$  converges to the sum  $f(z)$  in the whole of the open disc of centre  $a$  and of radius equal to the distance from  $a$  to  $\mathbf{C} - D$  (i.e. in the largest open disc of centre  $a$  contained in  $D$ ).*

If  $0 < r < d(a, \mathbf{C} - D)$ , Cauchy's formula (7.1.1) can be applied in the open disc  $|z-a| < r$ , by taking  $\gamma: t \rightarrow a + re^{it}$ ,  $0 \leq t \leq 2\pi$ , which gives  $j(z; \gamma) = 1$  (6.4). Applying (7.2), we do therefore obtain a development for  $f$  in a power series in  $z-a$ , convergent for  $|z-a| < r$ . This development is necessarily the Taylor series (VI, 6.3), and since  $r$  is arbitrarily near  $d(a, \mathbf{C} - D)$ , the proposition is proved.

It will be seen in Chap. VIII that if no further hypothesis on  $D$  or  $f$  is stipulated, the largest open disc of centre  $a$  contained in  $D$  is in general *smaller* than the disc of convergence of the Taylor series of  $f$  at the point  $a$ .

Proposition (7.3) implies the following corollary:

(7.4) *If  $f$  is an entire function, its Taylor series at each point of  $\mathbf{C}$  is convergent in the whole of  $\mathbf{C}$ .*

*Remarks (7.5)* If the formula (7.2.4) is applied to the situation of (7.1), it is seen that, under the hypotheses of (7.1), we have not only the formula (7.1.1), but also, for every integer  $n \geq 1$

$$(7.5.1) \quad j(x; \gamma) f^{(n)}(x) = \frac{n!}{2\pi i} \int_{\gamma} \frac{f(z) dz}{(z-x)^{n+1}}$$

for every  $x \notin \gamma(I)$ . In other words, at the points where  $j(x; \gamma) \neq 0$ , not only the value of  $f$ , but those of *all its derivatives*, are determined explicitly by the values of  $f$  on the set  $\gamma(I)$ .

(7.6) Only by considering complex values of the variable is it possible to understand phenomena which appear surprising when we confine ourselves to functions of a *real* variable. For example, consider the function  $f(z) = 1/(1 + z^2)$ : for real values of  $z$ , we obtain a function defined and continuous in the whole of  $\mathbf{R}$ , and, for each point  $a \in \mathbf{R}$ ,  $f(x)$ , in an open interval of centre  $a$ , is the sum of a power series in  $x - a$ , which is just its Taylor series at  $a$ . However, this series is not convergent *in the whole of  $\mathbf{R}$  for any point  $a \in \mathbf{R}$* : otherwise, by the Abel lemma (VI, 2.2), the series would also be convergent in the whole of  $\mathbf{C}$  and its sum would be an entire function  $g(z)$ , which would coincide with  $f(z)$  in an interval of  $\mathbf{R}$ , therefore also in the whole of the open connected set  $\mathbf{C} - \{i, -i\}$  where  $f$  is analytic (VI, 7.4). But this is absurd since  $|f(z)|$  tends to  $+\infty$  as  $z$  tends to  $i$  or  $-i$  in  $\mathbf{C} - \{i, -i\}$ , whereas  $g$  is continuous at these points. The reason for this behaviour of the Taylor series of the function of a real variable  $1/(1 + x^2)$ , which is nevertheless *analytic at every point of  $\mathbf{R}$* , arises from the fact that this function cannot be continued to an *entire* function of a complex variable (i.e. an analytic function at *every point of  $\mathbf{C}$* ). The function  $f(z)$  has, as we shall see in Chap. VIII, “singularities” at the points  $\pm i$ , which obviously are not apparent as long as we are considering only real values of the variable.

## 8. Cauchy inequalities; Liouville's theorem

(8.1) (Cauchy inequalities) *Let  $f$  be an analytic function in an open set  $D \subset \mathbf{C}$ ; let  $a$  be a point of  $D$ ,  $\Delta: |z - a| \leq r$  a closed disc of centre  $a$  contained in  $D$  and let  $M$  be the supremum of  $|f(z)|$  on the circle  $\Gamma: |z - a| = r$ , the boundary of  $\Delta$ . Then, for every integer  $n \geq 0$ ,*

$$(8.1.1) \quad |f^{(n)}(a)| \leq \frac{n! M}{r^n},$$

where, by convention,  $f^{(0)} = f$ .

Apply the formula (7.5.1) by taking for  $\gamma$  the loop

$$t \rightarrow a + re^{it} \quad (0 \leq t \leq 2\pi) \quad \text{and } x = a;$$

so obtaining

$$f^{(n)}(a) = \frac{n!}{2\pi r^n} \int_0^{2\pi} f(a + re^{it}) e^{-nit} dt$$

and since  $|e^{-nit}| = 1$ , the inequality (8.1.1) is an immediate consequence of the theorem of the mean (I, 3.3.2).

(8.2) (Liouville's theorem) *An entire function bounded in the whole of  $\mathbf{C}$  is necessarily constant.*

Let  $f(z) = \sum_{n=0}^{\infty} c_n z^n$  be an entire function, the power series converging in the whole

of  $\mathbf{C}$  (7.4). If  $|f(z)| \leq M$  for every  $z \in \mathbf{C}$ , (8.1.1) can be applied by taking for  $\Delta$  an open disc of centre 0 and *arbitrarily large* radius  $r$ ; then, for  $n \geq 1$ ,

$$|c_n| \leq M \cdot r^{-n}.$$

But as  $r$  tends to  $+\infty$ , the second member tends to 0 (since  $n \geq 1$ ) and as the first member is independent of  $r$ , we necessarily have  $c_n = 0$  for every  $n \geq 1$ , so  $f(z) = c_0$  in  $\mathbf{C}$ .

*Remark (8.3)* Here again, there is no result analogous to (8.1) or (8.2) if we confine ourselves to *real* values of the variable. For example, the entire function  $\sin kz$  satisfies  $|\sin kx| \leq 1$  for every *real*  $x$ , but there is no majorization of its derivative (in  $\mathbf{R}$ ) depending only on the supremum of the function *in*  $\mathbf{R}$ ; secondly, this function is bounded *in*  $\mathbf{R}$  without being constant.

Cauchy conditions

It is remarkable that just the fact of being *continuously differentiable* characterizes the analytic functions of a complex variable:

(9.1) *Every complex function defined and continuously differentiable in an open set  $D \subset \mathbf{C}$  is analytic in  $D$  (and therefore indefinitely differentiable in  $D$ ).*

Let  $f$  be a function continuously differentiable in  $D$  and let us prove that for every closed disc  $|z - a| \leq r$  contained in  $D$  (Fig. 33)  $f(z)$  is equal in the open disc  $|z - a| < r$  to the sum of a convergent power series in  $z - a$ . If we consider the loop

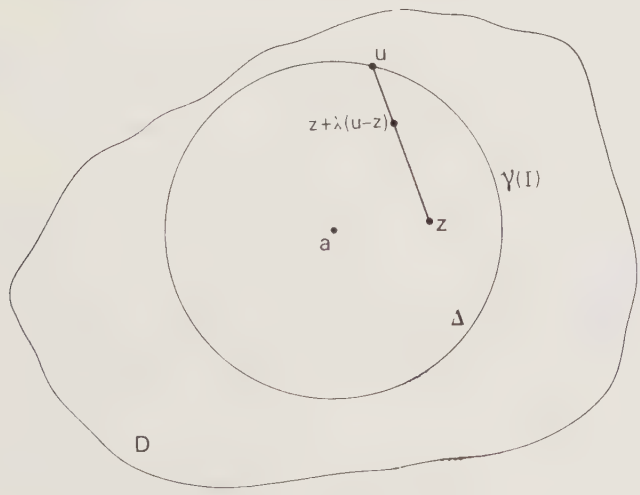


FIGURE 33

$\gamma: t \rightarrow a + re^{it}$  ( $0 \leq t \leq 2\pi$ ), it is sufficient, by virtue of (7.2), to show that for  $z$  satisfying  $|z - a| < r$

$$(9.1.1) \quad f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(u) du}{u - z}.$$

To do this, write, for  $0 \leq \lambda \leq 1$

$$(9.1.2) \quad g(\lambda) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z + \lambda(u - z))}{u - z} du$$

(Fig. 33); we have

$$\begin{aligned} g(1) &= \frac{1}{2\pi i} \int_{\gamma} \frac{f(u) du}{u - z}, \\ g(0) &= \frac{1}{2\pi i} \int_{\gamma} \frac{f(z) du}{u - z} = f(z) \cdot \frac{1}{2\pi i} \int_{\gamma} \frac{du}{u - z} = f(z) \end{aligned}$$

since  $j(z; \gamma) = 1$  (6.4). To prove (9.1.1), it is enough to show that the function  $g$  is constant in  $[0, 1]$ . We can write

$$g(\lambda) = \frac{r}{2\pi} \int_0^{2\pi} \frac{f(z + \lambda(a + re^{it} - z))}{a + re^{it} - z} e^{it} dt.$$

The properties of integrals dependent on a parameter and the hypothesis of differentiability made on  $f$  show that  $g$  is continuous in the closed interval  $[0, 1]$ , and *differentiable* in the open interval  $]0, 1[$ , its derivative being given by the formula for differentiating under the integral sign (0, 4.6.4)

$$(9.1.3) \quad g'(\lambda) = \frac{r}{2\pi} \int_0^{2\pi} f'(z + \lambda(a + re^{it} - z)) e^{it} dt.$$

However observe that

$$\frac{\partial}{\partial t} (f(z + \lambda(a + re^{it} - z))) = i\lambda re^{it} f'(z + \lambda(a + re^{it} - z)).$$

Thus, for  $0 < \lambda < 1$ , the formula (9.1.3) gives

$$g'(\lambda) = \frac{1}{2\pi i \lambda} f(z + \lambda(a + re^{it} - z)) \Big|_0^{2\pi} = 0$$

which shows that  $g$  is constant in the open interval  $]0, 1[$ , and as it is continuous in the closed interval  $[0, 1]$ ,  $g(0) = g(1)$ . Q.E.D.

(9.2) Note once again the difference between the behaviour of functions of a complex variable and functions of a real variable. A function of a real variable can be continuously differentiable without even having a second derivative, as is shown by the example  $f(x) = x|x|$ .

(9.3) Every complex function (analytic or not) defined in an open set  $D \subset \mathbf{C}$  can be written

$$(9.3.1) \quad f(x + iy) = P(x, y) + iQ(x, y)$$

where  $P$  and  $Q$  are two *real* functions defined in  $D$ , and conversely for every such pair of functions, the formula (9.3.1) defines a complex function in  $D$ ;  $f$  is *continuous* in  $D$  if, and only if, both  $P$  and  $Q$  are. Assuming  $f$  continuous, let us find under what conditions on  $P$  and  $Q$  the function  $f$  possesses a derivative with respect to the complex variable  $z$  at a point  $z_0 = x_0 + iy_0$ . By definition (VI, 6.1), the expression

$$(9.3.2) \quad \frac{P(x_0 + s, y_0 + t) + iQ(x_0 + s, y_0 + t) - P(x_0, y_0) - iQ(x_0, y_0)}{s + it}$$

must tend to a limit as  $(s, t)$  tends to  $(0, 0)$  in  $\mathbf{R}^2$  while remaining  $\neq (0, 0)$  and *a fortiori* must tend to the *same* limit along every line  $\alpha s + \beta t = 0$  passing through the origin. In particular take the two lines defined by the coordinate axes: when  $t = 0$ , (9.3.2) tends to a limit as  $s$  tends to 0, and therefore the partial derivatives  $(\partial P / \partial x)(x_0, y_0)$  and  $(\partial Q / \partial x)(x_0, y_0)$  must exist and the limit of (9.3.2) has the value

$$(9.3.3) \quad \frac{\partial P}{\partial x}(x_0, y_0) + i \frac{\partial Q}{\partial x}(x_0, y_0).$$

Similarly, when  $s = 0$ , (9.3.2) tends to a limit as  $t$  tends to 0, and so the partial derivatives  $(\partial P / \partial y)(x_0, y_0)$  and  $(\partial Q / \partial y)(x_0, y_0)$  exist and the limit of (9.3.2) has the value

$$(9.3.4) \quad -i \frac{\partial P}{\partial y}(x_0, y_0) + \frac{\partial Q}{\partial y}(x_0, y_0).$$

Comparing the two equal values of the limit of (9.3.2), it is seen that the partial derivatives of  $P$  and  $Q$  must satisfy the *Cauchy conditions*

$$(9.3.5) \quad \frac{\partial P}{\partial x}(x_0, y_0) = \frac{\partial Q}{\partial y}(x_0, y_0), \quad \frac{\partial P}{\partial y}(x_0, y_0) = -\frac{\partial Q}{\partial x}(x_0, y_0).$$

Conversely:

(9.4) *Suppose that the functions  $P$  and  $Q$  possess continuous partial derivatives of the first order in  $D$  and that these derivatives satisfy identically in  $D$  the Cauchy conditions*

$$(9.4.1) \quad \frac{\partial P}{\partial x} = \frac{\partial Q}{\partial y}, \quad \frac{\partial P}{\partial y} = -\frac{\partial Q}{\partial x};$$

*then the function  $f(x + iy) = P(x, y) + iQ(x, y)$  is analytic in  $D$ .*

It is sufficient to prove that at each point  $(x_0, y_0)$  the limit (9.3.2) exists: the function  $f(z)$  then has a derivative at every point of  $D$ , and the expression (9.3.3) for this derivative, together with the hypothesis of continuity of the partial derivatives of  $P$  and  $Q$ , shows that  $f'(z)$  is continuous in  $D$ , which enables us to apply (9.1) for the conclusion.

Consider then the difference

$$\begin{aligned} F(s, t) &= P(x_0 + s, y_0 + t) + iQ(x_0 + s, y_0 + t) - P(x_0, y_0) - iQ(x_0, y_0) \\ &\quad - (s + it) \left( \frac{\partial P}{\partial x}(x_0, y_0) + i \frac{\partial Q}{\partial x}(x_0, y_0) \right). \end{aligned}$$

We have

$$\Re F(s, t) = P(x_0 + s, y_0 + t) - P(x_0, y_0) - s \frac{\partial P}{\partial x}(x_0, y_0) + t \frac{\partial Q}{\partial x}(x_0, y_0)$$

which in view of the Cauchy conditions can be written

$$\Re F(s, t) = P(x_0 + s, y_0 + t) - P(x_0, y_0) - s \frac{\partial P}{\partial x}(x_0, y_0) - t \frac{\partial P}{\partial y}(x_0, y_0)$$

and similarly we have

$$\Im F(s, t) = Q(x_0 + s, y_0 + t) - Q(x_0, y_0) - s \frac{\partial Q}{\partial x}(x_0, y_0) - t \frac{\partial Q}{\partial y}(x_0, y_0).$$

Since the partial derivatives of the first order of  $P$  and  $Q$  are continuous, we can apply the theorem of the mean (I, 3.6.2): for each  $\varepsilon > 0$ , there exists a number  $r > 0$  such that for  $|s| \leq r$  and  $|t| \leq r$

$$|\Re F(s, t)| \leq \frac{\varepsilon}{2} |s + it|, \quad |\Im F(s, t)| \leq \frac{\varepsilon}{2} |s + it|$$

and hence

$$|F(s, t)| \leq \varepsilon |s + it|$$

which proves the existence of the limit of (9.3.2).

An equivalent and shorter way of writing (9.4.1) is the following:

$$(9.4.2) \quad \frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} = 0.$$

(9.5) When the real continuously differentiable functions  $P, Q$  satisfy the Cauchy conditions in  $D$ , it follows from (9.4) and from the fact that an analytic function is indefinitely differentiable, that  $P$  and  $Q$  are *ipso facto indefinitely differentiable*, and that we have

$$f^{(n)}(x + iy) = \frac{\partial^n P}{\partial x^n} + i \frac{\partial^n Q}{\partial x^n} = (-i)^n \left( \frac{\partial^n P}{\partial y^n} + i \frac{\partial^n Q}{\partial y^n} \right).$$

Furthermore, the functions  $\partial^n P / \partial x^n$  and  $\partial^n Q / \partial x^n$  (resp.  $\partial^n P / \partial y^n$  and  $\partial^n Q / \partial y^n$ ) satisfy in their turn the Cauchy conditions; in particular, for  $n = 1$ ,

$$\frac{\partial}{\partial x} \left( \frac{\partial P}{\partial x} \right) = \frac{\partial}{\partial y} \left( \frac{\partial Q}{\partial x} \right), \quad \frac{\partial}{\partial y} \left( \frac{\partial P}{\partial y} \right) = -\frac{\partial}{\partial x} \left( \frac{\partial Q}{\partial y} \right)$$

and since

$$\frac{\partial}{\partial y} \left( \frac{\partial Q}{\partial x} \right) = \frac{\partial}{\partial x} \left( \frac{\partial Q}{\partial y} \right),$$

$P$  satisfies the *Laplace equation*

$$(9.5.1) \quad \frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} = 0.$$

We verify similarly that  $Q$  (which is actually the real part of  $-if$ ) is also a solution of this equation.

Conversely, a twice-differentiable function  $P$  which satisfies the Laplace equation (9.5.1) in  $D$  is *not necessarily* the real part of a function analytic in  $D$  (cf. VIII, 9.3); however the following result holds:

(9.6) *Let  $P(x, y)$  be a twice continuously differentiable function in a square  $D: |x - x_0| < r, |y - y_0| < r$ , and satisfying Laplace's equation (9.5.1) in  $D$ . Then there exists a function  $f$  analytic in  $D$  such that  $\Re f(x + iy) = P(x, y)$ , and all the functions having this property are of the form  $f + c$ , where  $c$  is a constant.*

Note that if  $\Re f(x + iy) = P(x, y)$ , then by the Cauchy conditions and (9.3.3),  $f'(x + iy) = (\partial P / \partial x)(x, y) - i(\partial P / \partial y)(x, y)$  and the last assertion follows from (3.2),  $D$  being connected. To prove the existence of  $f$ , it is sufficient, in view of (9.4), to show that there exists a function  $Q(x, y)$  continuously differentiable in  $D$  satisfying the Cauchy conditions (9.4.1); such a function is defined by the formula

$$Q(x_0 + s, y_0 + t) = - \int_0^s \frac{\partial P}{\partial y}(x_0 + u, y_0) du + \int_0^t \frac{\partial P}{\partial x}(x_0 + s, y_0 + v) dv.$$

Using the formula for differentiating under the integral sign (0, 4.6.4),

$$\frac{\partial Q}{\partial y}(x_0 + s, y_0 + t) = \frac{\partial P}{\partial x}(x_0 + s, y_0 + t)$$

$$\frac{\partial Q}{\partial x}(x_0 + s, y_0 + t) = - \frac{\partial P}{\partial y}(x_0 + s, y_0) + \int_0^t \frac{\partial^2 P}{\partial x^2}(x_0 + s, y_0 + v) dv$$

By by virtue of Laplace's equation

$$\begin{aligned} \int_0^t \frac{\partial^2 P}{\partial x^2}(x_0 + s, y_0 + v) dv &= - \int_0^t \frac{\partial^2 P}{\partial y^2}(x_0 + s, y_0 + v) dv \\ &= - \frac{\partial P}{\partial y}(x_0 + s, y_0 + t) + \frac{\partial P}{\partial y}(x_0 + s, y_0) \end{aligned}$$

which proves the proposition.

The result of (9.6) is easily extended to the case where  $D$  is *simply connected*, by the same reasoning as in the Cauchy theorem (5.1).

## 10. Weierstrass convergence theorem

(10.1) (Weierstrass convergence theorem) *Let  $(f_n)$  be a sequence of functions analytic in an open set  $D \subset \mathbf{C}$ , and suppose that for every closed disc  $\Delta$  contained in  $D$ , the sequence  $(f_n(z))$  converges uniformly in  $\Delta$  to a limit  $f(z)$ . Then the function  $f$  is analytic in  $D$  and for every closed*

disc  $\Delta$  contained in  $D$ , and every integer  $k \geq 1$ , the sequence of derivatives  $(f_n^{(k)}(z))$  converges uniformly in  $\Delta$  to  $f^{(k)}(z)$  (compare (V, 3.6.4)).

Let  $\Delta: |z - z_0| \leq r$  be a closed disc contained in  $D$ , and let  $\gamma$  be the loop  $t \rightarrow z_0 + re^{it}$  ( $0 \leq t \leq 2\pi$ ). The function  $f$  being continuous in  $D$  (V, 3.1), to prove that it is analytic in the open disc  $|z - z_0| < r$ , it suffices to show that for every point  $z$  in this disc

$$(10.1.1) \quad f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(u) du}{u - z}$$

by virtue of (7.2). Now, by Cauchy's formula (7.1)

$$f_n(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f_n(u) du}{u - z} = \frac{r}{2\pi} \int_0^{2\pi} \frac{f_n(z_0 + re^{it}) e^{it} dt}{z_0 - re^{it} - z}$$

But for  $z$  fixed the sequence of functions

$$t \rightarrow \frac{f_n(z_0 + re^{it}) e^{it}}{z_0 + re^{it} - z}$$

converges uniformly in  $[0, 2\pi]$  to

$$t \rightarrow \frac{f(z_0 + re^{it}) e^{it}}{z_0 + re^{it} - z}.$$

Indeed,

$$|z - z_0 - re^{it}| \geq r - |z - z_0|$$

and by hypothesis, for each  $\varepsilon > 0$ , there exists  $n_0$  such that for  $n \geq n_0$  and for every  $t \in [0, 2\pi]$ , we have  $|f(z_0 + re^{it}) - f_n(z_0 + re^{it})| \leq \varepsilon$ , and so

$$(10.1.2) \quad \left| \frac{(f_n(z_0 + re^{it}) - f(z_0 + re^{it})) e^{it}}{z_0 + re^{it} - z} \right| \leq \frac{\varepsilon}{r - |z - z_0|}$$

which proves our assertion. The relation (10.1.1) follows at once from the passage to the uniform limit in an integral (V, 3.4).

The majorization

$$(10.1.3) \quad |f(z) - f_n(z)| \leq \frac{\varepsilon r}{r - |z - z_0|}$$

is thus obtained for  $n \geq n_0$ , which proves that it is enough to assume that the sequence  $(f_n(u))$  converges to  $f(u)$  uniformly on the circle  $|u - z_0| = r$  in order to imply the uniform convergence of the sequence  $(f_n(z))$  to  $f(z)$  in every disc

$$|z - z_0| \leq r' \quad \text{where } r' < r.$$

Similarly, starting from the relation (7.5.1)

$$f_n^{(k)}(z) = \frac{k!}{2\pi i} \int_{\gamma} \frac{f_n(u) du}{(u - z)^{k+1}}$$

and the analogous formula for  $f^{(k)}(z)$ , we obtain the majorization

$$(10.1.4) \quad |f^{(k)}(z) - f_n^{(k)}(z)| \leq \frac{k! \varepsilon r}{(r - |z - z_0|)^{k+1}}$$

for  $n \geq n_0$ , which proves the uniform convergence of  $f_n^{(k)}(z)$  to  $f^{(k)}(z)$  in every disc  $|z - z_0| \leq r'$  for  $r' < r$ .

The Weierstrass convergence theorem generalizes the fact that a power series is an analytic function in its disc of convergence (VI, 5.2). Similarly, the following result generalizes (7.2):

(10.2) *Let  $\gamma: \mathbf{I} = [a, b] \rightarrow \mathbf{C}$  be a path in  $\mathbf{C}$ ; let  $\mathbf{D}$  be an open set in  $\mathbf{C}$  and suppose we are given a complex function  $(z, u) \rightarrow g(z, u)$  in  $\mathbf{D} \times \gamma(\mathbf{I})$ , having the following properties:*

1. *For every  $u \in \gamma(\mathbf{I})$ , the function  $z \rightarrow g(z, u)$  is analytic in  $\mathbf{D}$ .*
2. *The functions  $(z, u) \rightarrow g(z, u)$  and  $(z, u) \rightarrow (\partial g / \partial z)(z, u)$  are continuous in  $\mathbf{D} \times \gamma(\mathbf{I})$  (note that there is not necessarily any relationship between the sets  $\mathbf{D}$  and  $\gamma(\mathbf{I})$  in  $\mathbf{C}$ ).*

*Under these conditions the function*

$$(10.2.1) \quad f(z) = \int_{\gamma} g(z, u) du$$

*is analytic in  $\mathbf{D}$ .*

If we put

$$\mathbf{R}(x, y) = f(x + iy) = \int_a^b g(x + iy, \gamma(t)) \gamma'(t) dt,$$

the properties of integrals depending on a parameter show that  $\mathbf{R}$  is continuous and has continuous partial derivatives of the first order  $\partial \mathbf{R} / \partial x$ ,  $\partial \mathbf{R} / \partial y$ , and that

$$\frac{\partial \mathbf{R}}{\partial x} + i \frac{\partial \mathbf{R}}{\partial y} = \int_a^b \left( \frac{\partial}{\partial x} g(x + iy, \gamma(t)) + i \frac{\partial}{\partial y} g(x + iy, \gamma(t)) \right) \gamma'(t) dt = 0$$

which proves (10.2) by virtue of the Cauchy conditions (9.4.2).

(10.3) Weierstrass's theorem (10.1) enables us to extend (10.2) to more general integrals. A *path without endpoints* in  $\mathbf{C}$  is a mapping  $\gamma: \mathbf{I} = ]a, b[ \rightarrow \mathbf{C}$  defined in an *open* interval  $\mathbf{I}$  of  $\mathbf{R}$  (bounded or not) and such that for each *closed* interval  $[c, d]$  contained in  $\mathbf{I}$ , the restriction of  $\gamma$  to  $[c, d]$  is a path (in the sense defined in section 1). *It is not supposed* that  $\gamma(t)$  tends to a limit as  $t$  tends to  $a$  or  $b$ . If  $f$  is a complex function *continuous* in  $\gamma(\mathbf{I})$ , the *integral of  $f$  along the path without endpoints  $\gamma$*  is the *improper integral* (III, 9.7)  $\int_{\gamma} f(\gamma(t)) \gamma'(t) dt$  (when it exists), and is again denoted by  $\int_{\gamma} f(z) dz$ .

(10.4) *Let  $\gamma: \mathbf{I} = ]a, b[ \rightarrow \mathbf{C}$  be a path without endpoints in  $\mathbf{C}$ ; let  $\mathbf{D}$  be an open set in  $\mathbf{C}$  and suppose given a complex function  $(z, u) \rightarrow g(z, u)$  in  $\mathbf{D} \times \gamma(\mathbf{I})$ , having the properties (1) and (2) of (10.2) and also satisfying the following condition:*

3. *Given any closed disc  $\Delta \subset \mathbf{D}$ , a closed interval  $\mathbf{J} \subset \mathbf{I}$  and a number  $\varepsilon > 0$ , there exists a closed*

interval  $[c_0, d_0] \subset I$  containing  $J$ , such that, if  $\gamma_0$  is the restriction of  $\gamma$  to  $[c_0, d_0]$ , we have, for every  $z \in \Delta$ ,

$$\left| \int_{\gamma} g(z, u) du - \int_{\gamma_0} g(z, u) du \right| \leq \varepsilon.$$

Under these conditions the function  $f(z) = \int_{\gamma} g(z, u) du$  is analytic in  $D$ .

By applying the hypothesis with  $\varepsilon = 1/n$  for all integers  $n > 1$ , we obtain a sequence of paths  $\gamma_n: [c_n, d_n] \rightarrow \mathbf{C}$ , which are restrictions of  $\gamma$  to the intervals  $[c_n, d_n] \subset I$ , where  $c_n$  tends to  $a$  and  $d_n$  tends to  $b$ , such that

$$\left| \int_{\gamma} g(z, u) du - \int_{\gamma_n} g(z, u) du \right| \leq 1/n$$

for every  $z \in \Delta$ . Since by virtue of (10.2) the function  $f_n(z) = \int_{\gamma_n} g(z, u) du$  is analytic in  $D$  and converges to  $f(z)$  uniformly in  $\Delta$ ,  $f$  is analytic in  $D$  by virtue of (10.1).

*Example (10.4.1)* The Eulerian integral of the second kind

$$(10.4.2) \quad \Gamma(z) = \int_0^{+\infty} e^{(z-1) \log t - t} dt$$

is defined, not only for  $z$  real and  $> 0$ , but also for  $\Re z > 0$ , since for  $z = x + iy$  and  $x > 0$ ,  $|e^{(z-1) \log t} e^{-t}| = t^{x-1} e^{-t}$  so the improper integral is absolutely convergent. The functions  $(z, t) \rightarrow e^{(z-1) \log t - t}$  and  $(z, t) \rightarrow \log t \cdot e^{(z-1) \log t - t}$  are continuous in the product of the half-plane  $D: \Re z > 0$  and of  $I = ]0, +\infty[$ , and for every  $t \in I$ ,  $z \rightarrow e^{(z-1) \log t - t}$  is analytic in  $D$ . Finally, for  $0 < h < 1$  and  $\Re z \geq a > 0$ ,

$$\left| \int_0^h e^{(z-1) \log t - t} dt \right| \leq \int_0^h t^{a-1} dt = \frac{1}{a} h^a$$

which (for  $a$  fixed) is arbitrarily small with  $h$ ; similarly, for  $N > 1$  and  $\Re z \leq b$

$$\left| \int_N^{+\infty} e^{(z-1) \log t - t} dt \right| \leq \int_N^{+\infty} t^{b-1} e^{-t} dt$$

which (for  $b$  fixed) is arbitrarily small with  $1/N$ . Condition (3) of (10.4) is thus satisfied, and  $\Gamma(z)$  is analytic for  $\Re z > 0$ .

*Remarks (10.5)* Under the hypotheses of (10.1), suppose that for every closed disc  $\Delta \subset D$ , the sets  $f_n(\Delta)$  are all contained in the same bounded and closed set  $F \subset \mathbf{C}$ , and let  $u$  be a function analytic in an open set  $E$  containing  $F$ . Then the sequence of composed functions  $u \circ f_n$  (analytic in  $D$ ) converges uniformly in every closed disc  $\Delta \subset D$  by virtue of (V, 2.6) and has for a limit the analytic function  $u \circ f$ .

(10.6) The Weierstrass convergence theorem (10.1) and the Weierstrass approximation theorem (V, 5.2) appear at first sight to be contradictory, since the first says that a uniform limit of polynomials is analytic and the second that a uniform limit of polynomials can be any continuous function whatever. The reason for this apparent paradox is that in (10.1) the uniform convergence is required to take place in an open non-empty subset of  $\mathbf{C}$ , whereas in (V, 5.2) only uniform convergence on a line segment is required: off this segment, the sequence is in general not even convergent.

## PROBLEMS

1. Let  $F$  be the union in the plane  $\mathbf{C}$  of a finite number of half-lines mutually disjoint, each being defined by equation of the form  $\mathcal{I}z = \beta_j$ ,  $\Re z \leq \alpha_j$  or  $\mathcal{I}z = \beta_j$ ,  $\Re z \geq \alpha_j$ . Show that the complement  $\mathbf{C} - F$  is simply connected (reason by induction on the number of half-lines).

2. If  $\gamma$  is a path in an open set  $D \subset \mathbf{C}$ , show that the loop  $\gamma \vee \gamma^0$  is homotopic to a point in  $D$ .

3. Let  $f$  be a function defined and bounded in the closed disc  $|z| \leq 1$ , analytic in the open disc  $|z| < 1$  and continuous in  $z \leq 1$ , except at a finite numbers of points. Show that

$$\frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) d\theta = f(0).$$

(Majorize the difference  $\int_0^{2\pi} (f(e^{i\theta}) - f(re^{i\theta})) d\theta$  when  $r$  tends to 1, by dividing the interval of integration at the points of discontinuity of  $\theta \rightarrow f(e^{i\theta})$  and use the uniform continuity of  $f$  in every closed bounded set not containing these points of discontinuity.)

Deduce from this that  $|f(0)| \leq (1/2\pi) \int_0^{2\pi} |f(e^{i\theta})| d\theta$  and that if  $f$  is analytic in an open set containing the disc  $|z| \leq 1$ , there is equality only if  $f$  is constant in the disc  $|z| \leq 1$  (use VI, problem 11).

4. Let  $f_1, \dots, f_r$  be analytic functions in a connected open set  $D \subset \mathbf{C}$ ; show that the function  $\varphi(z) = |f_1(z)| + |f_2(z)| + \dots + |f_r(z)|$  can attain a relative maximum at a point of  $D$  only if all the  $f_j$  are constant in  $D$ . (Observe that for  $r$  sufficiently small

$$\varphi(z_0) \leq \frac{1}{2\pi} \int_0^{2\pi} \varphi(z_0 + re^{i\theta}) d\theta$$

and that the equality can occur only if for every  $j$ , the function  $f_j$  satisfies the relation  $|f_j(z_0)| = (1/2\pi) \int_0^{2\pi} |f_j(z_0 + re^{i\theta})| d\theta$  (problem 3).)

Deduce from this that if there exists an analytic function such that

$$|f(z)| = |f_1(z)| + |f_2(z)| + \dots + |f_r(z)|$$

in  $D$ , the functions  $f_j$  and  $f$  are all proportional (consider the functions  $f_j/f$ ).

5. Let

$$f(z) = a_0 + a_1z + \dots + a_nz^n + \dots$$

be a convergent power series in the disc  $|z| < 1$ . Suppose that  $|f(z)| < 1/(1 - |z|)$ . Show that

$$|a_n| \leq \left(1 + \frac{1}{n}\right)^n (n+1) < e(n+1).$$

6. Let

$$f(z) = a_0 + a_1z + \dots + a_nz^n + \dots$$

be a convergent power series in the disc  $|z| < 1$ . Suppose that on some circle  $|z| = r < 1$ , we have  $|f(z)| \leq M$ . Show that if  $f(z_0) = 0$  for some  $z_0$  such that  $|z_0| < r$ , then

$$|z_0| \geq \frac{|a_0|r}{M + |a_0|}.$$

7. For every complex function  $f(x, y)$  of two real variables defined in an open set  $D \subset \mathbf{C}$ , let  $F(z, \bar{z}) = f(\frac{1}{2}(z + \bar{z}), (1/2i)(z - \bar{z}))$ . For  $f$  to be analytic in  $D$ , it is necessary and sufficient that  $\partial F / \partial \bar{z} = 0$ .

8. If  $f$  is an analytic function in  $D$  and if we put  $F(x, y) = |f(x + iy)|^2$ , show that

$$\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} = 4|f'(z)|^2.$$

Deduce from this that there exists no pair of non-constant analytic functions  $f, g$ , such that  $\Re g(z) = |f(z)|$ .

9. Let  $f(z) = a_0 + a_1 z + \cdots + a_n z^n + \cdots$  be a power series convergent in the disc  $|z| < R$ , and put, for  $0 < r < R$  and  $0 \leq \theta \leq 2\pi$

$$U(r, \theta) = \Re f(re^{i\theta}), \quad V(r, \theta) = \Im f(re^{i\theta}).$$

Show that for  $n \geq 1$

$$a_n = \frac{1}{\pi r^n} \int_0^{2\pi} U(r, \theta) e^{-ni\theta} d\theta = \frac{i}{\pi r^n} \int_0^{2\pi} V(r, \theta) e^{-ni\theta} d\theta.$$

Deduce from this that, if  $f(0)$  is *real*, then, for  $|z| < r$

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} U(r, \theta) \frac{r + ze^{-i\theta}}{r - ze^{-i\theta}} d\theta.$$

10. (a) Let  $f(z) = \frac{1}{2} + a_1 z + \cdots + a_n z^n + \cdots$  be a power series convergent in the disc  $|z| < 1$ . Show that if  $\Re f(z) \geq 0$  for  $|z| < 1$ , then  $|a_n| \leq 1$  for every  $n \geq 1$  (use problem 9).

(b) Let  $f(z) = a_0 + a_1 z + \cdots + a_n z^n + \cdots$  be a power series convergent in the disc  $|z| < R$ , and suppose that  $\Re f(z) \leq A$  in this disc. Show that for  $0 \leq r < R$

$$|a_0| + |a_1| r + \cdots + |a_n| r^n + \cdots \leq |a_0| + \frac{2r}{R-r} (A - \Re a_0)$$

(reduce to the case (a) by considering the function  $\alpha + \beta f(Rz)$  for suitable constants  $\alpha, \beta$ ).

11. Give an example of a power series  $\sum_{n=0}^{\infty} a_n z^n$  which converges absolutely and uniformly in the closed disc  $|z| \leq 1$ , but whose derived series  $\sum_{n=0}^{\infty} n a_n z^{n-1}$  does not converge uniformly in the open disc  $|z| < 1$ .

12. Show that for  $-1 \leq x \leq 0$ , the power series  $\sum_{n=0}^{\infty} x^n/n$  is uniformly convergent, but that the series of the absolute values is not uniformly convergent in the half-open interval  $-1 < x \leq 0$ .

13. Find the set of  $z \in \mathbf{C}$  in which each of the following series converges

$$\begin{aligned} \sum_{n=0}^{\infty} \left( z^n + \frac{1}{2^n z^n} \right), & \quad \sum_{n=1}^{\infty} \left( \frac{z(z+n)}{n} \right)^n, & \quad \sum_{n=1}^{\infty} \frac{\sin n z}{n} \\ \sum_{n=0}^{\infty} \frac{2^n}{z^{2n} + 1}, & \quad \sum_{n=1}^{\infty} \frac{z^n}{1 - z^n}, & \quad \sum_{n=0}^{\infty} \frac{z^n}{1 + z^{2n}}. \end{aligned}$$

(For the two last series, use the fact that if  $\theta$  is an irrational number, there exists an increasing sequence  $(n_k)$  of integers such that  $\lim_{k \rightarrow \infty} (n_k \theta - [n_k \theta]) = 0$ .) Where do these series converge uniformly?

14. Show that if the series  $a_1 + a_2 + \cdots + a_n + \cdots$  is convergent, the series  $\sum_{n=1}^{\infty} \frac{a_n z^n}{1 - z^n}$  is convergent for every  $z$  such that  $|z| \neq 1$ . If on the contrary the series  $a_1 + a_2 + \cdots + a_n + \cdots$  is not convergent, and if  $R \leq 1$  is the radius of convergence of the power series  $\sum_{n=1}^{\infty} a_n z^n$ , then the series  $\sum_{n=1}^{\infty} \frac{a_n z^n}{1 - z^n}$  is convergent for  $|z| < R$  and not convergent for  $|z| > R$ . (Observe that because of the Abel lemma, if the series of general term  $a_n z_0^n / (1 - z_0^n)$  is convergent for  $|z_0| > 1$ , then so is the series of general term  $a_n / (1 - z_0^n)$ .)

Where is the series  $\sum_{n=1}^{\infty} \frac{a_n z^n}{1 - z^n}$  uniformly convergent?

If  $f(z) = a_1 z + a_2 z^2 + \cdots + a_n z^n + \cdots$ , show that for  $|z| < R$ , the series of general term  $f(z^k)$  ( $k \geq 1$ ) is absolutely convergent and has the sum  $\sum_{n=1}^{\infty} \frac{a_n z^n}{1 - z^n}$ , which is itself equal to the power series

$$c_1 z + c_2 z^2 + \cdots + c_n z^n + \cdots$$

where for  $n \geq 1$ ,  $c_n = \sum_{d|n} a_d$ , where  $d$  runs through the set of the integers  $\geq 1$  which divide  $n$ .

15. The series

$$\frac{1}{1-z} + \frac{z}{1-z^2} + \frac{z^2}{1-z^4} + \cdots + \frac{z^{2^n-1}}{1-z^{2^n}} + \cdots$$

is convergent for  $|z| < 1$  and for  $|z| > 1$ ; its sum is equal to 1 for  $|z| < 1$  and to 0 for  $|z| > 1$ .

16. Show that the series of general term  $(-1)^n / (z + n)$  converges for every  $z \in \mathbf{C}$  distinct from the integers  $-n$  ( $n \in \mathbf{N}$ ), and converges uniformly in every closed bounded set not containing any of these points, but converges absolutely at no point of  $\mathbf{C}$ .

17. (a) Show that the series of general term  $z / (1 + z^2)^n$  is absolutely convergent in the set  $S$  of points  $z + re^{i\theta}$  with  $r \geq 0$  and  $|\theta| \leq \pi/4$ , but is not uniformly convergent in this set.

(b) Show that in  $S$  the series of general term  $(-1)^n z / (1 + z^2)^n$  is absolutely and uniformly convergent, but that the series of general term  $|z / (1 + z^2)^n|$  is not uniformly convergent.

18. Let  $(\lambda_n)$  be a strictly increasing sequence of real numbers such that  $\lim_{n \rightarrow \infty} \lambda_n = +\infty$ . Show that if a series  $\sum_{n=0}^{\infty} a_n e^{-\lambda_n z}$  ("Dirichlet series") converges (resp. converges absolutely) at a point  $z_0 = x_0 + iy_0$ , it converges (resp. converges absolutely) for every  $z = x + iy$  such that  $x > x_0$ . Also that it is uniformly convergent in the sector formed by the points  $z = z_0 + re^{i\theta}$  where  $r > 0$  and  $|\theta| \leq \alpha$ ,  $\alpha$  being a number such that  $0 < \alpha < \pi/2$ . (Write

$$a_n e^{-\lambda_n z} = a_n e^{-\lambda_n z_0} e^{-\lambda_n (z - z_0)};$$

use problem 3(a) of Chap. VI, as well as the inequality, for  $a < b$  real

$$|e^{-az} - e^{-bz}| = |z \int_a^b e^{-zt} dt| \leq \frac{|z|}{x} (e^{-ax} - e^{-bx}).$$

Case where  $a_n \geq 0$ .

19. Find for which values of  $z$  the Dirichlet series

$$\sum_{n=1}^{\infty} (-1)^n e^{-z \log n}, \quad \sum_{n=2}^{\infty} \frac{(-1)^n}{n} e^{-z \log \log n}$$

$$\sum_{n=2}^{\infty} (-1)^n e^{-z \log \log n}, \quad \sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n}} e^{-z \log \log n}$$

are convergent (resp. absolutely convergent).

20. For which values of  $z \in \mathbf{C}$  are the following integrals convergent, and where do they define analytic functions of  $z$ ?

$$\int_0^{+\infty} \frac{\sin t}{t^z} dt, \quad \int_0^{+\infty} \frac{\cos t}{t^z} dt, \quad \int_0^{+\infty} \frac{\sin tz}{t} dt$$

$$\int_c^{c+i\infty} \frac{e^{zt}}{t} dt, \quad \int_{c-i\infty}^{c+i\infty} \frac{e^{zt}}{t} dt, \quad \int_{c-i\infty}^{c+i\infty} \frac{z^t}{t} dt \quad (c \text{ real } \neq 0)$$

(In the three last integrals, the notation means that the path along which the integral is taken is in the first case the half-line  $t \rightarrow c + it$  ( $t \geq 0$ ), and in the remaining two cases the line  $t \rightarrow c + it$  ( $t \in \mathbf{R}$ ). For the definition of  $z^t = e^{t \log z}$ , see VIII, 9.6.)

21. Put  $g(z) = \int_0^z e^{-u^2/2} du$  (primitive of  $\exp(-z^2/2)$  in  $\mathbf{C}$ ).

(a) Show that for  $-\pi/4 \leq \theta \leq \pi/4$  (resp.  $3\pi/4 \leq \theta \leq 5\pi/4$ ),  $g(re^{i\theta})$  tends uniformly to  $\sqrt{\pi}/2$  (resp.  $-\sqrt{-\pi}/2$ ) as  $r$  tends to  $+\infty$ .

(b) Show that there exists a decreasing sequence  $(a_m)_{m \geq 0}$  of numbers  $> 0$  such that the series

$$f(z) = a_0 g(z) + a_1 g(z^8) + a_2 g(z^{64}) + \cdots + a_m g(z^{8^m}) + \cdots$$

is uniformly convergent in every disc  $|z| \leq R$ , and hence is an entire function. The sequence  $(a_m)$  can also be taken such that for every integer  $m$

$$a_{m+1} + a_{m+2} + \cdots \leq \frac{1}{2} a_m;$$

when this is the case, for every sequence  $(\varepsilon_m)_{m \geq 0}$  whose terms are equal to  $+1$  or to  $-1$ , the series  $\sum_{m=0}^{\infty} \varepsilon_m a_m$  is convergent, and two such series corresponding to different sequences  $(\varepsilon_m)$  have different sums.

Put  $\delta_m = 2(1 - \varepsilon_m)$  and  $\theta = 2 \sum_{m=0}^{\infty} \frac{\delta_m}{8^{m+1}}$ . Show by using (a), that as  $r$  tends to  $+\infty$ ,  $f(re^{i\theta})$  tends to  $\sum_{m=0}^{\infty} \varepsilon_m a_m$  (examine the value of  $8^m \theta$ ).

22. Let  $f$  be a complex function indefinitely differentiable in an interval  $]x_0 - c, x_0 + c[$  of  $\mathbf{R}$ . Show that in order that there exist in  $\mathbf{C}$  a disc  $|z - x_0| < r < c$  and an analytic function  $g$  in this disc such that  $g(x) = f(x)$  in the interval  $]x_0 - r, x_0 + r[$ , it is necessary and sufficient that there exist a number  $b < c$ , an integer  $k \geq 0$  and a number  $A \geq 0$  such that, for  $x_0 - b \leq x \leq x_0 + b$  and for every integer  $n \geq 0$

$$|f^{(n)}(x)| \leq A^n (n + k)!$$

(To see that the condition is necessary, apply the Cauchy inequalities; to see that it is sufficient, majorize the remainder of the Taylor formula for  $f$  at the point  $x_0$ .)

# Singular points of analytic functions: residues

## 1. Analytic continuation and singularities

Let  $f$  be a function analytic in an open connected set  $D \subset \mathbf{C}$ ; if  $z_0$  is a point of  $D$ , we know that the Taylor series of  $f$  at the point  $z_0$  converges in the largest open disc  $\Delta: |z - z_0| < r$  of centre  $z_0$  contained in  $D$  (VII, 7.3). However, it may happen (and this is frequently the case) that the *disc of convergence*  $\Delta_0$  (V, 2.3) of this series is *larger than*  $\Delta$  (Fig. 34). Since the boundary  $|z - z_0| = r$  of  $\Delta$  contains at least one point  $z_1$  of the

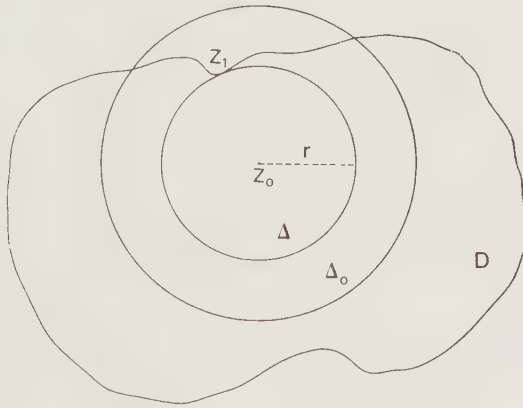


FIGURE 34

boundary of  $D$  (0, 5.5), there exists a function  $g$ , analytic in the open set  $\Delta_0$  containing  $z_1$ , which coincides with  $f$  in the open set  $\Delta_0 \cap D$  (not empty since  $z_1$  is a boundary point of  $D$ ).

Thus one distinguishes between two kinds of boundary points  $z_1$  of  $D$ . We say that  $z_1$  is a *regular point* (of  $f$ ) if there exists an open connected set  $\Delta_0$  containing  $z_1$  and a function  $g$  analytic in  $\Delta_0$ , which coincides with  $f$  in an open set  $D_1 \subset D \cap \Delta_0$ , of which  $z_1$  is a boundary point. Otherwise  $z_1$  is a *singular point* of  $f$ .

It is tempting to think that when  $z_1$  is a regular boundary point of  $D$  (for  $f$ ), it is possible (with the preceding notations) to *continue* the analytic function  $f$  into the open

set  $D \cup \Delta_0$ , which is larger than  $D$ , by considering the function which is equal to  $f$  in  $D$  and to  $g$  in  $\Delta_0$ . This is certainly possible when  $f$  and  $g$  coincide in the whole of  $D \cap \Delta_0$ ; it is automatically the case if  $D \cap \Delta_0$  is connected, by the principle of analytic continuation (VI, 7.3).

However it may happen that  $z_1$  is a regular point of  $f$ , that the intersection  $D \cap \Delta_0$  is not connected (Fig. 35), and that  $f$  and  $g$  do not coincide in the whole of this intersection, although

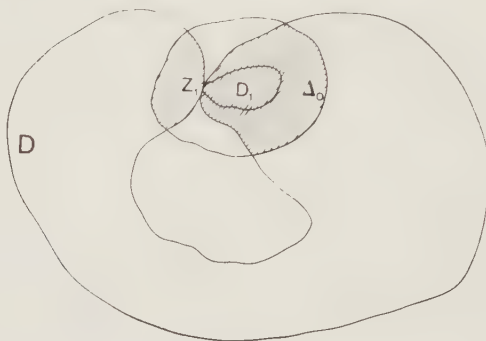


FIGURE 35

(by definition) they coincide in an open connected subset of  $D \cap \Delta_0$  of which  $z_1$  is a boundary point. Since a function can have only *one* value at each point, it is not possible to define a function in  $D \cup \Delta_0$  as suggested above. We shall see (8.7) examples of this phenomenon, where it is not possible to continue  $f$  no matter how small the open set  $\Delta_0$  containing the regular point  $z_1$ .

Thus the possibility of continuing  $f$  analytically into an open set larger than  $D$  depends on both  $f$  and the geometry of  $D$ . For example, if  $D$  is an open disc, an open half-plane or the interior of a rectangle,  $D \cap \Delta_0$  is *connected* for every *open disc*  $\Delta_0$  with centre *any* boundary point of  $D$ , and if such a point is regular for  $f$ ,  $f$  can be continued into  $D \cup \Delta_0$  for a sufficiently small disc  $\Delta_0$ . The following proposition is a consequence of this fact:

(1.1) *Let  $f(z) = c_0 + c_1z + \cdots + c_nz^n + \cdots$  be a power series in  $z$ , whose radius of convergence  $R$  is  $>0$  and finite. Then there exists at least one point on the boundary of the disc of convergence  $D: |z| < R$  of the series which is a singular point of  $f$ .*

If this were false, then for each boundary point  $u$  of  $D$  (satisfying  $|u| = R$ ), there would exist an open disc  $\Delta_u$  of centre  $u$  and radius  $>0$ , such that there is an analytic function  $g_u$  in  $\Delta_u$  coinciding with  $f$  in  $D \cap \Delta_u$ . Furthermore, for two distinct  $u, u'$  satisfying  $|u| = |u'| = R$  with  $\Delta_u \cap \Delta_{u'}$  non-empty, the functions  $g_u$  and  $g_{u'}$  coincide in  $\Delta_u \cap \Delta_{u'}$ . For the intersection  $D \cap \Delta_u \cap \Delta_{u'}$  is an open non-empty set (Fig. 36) in which by hypothesis the functions  $g_u$  and  $g_{u'}$  coincide with  $f$ , and since  $\Delta_u \cap \Delta_{u'}$  is connected, the conclusion follows from the principle of analytic continuation (VI, 7.3). Thus let  $D'$  be the open set which is the union of  $D$  and all the  $\Delta_u$  as  $u$  describes the circle  $|u| = R$ . In  $D'$  a function  $g$  can be defined equal to  $f$  in  $D$  and to  $g_u$  in each disc  $\Delta_u$ ; by the preceding *just one* value is obtained for  $g$  at each point of  $D'$ , and so a function in  $D'$  has indeed been defined which is clearly analytic and a continuation of  $f$ . By definition, the

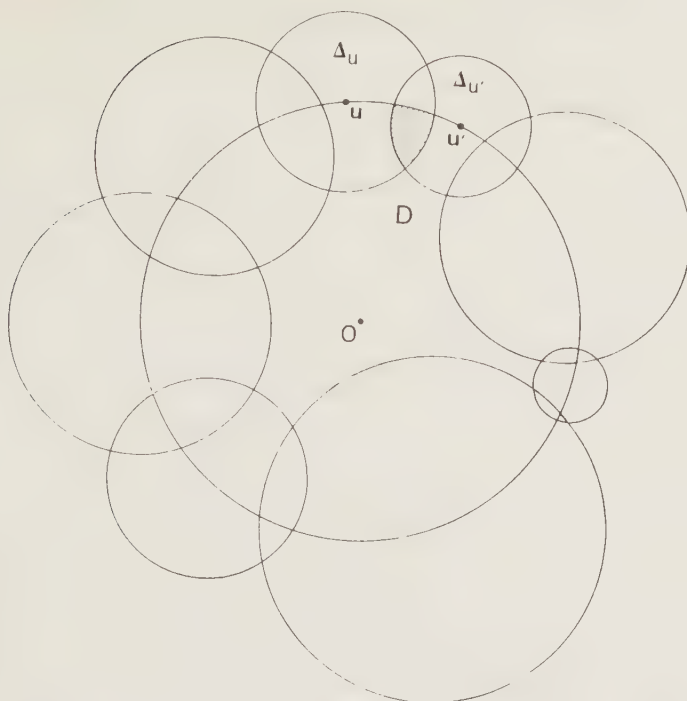


FIGURE 36

points of the circle  $|u| = R$  all belong to  $D'$ , therefore the largest open disc  $D_0$  of centre 0 contained in  $D'$  has radius  $R_0 > R$  (0, 5.6). The Taylor series of  $g$  at the point 0 must be convergent in  $D_0$  (VII, 7.3); however this series is the same as that of  $f$ , giving a contradiction, which proves the proposition.

(1.2) If  $f$  is an analytic function in an open connected set  $D$ , it may happen that *all the boundary points* of  $D$  are *singular points* of  $f$  (problem 2); in this case there exists *no* analytic continuation of  $f$  into an open set larger than  $D$ . It is sometimes said that  $D$  is the “natural domain of existence” of the function  $f$ . However, usually some of the boundary points of  $D$  are regular and some are singular; we return later (8.7) to the difficulties caused by the problem of analytic continuation. Meanwhile, we shall study in detail a particularly simple case, that of *isolated* boundary points.

## 2. Isolated singular points: Laurent series

(2.1) Let  $D$  be an open set in  $\mathbf{C}$ ; a boundary point  $a$  of  $D$  is said to be *isolated* if there exists an open disc  $\Delta: |z - a| < r$  of centre  $a$  *all of whose points, except  $a$ , belong to  $D$* . It can be seen that this is the same as saying that in this disc there are no boundary points of  $D$  distinct from  $a$ . We propose to study the behaviour of the analytic functions in  $D$  in the

$$(2.3.1) \quad f(x) = \frac{1}{2\pi i} \int_{\gamma'} \frac{f(z) dz}{z - x} - \frac{1}{2\pi i} \int_{\gamma} \frac{f(z) dz}{z - x}.$$

Again define in  $S$  the function  $g(z)$  as in (VII, 7.1)

$$\begin{cases} g(z) = \frac{f(z) - f(x)}{z - x} & \text{for } z \neq x \\ g(x) = f'(x) \end{cases}$$

The same reasoning as in (VII, 7.1) shows that  $g$  is analytic in  $S$ ; applying (2.2) to this function gives

$$\int_{\gamma} \frac{f(z) dz}{z - x} - f(x) \int_{\gamma} \frac{dz}{z - x} = \int_{\gamma'} \frac{f(z) dz}{z - x} - f(x) \int_{\gamma'} \frac{dz}{z - x}.$$

But by virtue of the hypothesis on  $x$ ,

$$\int_{\gamma} \frac{dz}{(z - x)} = 0 \quad \text{and} \quad \int_{\gamma'} \frac{dz}{z - x} = 2\pi i$$

(VII, 6.4), hence the formula (2.3.1).

From this a series development for  $f(z)$  is deduced which replaces the Taylor development:

(2.4) *With the notations of (2.2) we have, for every function  $f$  analytic in  $S$  and every  $z \in S$*

$$(2.4.1) \quad f(z) = \sum_{n=0}^{\infty} c_n (z - a)^n + \sum_{n=1}^{\infty} \frac{d_n}{(z - a)^n}$$

where the power series  $\sum_{n=0}^{\infty} c_n (z - a)^n$  is convergent for  $|z - a| < r_2$ , the power series (in  $1/(z - a)$ )  $\sum_{n=1}^{\infty} \frac{d_n}{(z - a)^n}$  is convergent for  $|z - a| > r_1$ , and the coefficients are given by the formulae

$$(2.4.2) \quad c_n = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z) dz}{(z - a)^{n+1}}, \quad d_n = \frac{1}{2\pi i} \int_{\gamma} f(z) (z - a)^{n-1} dz$$

valid for every closed path  $\gamma: t \rightarrow a + re^{it}$  ( $0 \leq t \leq 2\pi$ ) with  $r_1 < r < r_2$  ("Laurent development" of  $f$  in  $S$ ).

Let  $z$  be any point of  $S$ ; since  $r_1 < |z - a| < r_2$ , two numbers  $r, r'$  can be found such that  $r_1 < r < |z - a| < r' < r_2$ . It follows from (VII, 7.2) that

$$\frac{1}{2\pi i} \int_{\gamma'} \frac{f(u) du}{u - z} = \sum_{n=0}^{\infty} c_n (z - a)^n$$

with

$$c_n = \frac{1}{2\pi i} \int_{\gamma'} \frac{f(u) du}{(u - a)^{n+1}}$$

the series being convergent for  $|z - a| < r'$ .

On the other hand, for  $|u - a| = r$

$$(2.4.3) \quad \frac{1}{u - z} = -\frac{1}{z - a} \left( \frac{1}{1 - \frac{u - a}{z - a}} \right) \\ = - \left( \frac{1}{z - a} + \dots + \frac{(u - a)^{n-1}}{(z - a)^n} + \dots \right)$$

where the series is convergent with

$$(2.4.4) \quad \left| \frac{(u - a)^{n-1}}{(z - a)^n} \right| = \frac{r^{n-1}}{|z - a|^n} = \frac{1}{r} \left| \frac{r}{z - a} \right|^n.$$

Since the function  $f$  is continuous (therefore bounded) on the circle  $|u - a| = r$ , the series with general term

$$\frac{r^n f(a + re^{it}) e^{nit}}{(z - a)^n}$$

is *normally convergent* (V, 2.5) for  $0 \leq t \leq 2\pi$  by virtue of (2.4.4). Applying (V, 3.5)

$$-\frac{1}{2\pi i} \int_{\gamma} \frac{f(u) du}{u - z} = \sum_{n=1}^{\infty} \frac{d_n}{(z - a)^n}$$

with

$$d_n = \frac{1}{2\pi i} \int_0^{2\pi} i r^n f(a + re^{it}) e^{nit} dt = \frac{1}{2\pi i} \int_{\gamma} f(u) (u - a)^{n-1} du,$$

the series being convergent for  $|z - a| > r$ . Taking into account (2.2) and the fact that the functions  $f(z)/(z - a)^{n+1}$  and  $f(z)(z - a)^{n-1}$  are analytic in  $S$ , it is seen that in the expressions given above for  $c_n$  and  $d_n$  the radii  $r$  and  $r'$  can be replaced by any number  $r''$  such that  $r_1 < r'' < r_2$ . Therefore the series  $\sum_{n=0}^{\infty} c_n (z - a)^n$  and  $\sum_{n=1}^{\infty} d_n (z - a)^{-n}$  converge for  $|z - a| < r_2$  and  $|z - a| > r_1$  respectively. The formula (2.4.1) follows from the preceding and from (2.3.1).

(2.5) Note that there is *only one* development of  $f(z)$  in the form (2.4.1) with two power series in  $z - a$  and  $1/(z - a)$  respectively, convergent for  $r_1 < |z - a| < r_2$ . For if we have such a development

$$f(z) = \sum_{n=0}^{\infty} c'_n (z - a)^n + \sum_{n=1}^{\infty} \frac{d'_n}{(z - a)^n}$$

the assumed convergence of the two power series implies, by (VI, 2.2), that for  $r_1 < r' < r'' < r_2$ , the series  $\sum_{n=1}^{\infty} c'_n (z - a)^n$  is normally convergent for  $|z - a| < r''$  and the series  $\sum_{n=1}^{\infty} d'_n (z - a)^{-n}$  is normally convergent for  $|z - a| > r'$ . For each number  $r$  such

that  $r_1 < r < r_2$  and each integer  $m$  (positive or negative) therefore, by virtue of (V 3.5),

$$\int_{\gamma} f(z)(z-a)^m dz = \sum_{n=0}^{\infty} c'_n \int_{\gamma} (z-a)^{m+n} dz + \sum_{n=1}^{\infty} d'_n \int_{\gamma} (z-a)^{m-n} dz$$

where  $\gamma$  is defined as in (2.4). But

$$(2.5.1) \quad \int_{\gamma} (z-a)^p dz = r^{p+1} i \int_0^{2\pi} e^{(p+1)it} dt = \begin{cases} 0 & \text{if } p \neq -1 \\ 2\pi i & \text{if } p = -1 \end{cases}$$

so, because of (2.4.2), we obtain  $c'_n = c_n$  for  $n \geq 0$  and  $d'_n = d_n$  for  $n \geq 1$ .

### 3. Behaviour of an analytic function in the neighbourhood of an isolated singularity

(3.1) We revert to the study of a function  $f$  analytic in a “punctured disc”  $\Delta - \{a\}$ :  $0 < |z-a| < r$ . Because of the preceding we can associate with the function its Laurent development (2.4.1), where the two series converge in  $\Delta - \{a\}$ . The function

$$(3.1.1) \quad u(z) = \sum_{n=1}^{\infty} \frac{d_n}{(z-a)^n}$$

is called the *singular part* of  $f$  at the point  $a$ ; since this series converges for  $z-a \neq 0$ , the function

$$(3.1.2) \quad v(x) = u\left(a + \frac{1}{x}\right) = \sum_{n=1}^{\infty} d_n x^n$$

is an *entire function* of the complex variable  $x$ . The function  $f(z) - u(z)$  is thus the restriction to  $\Delta - \{a\}$  of a function *analytic in*  $\Delta$ . The isolated boundary points (for given  $f$ ) are classified according to the nature of the corresponding singular part:

1. Suppose first that  $u$  is identically zero, i.e.  $d_n = 0$  for every  $n \geq 1$ . Then for  $0 < |z-a| < r$

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n.$$

But the function of the second member is *analytic in the whole of the open disc*  $\Delta$ :  $|z-a| < r$  and so *continues*  $f$  into  $\Delta$ . Conversely, if  $f$  can be continued analytically into  $\Delta$ , Cauchy's theorem (VIII, 5.1) shows, granted the expression (2.4.2) for  $d_n$ , that  $d_n = 0$  for every  $n \geq 1$ . The case under examination is therefore that where  $a$  is a *regular* boundary point of  $f$  (Section 1). When  $f$  is not identically zero, the smallest number  $m \geq 0$  such that  $c_m \neq 0$  is called the *order of*  $f$  at the point  $a$  and denoted by  $\omega(a; f)$ . To say that  $\omega(a; f) = 0$  signifies that  $f$  (more precisely the continuation of  $f$  into  $\Delta$ ) does not vanish at the point  $a$ . If  $\omega(a; f) = m \geq 1$ , one can write  $f(z) = (z-a)^m f_1(z)$  where  $f_1(a) \neq 0$ , and  $a$  is called a *multiple zero of order*  $m$  of  $f$  (*simple, double, triple, etc.* . . . for  $m = 1, 2, 3, \dots$ ).

2. Suppose secondly that the entire function (3.1.2) is a polynomial not identically zero, so that

$$(3.1.3) \quad u(z) = \frac{d_1}{z-a} + \frac{d_2}{(z-a)^2} + \cdots + \frac{d_n}{(z-a)^n}$$

with  $d_n \neq 0$ . It is then said that  $a$  is a *multiple pole of order  $n$*  of  $f$  (*simple pole*, *double*, *triple*, ..., for  $n = 1, 2, 3, \dots$ ). The number  $-n$  is called the *order of  $f$*  at the point  $a$  and denoted by  $\omega(a; f)$ ; one has  $f(z) = (z-a)^{-n}f_1(z)$ , with

$$f_1(z) = d_n + d_{n-1}(z-a) + \cdots + d_1(z-a)^{n-1} + \sum_{k=0}^{\infty} c_k(z-a)^{n+k}$$

the series of the second member being convergent in  $\Delta$ . Thus  $f_1$  is *analytic in  $\Delta$*  and  $f_1(a) \neq 0$ .

3. Suppose finally that the entire function (3.1.2) is not a polynomial (or, as is also said, is a *transcendental* entire function); in other words, there are infinitely many values of  $n$  such that  $d_n \neq 0$ . In this case we say that  $a$  is an *essential isolated singularity of  $f$* . For every transcendental entire function  $v$ , the function  $v(1/z)$  has an essential singularity at  $z = 0$ .

(3.2) When  $f$  is an analytic function defined in an open connected set  $D \subset \mathbf{C}$ , the order of  $f$  at an *isolated* boundary point  $a$  of  $D$  is thus defined when  $f$  is not identically zero, and when  $a$  is not an essential singularity of  $f$ . This integer  $\omega(a; f)$  (positive or negative) can also be characterized in the following way:

(3.3) A function  $f$  analytic in  $D$  has order equal to  $m$  at the point  $a$  if, and only if, as  $z$  tends to  $a$  while remaining in  $D$ ,  $|(z-a)^kf(z)|$  tends to 0 for  $k > -m$ , to  $+\infty$  for  $k < -m$ . We may also say that  $-m$  is the *smallest integer  $k$  (positive or negative) such that  $(z-a)^kf(z)$  remains bounded as  $z$  tends to  $a$  while remaining in  $D$* .

The necessity of the conditions follows at once from the fact that

$$|(z-a)^kf(z)| = |z-a|^{k+m}|f_1(z)|$$

where  $f_1$ , being analytic and  $\neq 0$  at the points  $a$ , tends to the finite limit  $f_1(a) \neq 0$ . To see the sufficiency of the conditions, first note that they imply that  $f$  is not identically zero in  $D$ . Thus we only need to show that  $a$  is not an essential singularity of  $f$ , and for this it is enough to show that the coefficients  $d_n$  of the Laurent development of  $f$  at  $a$  all vanish for  $n > -m$ . Now, with the notations of (2.4)

$$(3.3.1) \quad d_n = \frac{r^n}{2\pi} \int_0^{2\pi} f(a + re^{it}) e^{-nit} dt$$

where  $r > 0$  can be taken arbitrarily small. By hypothesis, for each  $\varepsilon > 0$ , there exists a number  $r_0 > 0$  such that, for  $0 < r < r_0$ ,

$$|f(a + re^{it})| \leq \varepsilon r^{-n}.$$

Thus by virtue of (3.3.1) and the theorem of the mean,  $|d_n| \leq \varepsilon$  and since  $\varepsilon$  is arbitrary,  $d_n = 0$ .

The interest of the criterion (3.3) is that it enables us to determine the order of  $f$  at the point  $a$  without knowing in advance the Laurent development of  $f$  at this point.

*Remarks (3.3.2)* Note that if  $a$  is an isolated *essential* singularity of  $f$ ,  $|(z - a)^k f(z)|$  cannot remain bounded as  $z$  tends to  $a$  for any value of the integer  $k$ , because of (3.3). On the other hand, the function  $\sin(1/z)$  (for  $a = 0$ ) shows that  $f$  may have *infinitely many* zeros in the neighbourhood of  $a$ , so that  $a$  is not necessarily an *isolated* singularity of the function  $1/f$ . When further  $f(z) \neq 0$  in  $D$ , the point  $a$  is an *essential* isolated singularity of  $1/f$  (otherwise  $a$  would be a pole or a zero of  $f$ ); therefore  $|(z - a)^k/f(z)|$  also cannot remain bounded for *any* value of  $k$ .

(3.3.3) The differential properties of an analytic function  $f$  (not identically zero) at a point  $a$  (VI, 6.2.3) show immediately that the order  $\omega(a; f)$  is the *smallest integer*  $m$  such that  $f^{(k)}(a) = 0$  for  $0 \leq k \leq m - 1$  and  $f^{(m)}(a) \neq 0$ .

(3.4) Let  $D$  be an open connected set in  $\mathbf{C}$ ,  $a$  an isolated boundary point of  $D$ ,  $f, g$  two functions analytic in  $D$  each possessing an order at the point  $a$ . Then:

(i) The function  $fg$  is analytic in  $D$  and possesses at the point  $a$  an order satisfying

$$(3.4.1) \quad \omega(a; fg) = \omega(a; f) + \omega(a; g).$$

(ii) The function  $f + g$  is analytic in  $D$ ; if it is not identically zero, it possesses at the point  $a$  an order satisfying

$$(3.4.2) \quad \omega(a; f + g) \geq \inf(\omega(a; f), \omega(a; g)).$$

If further  $\omega(a; f) \neq \omega(a; g)$ , then

$$(3.4.3) \quad \omega(a; f + g) = \inf(\omega(a; f), \omega(a; g)).$$

(iii) There exists an open disc  $\Delta$  of centre  $a$  such that  $1/f$  is analytic in  $\Delta - \{a\}$ ;  $1/f$  has an order at the point  $a$  satisfying

$$(3.4.4) \quad \omega(a; 1/f) = -\omega(a; f).$$

Everything follows from (3.3) and from the fact that, if  $\omega(a; f) = m$ ,  $\omega(a; g) = n$ , then

$$f(z) = (z - a)^m f_1(z), \quad g(z) = (z - a)^n g_1(z)$$

where  $f_1$  and  $g_1$  are analytic in an open disc of centre  $a$  and  $f_1(a) \neq 0$ ,  $g_1(a) \neq 0$ . Assertion (i) follows immediately; if for example  $n \geq m$

$$f(z) + g(z) = (z - a)^m (f_1(z) + (z - a)^{n-m} g_1(z))$$

and the function  $f_1(z) + (z - a)^{n-m} g_1(z)$  being analytic in the neighbourhood of  $a$ , tends to the limit  $f_1(a) \neq 0$  if  $n > m$ , to  $f_1(a) + g_1(a)$  if  $n = m$ . Finally, by virtue of the principle of isolated zeros (VI, 3.2), there exists an open disc  $\Delta$  of centre  $a$  such that  $f_1(z) \neq 0$  in  $\Delta$ ; hence (iii).

(3.5) Given an open set  $D \subset \mathbf{C}$  and a sequence (finite or infinite) of points  $a_n$  which are *isolated* boundary points of  $D$ , the union  $D'$  of  $D$  and the set of points  $a_n$  is an open set

in  $\mathbf{C}$ . A complex function  $f$  analytic in  $D$  is said to be *meromorphic* in  $D'$  if each  $a_n$  is a regular point or a *pole* of  $f$ . It is clear that the sum and the product of two functions meromorphic in  $D'$  is again meromorphic in  $D'$ . It can be shown that if  $D'$  is connected, the zeros of a function  $f$  meromorphic and not identically zero in  $D'$  can also be arranged in a sequence  $(b_n)$  (finite or not). If  $D''$  is the complement in  $D$  of the set of the  $b_n$ , the  $a_n$  and the  $b_n$  are again *isolated* boundary points of  $D''$ . The fact that the  $b_n$  are isolated is just the principle of isolated zeros (VI, 3.2) and, since  $|f(z)|$  tends to  $+\infty$  as  $z$  tends to a pole  $a_n$  (3.3), there is a neighbourhood of this pole not containing any of the zeros  $b_m$ . It is therefore concluded that the function  $1/f$  is also *meromorphic* in  $D'$ .

#### 4. Residue theorem

(4.1) Let  $v(z) = \sum_{n=0}^{\infty} d_n z^n$  be an entire function in  $\mathbf{C}$ . For each  $a \in \mathbf{C}$  and each closed path  $\gamma: I \rightarrow \mathbf{C}$  such that  $a \notin \gamma(I)$ , we have

$$(4.1.1) \quad \int_{\gamma} v\left(\frac{1}{z-a}\right) dz = 2\pi i d_1 j(a; \gamma).$$

Let  $\delta > 0$  be the distance from  $a$  to  $\gamma(I)$ ; since the series  $\sum_{n=0}^{\infty} d_n z^n$  is normally convergent for  $|z| < 2/\delta$ , the series with general term

$$\frac{d_n \gamma'(t)}{(\gamma(t) - a)^n}$$

is normally convergent for  $t \in I$ . Thus by virtue of (V, 3.5)

$$\int_{\gamma} v\left(\frac{1}{z-a}\right) dz = \sum_{n=0}^{\infty} d_n \int_{\gamma} \frac{dz}{(z-a)^n}$$

Now, for  $n = 0$ , and  $n \geq 2$ , the function  $1/(z-a)^n$  possesses in  $\mathbf{C} - \{a\}$  a primitive  $(1-n)(z-a)^{1-n}$ , hence the integral  $\int_{\gamma} dz/(z-a)^n$  is zero (VII, 3.2) and the formula (4.1.1) follows from the definition of the index (VII, 6.1).

(4.2) Because of (4.1), when  $a$  is an isolated boundary point of an open connected set  $D \subset \mathbf{C}$ , and  $f$  an analytic function in  $D$ , the coefficient  $d_1$  of  $1/(z-a)$  in the Laurent development of  $f$  at the point  $a$  is called the *residue* at  $a$  of  $f$  and is denoted by  $\text{Res}_a f$ . It is the *only* term whose contribution to the integral  $\int_{\gamma} f(z) dz$  need not be zero, by virtue of (4.1) and (2.4), hence its name.

(4.3) (Residue theorem) Let  $D \subset \mathbf{C}$  be an open simply connected set,  $a_1, a_2, \dots, a_n$  distinct points of  $D$ , so that the  $a_k$  are isolated boundary points of the open set  $D' = D - \{a_1, a_2, \dots, a_n\}$ . For every complex function  $f$  analytic in  $D'$  and every closed path  $\gamma$  contained in  $D'$ , we have

$$(4.3.1) \quad \int_{\gamma} f(z) dz = 2\pi i \sum_{k=1}^n j(a_k; \gamma) \text{Res}_{a_k} f.$$

The function  $f$  has a Laurent development in the neighbourhood of each of the points  $a_k$ ; let  $u_k(z)$  be the *singular part* of  $f$  at the point  $a_k$  (3.1); it is an entire function of  $1/(z - a_k)$  (which may actually be identically zero). Consider the function

$$g(z) = f(z) - u_1(z) - u_2(z) - \cdots - u_n(z)$$

which is analytic in  $D'$ , and let us show that all the  $a_k$  are *regular* points for  $g$ . Let  $\Delta$  be an open disc of centre  $a_k$  contained in  $D$  and not containing any  $a_j \neq a_k$ ; then in  $\Delta - \{a_k\}$

$$g(z) = (f(z) - u_k(z)) - \sum_{j \neq k} u_j(z).$$

Since for  $j \neq k$  the point  $a_k$  is regular for  $u_j(z)$ , it is also regular for  $\sum_{j \neq k} u_j(z)$ . By definition the point  $a_k$  is regular for  $f(z) - u_k(z)$  (3.1), hence it is regular for  $g$ . We can thus continue  $g$  analytically into the whole of  $D$ ; since  $D$  is simply connected, Cauchy's theorem (VII, 5.1) gives

$$\int_{\gamma} g(z) dz = 0.$$

On the other hand, it follows from (4.1) that for each  $k$ ,

$$\int_{\gamma} u_k(z) dz = 2\pi i j(a_k; \gamma) \operatorname{Res}_{a_k} u_k$$

and by definition  $\operatorname{Res}_{a_k} u_k = \operatorname{Res}_{a_k} f$ ; hence the formula (4.3.1).

(4.4) To calculate the residue of a function  $f$  at an isolated singular point  $a$ , is the same as finding the first term of the Taylor development of the entire function  $v$  such that  $v(1/(z - a))$  is the singular part of  $f$  at the point  $a$ . In the particular case where  $f$  has a *pole* of order  $m$  at  $a$

$$f(z) = \frac{f_1(z)}{(z - a)^m}$$

where  $f_1$  is analytic in a neighbourhood of  $a$  and  $f_1(a) \neq 0$  (3.1); replacing  $f(z)$  by its Laurent development at the point  $a$  it is seen that  $\operatorname{Res}_a f$  is the *coefficient of  $(z - a)^{m-1}$  in the Taylor development of  $f_1(z)$  at the point  $a$* .

A frequently occurring particular case is that of a *simple pole*, where

$$(4.4.1) \quad f(z) = \frac{f_1(z)}{z - a}$$

with  $f_1$  analytic at the point  $a$  and  $f_1(a) \neq 0$ ; then

$$(4.4.2) \quad \operatorname{Res}_a f = f_1(a).$$

The function  $f$  often occurs in the form

$$(4.4.3) \quad f(z) = \frac{P(z)}{Q(z)}$$

where  $P$  and  $Q$  are analytic at  $a$ ,  $P(a) \neq 0$  and  $a$  is a *simple zero* of  $Q$ . In this case

$$(4.4.4) \quad \operatorname{Res}_a f = \frac{P(a)}{Q'(a)}$$

for we can write  $Q(z) = (z - a)Q_1(z)$ , where  $Q_1$  is analytic at  $a$  and  $Q_1(a) \neq 0$ , and  $Q'(z) = Q_1(z) + (z - a)Q_1'(z)$ , hence

$$Q'(a) = Q_1(a).$$

From the formula (4.4.1) with  $f_1 = P/Q_1$ , (4.4.4) is deduced from (4.4.2).

One should take care not to apply these formulae at random, and not to think that when  $f(z) = f_1(z)/(z - a)^m$  with  $m \geq 2$  and  $f_1(a) \neq 0$ , the residue of  $f$  at the point  $a$  is equal to  $f_1(a)$ .

## 5. Application of the residue theorem to the calculation of integrals

(5.1) The *method of residues* for calculating certain improper integrals

$$(5.1.1) \quad \int_{-\infty}^{+\infty} f(x) dx$$

is as follows. Suppose that  $f$  is the restriction to  $\mathbf{R}$  of a function (still denoted by  $f$ ) which is, for example, analytic in an open set of the form  $D' = D - \{a_1, a_2, \dots, a_n\}$ , where  $D$  contains the closed half-plane  $\mathcal{I}z \geq 0$ , and the  $a_k$  are points of the open half-plane  $\mathcal{I}z > 0$  (these half-planes can of course be replaced by  $\mathcal{I}z \leq 0$  and  $\mathcal{I}z < 0$  respec-

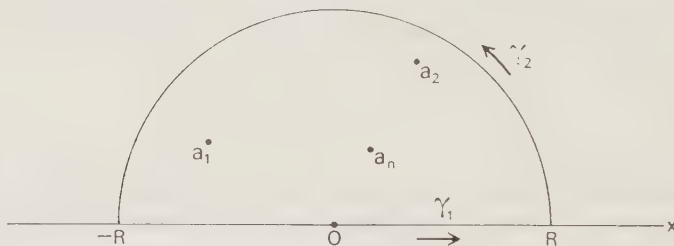


FIGURE 38

tively) (Fig. 38). Consider now a loop  $\gamma$ , which is the juxtaposition  $\gamma_1 \vee \gamma_2$  of the following two paths:

$$\gamma_1: t \rightarrow t \quad \text{for } -R \leq t \leq R$$

$$\gamma_2: t \rightarrow Re^{it} \quad \text{for } 0 \leq t \leq \pi \quad (\text{also denoted by } \gamma_{2,R})$$

where the number  $R$  satisfies  $R > |a_k|$  for all the indices  $k$ . It follows immediately that for each  $k$ ,

$$j(a_k; \gamma) = 1$$

(VII, 6.6), so that the residue theorem allows us to write

$$(5.1.2) \quad \int_{-R}^R f(x) dx + \int_{\gamma_2} f(z) dz = \int_{\gamma} f(z) dz = 2\pi i \sum_{k=1}^n \text{Res}_{a_k} f.$$

If in addition

$$(5.1.3) \quad \lim_{R \rightarrow +\infty} \int_{\gamma_{2,R}} f(z) dz = 0$$

from (5.1.2), by passage to the limit, it follows that

$$(5.1.4) \quad \int_{-\infty}^{+\infty} f(x) dx = 2\pi i \sum_{k=1}^n \text{Res}_{a_k} f.$$

*Examples* (5.2) Suppose first that  $f(z) = P(z)/Q(z)$  is a *rational function*, where  $P$  and  $Q$  are relatively prime polynomials and none of the zeros of  $Q$  are real. Suppose further that

$$(5.2.1) \quad \deg Q \geq \deg P + 2.$$

Then the formula (5.1.4) is valid, the  $a_k$  being the zeros of  $Q$  such that  $\mathcal{I}a_k > 0$ . For if

$$P(z) = c_0 z^m + \cdots + c_m, \quad Q(z) = b_0 z^n + \cdots + b_n$$

with  $c_0 \neq 0$ ,  $b_0 \neq 0$ , the calculations of (VI, 9.4) show that there exists a number  $R_0 > 0$  such that, for  $R \geq R_0$

$$|P(Re^{it})| \leq 2|c_0|R^m, \quad |Q(Re^{it})| \geq \frac{1}{2}|b_0|R^n.$$

Hence, with the notations of (5.1)

$$\left| \int_{\gamma_{2,R}} f(z) dz \right| \leq \int_0^\pi \frac{|P(Re^{it})|}{|Q(Re^{it})|} R dt \leq 4\pi \frac{|c_0|}{|b_0|} R^{m+1-n}$$

which shows that condition (5.1.3) is satisfied.

For example, let us show that

$$(5.2.2) \quad \int_{-\infty}^{+\infty} \frac{dx}{(x^2 + 1)^3} = \frac{3\pi}{8}.$$

There is here just one pole in the half-plane  $\mathcal{I}z > 0$ , the point  $i$ , and thus

$$\int_{-\infty}^{+\infty} \frac{dx}{(x^2 + 1)^3} = 2\pi i \text{Res}_i f.$$

To obtain the residue at the point  $i$ , put  $z = i + t$  and develop the function in the neighbourhood of  $t = 0$  up to the term in  $1/t$ :

$$\frac{1}{(z^2 + 1)^3} = \frac{1}{i^3(2i + t)^3} = -\frac{1}{8it^3} \left(1 + \frac{t}{2i}\right)^{-3} = -\frac{1}{8it^3} \left(1 - \frac{3t}{2i} - \frac{3t^2}{2} + o(t^2)\right)$$

The residue is thus equal to  $3/16i$ , hence the formula (5.2.2).

(5.3) Suppose now that

$$f(z) = g(z) e^{miz} \quad \text{with } m > 0,$$

$g$  being analytic in  $D' = D - \{a_1, \dots, a_n\}$ . The following lemma is then used:

(5.3.1) (Jordan's lemma) *As  $R$  tends to  $+\infty$ , the integral*

$$(5.3.2) \quad \int_0^\pi R |e^{imRe^{it}}| dt \quad (\text{for } m > 0)$$

*remains bounded. If there exists a sequence  $(R_n)$  tending to  $+\infty$  and such that the sequence of functions  $g(R_n e^{it})$  tends uniformly to 0 in the interval  $[0, \pi]$ , then the corresponding sequence of integrals*

$$\int_{\gamma_{2, R_n}} f(z) dz = i \int_0^\pi g(R_n e^{it}) R_n e^{imR_n e^{it}} dt$$

*tends to 0.*

Indeed, we have  $|e^{imR e^{it}}| = e^{-mR \sin t}$ , and since  $\sin t = \sin(\pi - t)$ ,

$$\int_0^\pi e^{-mR \sin t} dt = 2 \int_0^{\pi/2} e^{-mR \sin t} dt.$$

Now, in the interval  $[0, \pi/2]$ ,  $\tan t \geq t$ , which shows that the function  $\sin t/t$  is decreasing, thus  $\sin t/t \geq 2/\pi$ , or again  $\sin t \geq 2t/\pi$ , which permits the majorization

$$\int_0^{\pi/2} e^{-mR \sin t} dt \leq \int_0^{\pi/2} e^{-2mRt/\pi} dt \leq \frac{\pi}{2mR}$$

hence the lemma.

For example

$$(5.3.3) \quad \int_0^{+\infty} \frac{\cos x \, dx}{x^2 + a^2} = \frac{\pi}{2a} e^{-a} \quad \text{for } a > 0.$$

Since the integrand is even and the function  $\sin x/(x^2 + a^2)$  is odd, the integral is also equal to  $\frac{1}{2} \int_{-\infty}^{+\infty} e^{ix} dx/(x^2 + a^2)$ . We therefore have the conditions for the application of (5.3.1). The only pole of  $g(z) = 1/(z^2 + a^2)$  in the half-plane  $\Re z > 0$  is the point  $ai$ , which is a simple pole. Thus  $\text{Res}_{ai} f = e^{-a}/2ai$  (4.4.4), hence the formula (5.3.3).

(5.4)  $\int_{-\infty}^{+\infty} f(x) dx$  can also be calculated by the preceding method when, for a sequence  $(R_n)$  of numbers  $> 0$  tending to  $+\infty$ , the sequence of integrals  $\int_{\gamma_{2, R_n}} f(z) dz$  tends to a limit not necessarily zero (problem 9).

## 6. Applications of the residue theorem to the solution of equations

Let  $f$  be a function analytic in an open set  $D \subset \mathbf{C}$ ; solving the equation  $f(z) = 0$  consists in approximating arbitrarily the zeros of  $f$  in  $D$ ; this is of course a problem of even greater difficulty than the similar problem for functions of a real variable (II, 1).

However we do at least have a theoretical way of “separating” the roots, i.e. determining open sets  $A \subset D$  for which we know exactly the *number* of roots of the equation belonging to  $A$ ; this is due to the following consequence of the residue theorem:

(6.1) *Let  $D \subset \mathbf{C}$  be a simply connected open set,  $f$  a function meromorphic in  $D$  (3.5) having only finitely many poles  $b_1, \dots, b_n$  and finitely many zeros  $a_1, \dots, a_m$  in  $D$ . Then, for every loop  $\gamma: I \rightarrow D$  contained in  $D$  such that  $\gamma(I)$  does not contain any poles nor zeros of  $f$ , and for every function  $g$  analytic in  $D$*

$$(6.1.1) \quad \int_{\gamma} g(z) \frac{f'(z)}{f(z)} dz = 2\pi i \sum_{k=1}^m j(a_k; \gamma) \omega(a_k; f) g(a_k) + 2\pi i \sum_{h=1}^n j(b_h; \gamma) \omega(b_h; f) g(b_h).$$

Note that a point  $c \in D$  is *regular* for the function  $f'/f$  when  $c$  is neither a zero nor a pole of  $f$ ; if on the contrary  $c$  is a zero or a pole of  $f$ ,  $c$  is a *simple pole* for  $f'/f$  with a residue equal to the order  $\omega(c; f)$  (3.1). Indeed, in the neighbourhood of  $c$

$$f(z) = (z - c)^r f_1(z)$$

where  $r = \omega(c; f)$ ,  $f_1$  is analytic at  $c$  and  $f_1(c) \neq 0$ ; we deduce that

$$\frac{f'(z)}{f(z)} = \frac{r}{z - c} + \frac{f'_1(z)}{f_1(z)}$$

hence the conclusion, since  $c$  is a regular point for  $f'_1/f_1$ . The formula (6.1.1) is then an immediate consequence of this result and the residue theorem.

In particular, taking for  $g(z)$  the constant 1, we obtain from (VII, 2.2.3) and the definition of the index:

(6.2) *Under the conditions of (6.1), let  $\Gamma: I \rightarrow \mathbf{C}$  be the loop  $t \rightarrow f(\gamma(t))$ . Then*

$$(6.2.1) \quad j(0; \Gamma) = \sum_{k=1}^m j(a_k; \gamma) \omega(a_k; f) + \sum_{h=1}^n j(b_h; \gamma) \omega(b_h; f).$$

Consider the particular case where  $f$  is *analytic* in  $D$  and has only finitely many zeros in  $D$ , and where the loop  $\gamma$  is chosen so that the set  $D - \gamma(I)$  is the union of two open sets  $A, B$  with  $j(z; \gamma) = 1$  in  $A$  and  $j(z; \gamma) = 0$  in  $B$ . Then

$$(6.2.2) \quad \sum_{a_k \in A} \omega(a_k; f) = j(0; \Gamma)$$

where the sum of the first member is taken over all the indices  $k$  such that  $a_k \in A$ ; this is what is called the “number of zeros of  $f$  in  $A$  counted with their order of multiplicity”. The determination of this number thus reduces to the calculation of the *index*, using the general method described in (VII, 6.6).

*Example (6.3)* The function  $f(z) = e^z + z$  is an entire function; we propose to show that in the “strip”  $B$  defined by  $0 < \mathcal{I}z < \pi$ , the equation

$$(6.3.1) \quad e^z + z - a = 0$$

has *exactly one root*, provided the complex number  $a$  satisfies the conditions: 1.  $\mathcal{I}a > 0$ ; 2. if  $\mathcal{I}a = \pi$ , then  $\mathcal{R}a > -1$ .

The formula (6.2.2) will be applied, taking for  $\gamma$  the "rectangle" which is the juxtaposition of the four paths (Fig. 39)

$$\begin{aligned}\gamma_1: t &\rightarrow t, & -R \leq t \leq R; \\ \gamma_2: t &\rightarrow R + it, & 0 \leq t \leq \pi; \\ \gamma_3: t &\rightarrow \pi i - t, & -R \leq t \leq R; \\ \gamma_4: t &\rightarrow -R + \pi i - it, & 0 \leq t \leq \pi.\end{aligned}$$

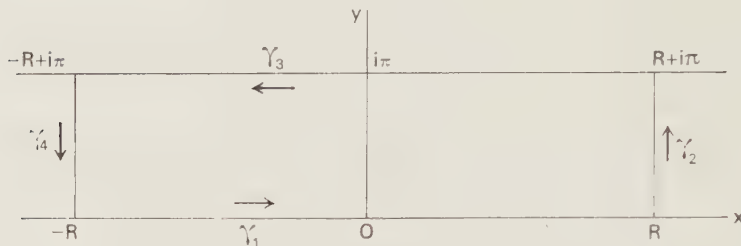


FIGURE 39

We have to prove that if  $R$  is taken sufficiently large,  $j(a; \Gamma) = 1$ , and to do this (VII, 6.6), it will be enough to show that there exists a half-line  $H$  with initial point  $a$  which meets at just one point the image  $L_0$  of the closed path  $f \circ \gamma$ . *Except* when both  $\mathcal{I}a > \pi$  and  $\Re a < -1$ , we take for  $H$  the half-line  $t \rightarrow a - it$ ,  $0 \leq t < +\infty$  (Fig. 40). Since  $t \rightarrow f(\gamma_1(t))$  is a strictly increasing real function,  $t \rightarrow f(\gamma_3(t))$  has an imaginary part  $\pi$  and a real part  $\leq -1$  and the real part of  $t \rightarrow f(\gamma_4(t))$  is at most equal to  $-R + e^{-R}$ , we need only verify that the curve  $t \rightarrow f(\gamma_2(t))$  does not meet  $H$  for  $0 \leq t \leq \pi$ . Now

$$f(\gamma_2(t)) = R + e^R \cos t + i(t + e^R \sin t)$$

and since the function  $t \rightarrow R + e^R \cos t$  is strictly decreasing for  $0 \leq t \leq \pi$  and  $t + e^R \sin t = (\pi/2) + R$  for  $t = \pi/2$ , it is enough to show that when this function takes the value  $-1$ , we have, for the corresponding value  $t_0$  of  $t$ ,  $t_0 + e^R \sin t_0 > \sup(\mathcal{I}a, \pi)$  for  $R$  sufficiently large (the function  $t + e^R \sin t$  being decreasing for  $t_0 \leq t \leq \pi$ ). Now  $\cos t_0 = -(R + 1)e^{-R}$ , hence as soon as  $R$  is sufficiently large

$$\cos t_0 \geq -\frac{\sqrt{2}}{2}, \quad \sin t_0 \geq \frac{\sqrt{2}}{2} \quad \text{and} \quad t_0 + e^R \sin t_0 \geq \frac{\sqrt{2}}{2} e^R,$$

which proves our assertion. If  $\mathcal{I}a > \pi$  and  $\Re a < -1$ , we take for  $H$  the half-line  $t \rightarrow a + it$ ,  $0 \leq t < +\infty$  (Fig. 40). This time the result reduces to showing that  $H$  meets the curve  $t \rightarrow f(\gamma_2(t))$  at only one point, which is done similarly to the above by using the fact that the function  $t \rightarrow R + e^R \cos t$  is decreasing.

A deeper study would enable us to "localize" the root of (6.3.1) more precisely, depending on the value of  $a$ , and one could even succeed in obtaining a numerical value approximating this root as closely as one pleased (when the numerical value of  $a$  is given), or an asymptotic development as  $|a|$  tends to  $+\infty$ .

(6.4) In a number of important cases, we can "separate" the roots of an equation by replacing it by an "approximate" equation, using the following theorem:

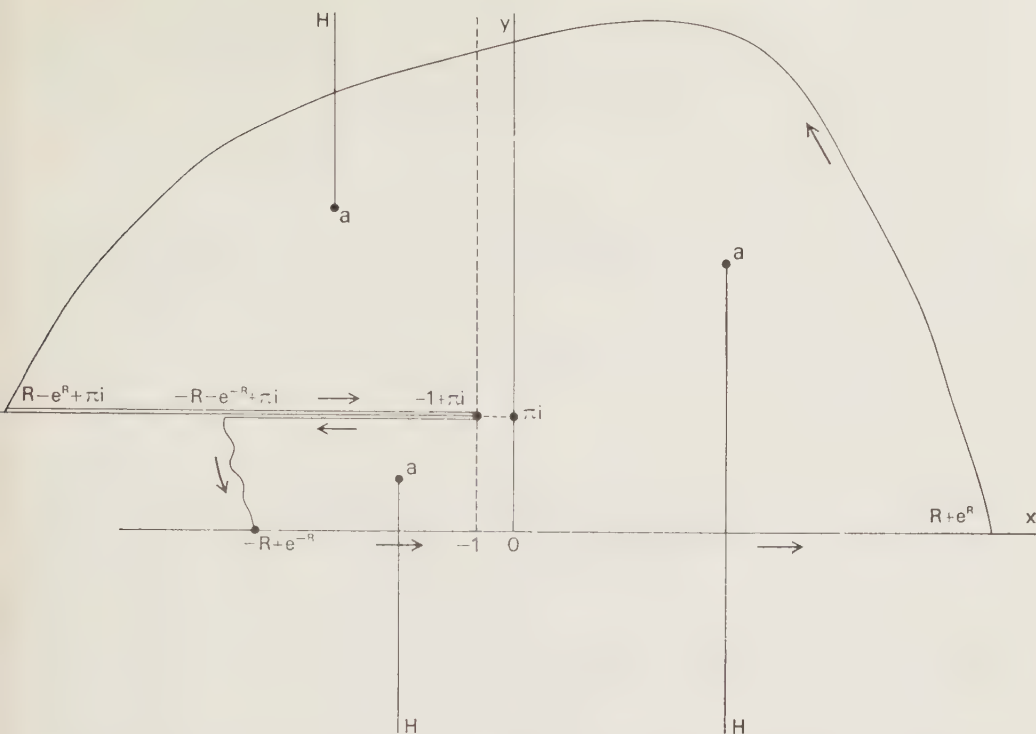


FIGURE 40

(6.5) (Rouché's theorem) Let  $D \subset \mathbf{C}$  be an open simply connected set,  $f, g$  two functions analytic in  $D$ ,  $\gamma: I \rightarrow D$  a loop contained in  $D$  such that  $\gamma(I)$  does not contain any of the zeros of  $f$ . Suppose further that  $|g(z)| < |f(z)|$  on the set  $\gamma(I)$ ; then  $\gamma(I)$  does not contain any of the zeros of  $f + g$  and if  $a_1, \dots, a_r$  (resp.  $b_1, \dots, b_s$ ) are the zeros of  $f$  (resp. of  $f + g$ ) of index  $\neq 0$  with respect to  $\gamma$ , we have

$$(6.5.1) \quad \sum_{h=1}^r j(a_h; \gamma) \omega(a_h; f) = \sum_{k=1}^s j(b_k; \gamma) \omega(b_k; f + g).$$

The first assertion is evident, since the relation  $f(z) + g(z) = 0$  implies  $|f(z)| = |g(z)|$ . To prove the second, we consider the function  $h = (f + g)/f$ , which is meromorphic in  $D$ ; then

$$\frac{h'}{h} = \frac{(f + g)'}{f + g} - \frac{f'}{f}$$

and by virtue of (6.2.1) it is enough to show that if  $\Gamma: I \rightarrow \mathbf{C}$  is the loop  $t \mapsto h(\gamma(t))$ , we have  $j(0; \Gamma) = 0$ . Now, the function  $|g(\gamma(t))/f(\gamma(t))|$  is defined and continuous in  $I$ , and so attains its maximum  $r$  at a point of  $I$  (0, 3.2). Therefore  $|h(\gamma(t)) - 1| \leq r < 1$ ; hence  $j(0; \Gamma) = 0$  by virtue of (VII, 6.5), the disc  $|z - 1| < 1$  being simply connected (VII, 4.4).

Rouché's theorem is most often applied under the conditions of (6.2.2): the theorem then says that the number of zeros of  $f + g$  in  $A$ , counted with their order of multiplicity, is the same as the number of zeros of  $f$ . The interest here is that the verification of the hypothesis requires only a majorization of  $|g|$  (and a minorization of  $|f|$ ) on the set  $\gamma(I)$ , and not in the whole of  $D$ .

*Example (6.6)* We propose to "localize" the roots of the equation

$$(6.6.1) \quad \tan z = a(z - \alpha)$$

where  $a \neq 0$  and  $\alpha$  are complex numbers; these zeros are the same as those of the equation

$$(6.6.2) \quad a(z - \alpha) \cos z - \sin z = 0$$

and we shall "compare" them to those of the equation

$$(6.6.3) \quad a(z - \alpha) \cos z = 0$$

which are known, namely the point  $z = \alpha$  and the zeros  $(2n + 1)\pi/2$  of  $\cos z$  ( $n$  integer positive or negative). Rouché's theorem will be applied with  $D = \mathbf{C}$ ,  $\gamma$  being the perimeter of the "rectangle" which is the juxtaposition of the four paths

$$\gamma_1: t \rightarrow n\pi + it, \quad -n\pi \leq t \leq n\pi,$$

$$\gamma_2: t \rightarrow n\pi - t, \quad -n\pi \leq t \leq n\pi,$$

$$\gamma_3: t \rightarrow -n\pi - it, \quad -n\pi \leq t \leq n\pi,$$

$$\gamma_4: t \rightarrow -n\pi + t, \quad -n\pi \leq t \leq n\pi.$$

Since for  $n$  sufficiently large,  $1/|a(z - \alpha)|$  is *arbitrarily small* on  $\gamma(I)$ , we need only prove that on  $\gamma(I)$  the function  $\tan z$  is *bounded by a number independent of  $n$* . Taking into account

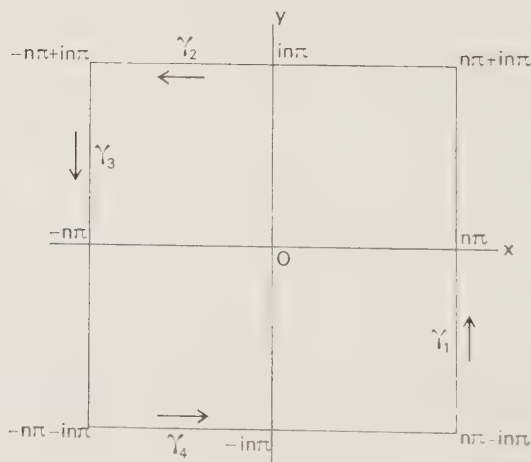


FIGURE 41

the periodicity of  $\tan z$ , the choice of  $\gamma_1$  and  $\gamma_3$ , and the formula (VI, 8.6.14)  $\tan z = \frac{2i}{1 + e^{2iz}} - i$ , everything reduces to proving the following lemma:

(6.6.4) *For each number  $\delta > 0$ , the function  $1/(1 + e^{iz})$  is bounded in the half-plane  $\Im z \geq \delta$ .*

For if  $z = x + iy$ , we have  $|1 + e^{iz}| \geq 1 - |e^{iz}| = 1 - e^{-y} \geq 1 - e^{-\delta}$ .

Rouché's theorem thus shows that the equation (6.6.1) has *exactly*  $2n + 1$  roots satisfying  $|\Re z| < n$  and  $|\Im z| < n$ , as soon as  $n$  is sufficiently large. This result can be improved (when  $a$  and  $\alpha$  are fixed) for all the roots of (6.6.1) except a finite number of them. Indeed it is enough to apply Rouché's theorem to a square  $Q_n$ , with sides parallel to the axes and of length equal to an arbitrary number  $\delta$  satisfying  $0 < \delta < \pi/2$ , and of centre  $(2n + 1)\pi/2$ . On the perimeter of this square the maximum  $M(\delta)$  of the values of  $|\tan z|$  is *independent of*  $n$  by periodicity; as soon as  $|n|$  is sufficiently large so that

$$\frac{M(\delta)}{|a| \left( |2n + 1| \frac{\pi}{2} - |\alpha| \right)} < 1$$

it can be affirmed that the equation (6.6.2) has *exactly one root* in  $Q_n$ .

*Remark (6.7)* The methods developed above often enable one, not only to determine open sets  $A \subset D$  in which it is known that the equation  $f(z) = 0$  has *exactly one simple root*, but even to choose the set  $A$  so small that a numerical approximation to the root can be obtained with arbitrarily small error, provided the existence of the root has been proved. This follows from the methods developed in (II, 3) and (II, 4), whose results are valid *without modification* (and with the same proofs) when the interval  $[x_0 - c, x_0 + c]$  is replaced by the disc  $|z - z_0| \leq c$ ; indeed, these methods require *only* the theorem of the mean, which generalizes to functions analytic in a disc, as has been seen (VI, 6.6).

## 7. Inversion of analytic functions: (I) The local problem

(7.1) When we consider a real function  $f(x)$  of a *real* variable  $x$ , continuously differentiable in an interval  $I: |x - x_0| \leq r$ , and whose derivative  $f'(x)$  has *constant sign* in  $I$  (so that  $f$  is *strictly monotone* in  $I$ ), we know (0, 3.3) that for each  $y$  belonging to the interval  $J$  with endpoints  $f(x_0 - r), f(x_0 + r)$ , there exists *one and only one root* of the equation  $f(x) = y$  belonging to  $I$ . If we denote it by  $h(y)$ , the function  $h$  is *continuously differentiable* in  $J$  and has the derivative  $1/f'(h(y))$  at every point of this interval. It will be seen that this result generalizes to analytic functions of a complex variable.

(7.2) *Let  $f$  be a function analytic in an open disc  $D: |z - z_0| < r$  and satisfying  $f'(z_0) \neq 0$ . Then there exists an open disc  $D': |z - z_0| < r'$  (with  $r' \leq r$ ) and an open disc  $\Delta: |w - w_0| < \rho$  of centre  $w_0 = f(z_0)$  such that for each  $w \in \Delta$ , the equation  $f(z) = w$  has *one and only one root*  $z = h(w)$  belonging to  $D'$ ; moreover the function  $h$  thus defined is analytic in  $\Delta$  and*

$$h'(w) = 1/f'(h(w))$$

*for every  $w \in \Delta$ .*

Since  $f'(z_0) \neq 0$ , we can find  $r' > 0$  such that  $|f(z) - f(z_0)| \geq \frac{1}{2}|f'(z_0)| \cdot |z - z_0|$  for  $z \in D'$ :  $|z - z_0| < r'$ . Then the function  $g(z)$  equal to  $(z - z_0)/(f(z) - f(z_0))$  for  $z \neq z_0$  and to  $1/f'(z_0)$  for  $z = z_0$ , is analytic in  $D'$ . The equation  $f(z) = w$  can then be written in the form

$$(7.2.1) \quad z - z_0 - (w - w_0)g(z) = 0.$$

It may be supposed, by translations of  $z$  and  $w$ , that  $z_0 = w_0 = 0$ . The conclusion of (7.2) is then a consequence of the following more precise theorem:

(7.3) *Let  $g$  be a function analytic in an open set containing a closed disc  $D: |z| \leq r$  ( $r > 0$ ), and put  $M = \sup_{|z|=r} |g(z)|$ . Then for each  $w$  satisfying  $|w| < r/M$ , the equation*

$$(7.3.1) \quad z - wg(z) = 0$$

*possesses one, and only one, root  $z = h(w)$  in the open disc  $\dot{D}: |z| < r$ . This function  $h$  is analytic in the disc  $\Delta: |w| < r/M$  and to be precise, for each function  $F$  analytic in  $D$ , we have the Taylor development convergent in  $\Delta$*

$$(7.3.2) \quad F(h(w)) = F(0) + \sum_{n=1}^{\infty} \frac{w^n}{n!} \left( \frac{d^{n-1}}{dz^{n-1}} (F'(z)(g(z))^n) \right)_{z=0}$$

*the coefficients being the values of the derivatives at  $z = 0$ . In particular*

$$(7.3.3) \quad h(w) = \sum_{n=1}^{\infty} \frac{w^n}{n!} \left( \frac{d^{n-1}}{dz^{n-1}} ((g(z))^n) \right)_{z=0}$$

(“Lagrange inversion formula”).

If  $|w| < r/M$ , on the circle  $|z| = r$  we have the majorization  $|wg(z)/z| < 1$ , hence, by virtue of Rouché's theorem (6.5), the equation (7.3.1) has one, and only one, simple root  $z = h(w)$  such that  $|h(w)| < r$ . If  $\gamma$  is the loop  $t \rightarrow re^{it}$  ( $0 \leq t \leq 2\pi$ ), the residue theorem gives the formula

$$(7.3.4) \quad F(h(w)) = \frac{1}{2\pi i} \int_{\gamma} \frac{(1 - wg'(z))F(z) dz}{z - wg(z)}$$

by virtue of the expression for the residue of the integrand at the simple pole  $z = h(w)$  (4.4.4). Moreover, for  $|z| = r$ , we can write

$$\frac{1}{z - wg(z)} = \sum_{n=0}^{\infty} \frac{w^n (g(z))^n}{z^{n+1}}$$

where the series is normally convergent, since  $|wg(z)/z| \leq M|w|/r < 1$ . We therefore have (V, 3.5)

$$(7.3.5) \quad F(h(w)) = \sum_{n=0}^{\infty} \frac{w^n}{2\pi i} \int_{\gamma} \frac{F(z)(g(z))^n}{z^{n+1}} dz = \sum_{n=0}^{\infty} \frac{w^{n+1}}{2\pi i} \int_{\gamma} \frac{F(z)g'(z)(g(z))^n}{z^{n+1}} dz$$

On the other hand, the formula (VIII, 7.5.1) gives

$$\begin{aligned} \frac{1}{2\pi i} \int_{\gamma} \frac{F(z)(g(z))^n}{z^{n+1}} dz &= \frac{1}{n!} \left( \frac{d^n}{dz^n} (F(z)(g(z))^n) \right)_{z=0} \\ &= \frac{1}{n!} \left( \frac{d^{n-1}}{dz^{n-1}} (F'(z)(g(z))^n + nF(z)g'(z)(g(z))^{n-1}) \right)_{z=0} \end{aligned}$$

and similarly

$$\int_{\gamma} \frac{F(z)g'(z)(g(z))^n}{z^{n+1}} dz = \frac{1}{n!} \left( \frac{d^n}{dz^n} (F(z)g'(z)(g(z))^n) \right)_{z=0}$$

hence, substituting into (7.3.5), the formula (7.3.2).

Formula (7.3.3), requiring the calculation of derivatives of arbitrarily large order, is in general of more theoretical than practical interest.

## 8. Inversion of analytic functions: (II) The global problem

(8.1) *Let  $f$  be a function analytic in an open connected set  $D$ ; if  $f$  is not constant in  $D$ , the set  $f(D)$  is open in  $\mathbf{C}$ . If further  $f$  is injective, then  $f'(z) \neq 0$  in  $D$  and there exists one, and only one, function  $g$  analytic in  $f(D)$  such that  $f(g(w)) = w$  for every  $w \in D$  and  $g'(w) = 1/f'(g(w))$ .*

Let  $z_0 \in D$  and put  $w_0 = f(z_0)$ ; we have to show that there exists a number  $\rho > 0$  such that for each  $w$  satisfying  $|w - w_0| < \rho$ , the equation  $f(z) = w$  has at least one root in  $D$ . Since  $f$  is not constant,  $f(z) - f(z_0) = (z - z_0)^k h(z)$  for some integer  $k > 1$ , where  $h$  is analytic in  $D$  and  $h(z_0) \neq 0$  (VI, 7.3). Because of the continuity of  $h$ , there exists  $r > 0$  such that for  $|z - z_0| \leq r$ , we have  $|(h(z) - h(z_0))/h(z_0)| \leq \frac{1}{2}$ , so  $|h(z)| \geq \frac{1}{2}|h(z_0)|$ . Note then that the equation  $f(z) = w$  can also be written  $u(z) = 0$ , where

$$u(z) = (z - z_0)^k - \frac{w - w_0}{h(z)},$$

put  $v(z) = (z - z_0)^k$  and compare the equations  $u(z) = 0$  and  $v(z) = 0$  with the help of Rouché's theorem (6.5). Determine the number  $\rho > 0$  so that  $\rho < \frac{1}{2}r^k|h(z_0)|$ ; then, for  $|z - z_0| = r$  and  $|w - w_0| < \rho$

$$\frac{|w - w_0|}{|(z - z_0)^k h(z)|} \leq \frac{2\rho}{r^k|h(z_0)|} < 1$$

which shows that the equation  $u(z) = 0$  and the equation  $v(z) = 0$  have the same number of roots in the disc  $|z - z_0| < r$ , counted with their order of multiplicity. This proves the first assertion. To show that if  $f$  is injective,  $f'(z) \neq 0$  in  $D$ , assume on the contrary that  $f'(z_0) = 0$ , so that with the preceding notations,  $k \geq 2$ ; since  $f'$  is not identically zero in  $D$  (otherwise  $f$  would be constant), it may be supposed, by virtue of the principle of isolated zeros, that  $f'(z) \neq 0$  for  $|z - z_0| < r$  and  $z \neq z_0$ . Now, for  $|w_1 - w_0| < \rho$  and  $w_1 \neq w_0$ , it has just been seen that the equation  $f(z) = w_1$  has  $k$  roots (counted with their order of multiplicity) in the disc  $|z - z_0| < r$ . Since these roots are distinct from  $z_0$ , we have  $f'(z) \neq 0$  at each of them, therefore they are simple roots and hence distinct, which contradicts the hypothesis that  $f$  is injective in  $D$ . The last assertion is then immediate, since for each  $w \in f(D)$ , there exists a unique number  $g(w) \in D$  such that  $f(g(w)) = w$ , and since  $f'(g(w)) \neq 0$ ,  $g$  is analytic at the point  $w$  by virtue of (7.2).

(8.2) The problem of the inversion of an analytic function  $f$  in  $D$  is therefore solved when  $f$  is injective. When  $f$  is no longer injective, there are values of  $w \in f(D)$  such that

the equation  $f(z) = w$  has *several* distinct roots; since a function can have *only one value* at each point, we might expect the existence of *several* analytic functions  $g_1(w), g_2(w), \dots$  defined in  $f(D)$  and satisfying  $f(g_1(w)) = w, f(g_2(w)) = w$ , etc. There is indeed a result of this kind in the real domain: for example, for the mapping  $f: x \rightarrow x^2$  of  $\mathbf{R}$  into itself,  $f(\mathbf{R}) = [0, +\infty[$  and there are *two continuous functions*  $x \rightarrow \sqrt{x}$  and  $x \rightarrow -\sqrt{x}$  defined in  $f(\mathbf{R})$  and “inverses” of  $f$ .

(8.3) It will be seen on the contrary that it is not in general possible in the *complex* domain (when the analytic function  $f$  is not injective in  $D$ ) to define a function  $w \rightarrow g(w)$  *continuous* in the *whole* of  $f(D)$  and satisfying  $f(g(w)) = w$  in  $f(D)$ ; *a fortiori* there will be no *analytic* “inverse” of  $f$  in the *whole* of the set  $f(D)$ . The simplest example showing the impossibility of an analytic inverse is given by the function  $f(z) = z^2$  in  $\mathbf{C}$ . We then have  $f(\mathbf{C}) = \mathbf{C}$  and for each  $w = re^{it}$  with  $r > 0, -\pi \leq t \leq \pi$ , the equation  $z^2 = w$  has two roots  $z_1 = \sqrt{r}e^{it/2}$  and  $z_2 = \sqrt{r}e^{i(\pi+t/2)}$ .

Assume that there exists a function  $g$  *continuous* in  $\mathbf{C}$  such that  $(g(w))^2 = w$  for every  $w \in \mathbf{C}$ . In particular  $g(e^{i\theta}) = e^{i\varphi(\theta)}$ , for  $-\pi < \theta < \pi$ , where  $\varphi$  is a function which takes at each point of  $]-\pi, \pi[$  one of the values  $\frac{1}{2}\theta, \pi + \frac{1}{2}\theta$ , which has the property that  $e^{i\varphi(\theta)}$  is *continuous* in  $]-\pi, \pi[$  and tends to the *same* limit at the points  $\pi$  and  $-\pi$ . Suppose for example that  $\varphi(0) = 0$ ; if at a point  $\theta_0 \in ]-\pi, \pi[$  we have  $\varphi(\theta_0) = \frac{1}{2}\theta_0$ , we must also have  $\varphi(\theta) = \frac{1}{2}\theta$  in an *interval* of centre  $\theta_0$  and of arbitrarily small length, since in such an interval  $e^{i\theta_0/2} - e^{i(\pi+\frac{1}{2}\theta)}$  is arbitrarily near to  $2e^{i\theta_0/2}$ , whereas by hypothesis the difference  $e^{i\varphi(\theta)} - e^{i\varphi(\theta_0)}$  must be arbitrarily small. It therefore follows from (0, 5.9) applied to the *locally constant* function  $\varphi(\theta) - \frac{1}{2}\theta$ , that this function must be identically zero in  $]-\pi, \pi[$ . But then, when  $\theta$  tends to  $\pi$  and  $\theta'$  to  $-\pi$ ,  $e^{i\varphi(\theta)}$  tends to  $i$ , whereas  $e^{i\varphi(\theta')}$  tends to  $-i$ , and by hypothesis these two limits should be equal, thus giving a contradiction.

(8.4) If therefore we wish to speak of the “inverse function” of the function  $f: z \rightarrow z^2$ , we must *restrict* its domain of definition. This can be done, for example, in the following way: let  $D_0$  be the complement in  $\mathbf{C}$  of the half-line  $L_0 = -\mathbf{R}_+ = ]-\infty, 0]$ , i.e. the set of points  $re^{i\theta}$  with  $r > 0, -\pi < \theta < \pi$  (*equalities excluded*) (Fig. 42). Note that for each  $w \in D_0$ , the values of  $r$  and  $\theta$  such that  $re^{i\theta} = w, r > 0$  and  $-\pi < \theta < \pi$  are determined in a *unique* way: if  $w = s + it$  with  $s, t$  real, we have  $r = (s^2 + t^2)^{1/2}$  and  $\theta = 2 \arctan(t/(r+s))$ . We can thus put  $g_0(w) = \sqrt{r}e^{i\theta/2}$ , and the preceding formulae show that  $g_0$  is continuous in  $D_0$ , and that  $g_0(D_0)$  is the *open half-plane*  $H: \Re z > 0$ . Since  $f(H) = D_0$  and  $f$  is injective in  $H$ , it is seen that  $g_0$  is an *analytic* function in  $D_0$  satisfying  $f(g_0(w)) = w$  (8.1). Also,  $f(-g_0(w)) = w$ , and the two analytic functions  $g_0$  and  $-g_0$  are said to be the *two determinations* of the inverse function of  $f$  in  $D_0$ . Note that either of these functions is completely determined if the value of the function is known at *just one* point of  $D_0$  (since it is determined by its sign).

The set  $D_0$  is called the *plane cut along the real negative half-axis*  $L_0 = -\mathbf{R}_+$ . Note that at a point  $-s_0 < 0$  of  $-\mathbf{R}_+$  the function  $g_0$  *does not have a limit*: as  $t$  tends to 0 in  $\mathbf{R}$  through values  $> 0$ ,  $g_0(-s_0 + it)$  tends to the limit  $i\sqrt{s_0}$  and  $g_0(-s_0 - it)$  tends to the limit  $-i\sqrt{s_0}$ .

(8.5) Instead of “cutting” the plane along the negative real axis, we can consider, for each  $\alpha$  satisfying  $0 \leq \alpha < 2\pi$ , the complement  $D_\alpha = e^{i\alpha}D_0$  of the half-line  $L_\alpha = e^{i\alpha}L_0$ :

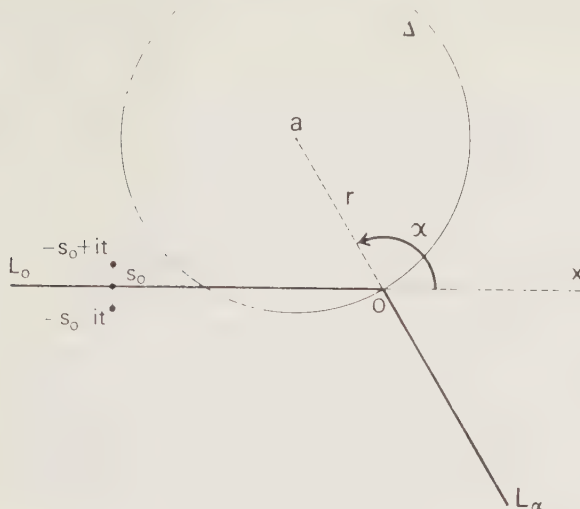


FIGURE 42

$r \rightarrow -re^{i\alpha}$  ( $0 \leq r < +\infty$ ) (the “plane cut along the line  $L_\alpha$ ”). Two determinations  $g_\alpha$  and  $-g_\alpha$  of the inverse function of  $f$  in  $D_\alpha$  can then be defined such that  $g_\alpha(re^{i\alpha}) = g_0(re^{i\alpha})$  for  $r > 0$ . They are not defined at the points of the “cut”  $L_\alpha$ ; the intersection  $D_\alpha \cap D_0$  is the union of two open sets with no common points, the “angular sectors” defined by the half-lines  $L_0$  and  $L_\alpha$ ; if  $g_\alpha$  coincides with  $g_0$  in one of these sectors, it coincides with  $-g_0$  in the other, since  $g_0$  (resp.  $g_\alpha$ ) tends to different limits on each side of  $L_0$  (resp.  $L_\alpha$ ). More generally it can be shown that in every open simply connected set not containing the point 0, there exist two analytic functions  $g$  and  $-g$  such that  $(g(w))^2 = w$ . It is thus sensible in the theory of analytic functions to speak of a “determination”  $g$  of the inverse function of  $f: z \rightarrow z^2$ , only when we have fixed: first, the open connected set where  $g$  is defined; secondly, the value taken by  $g$  at a point of  $D$ . There are thus *infinitely many* functions (such as the  $g_\alpha$ ) each defined in different open sets, but *maximal*, in the sense that the function *cannot be continued* into a larger open set, the function  $g_\alpha$  having no limit at the boundary points of  $D_\alpha$  except at 0.

(8.6) This situation may seem shocking; however it is no remedy to say, as unfortunately many books do on this problem, that we can define a so-called “multivalued function” as the inverse of the function  $z \rightarrow z^2$  by admitting that this so-called “function” has *two* values at each point  $\neq 0$ . Such a “definition” is mere *verbiage*, for the authors take good care not to give the slightest rule for calculating with these new mathematical objects which they pretend to define. This of course makes the so-called definition useless; it is easy to see why they avoid this problem, since by admitting two values, between which no distinction is made, for the so-called function  $\sqrt{z}$ , one is compelled to admit four values for  $\sqrt{z} + 2\sqrt{z}$ , eight values for  $\sqrt{z} + 2\sqrt{z} + 4\sqrt{z}$ , and so on, which makes all calculations impossible.

There is in fact a solution to the paradox of “multivalued functions”, the profound and powerful theory of “Riemann Surfaces”, which is beyond the level of this book.

(8.7) The problem of the inversion of the function  $f: z \rightarrow z^2$  illustrates the difficulties of the problem of analytic continuation mentioned in no. 1 of this Chapter. Consider a point  $a = re^{i\alpha}$  of  $D_0$  and let us seek the *disc of convergence*  $\Delta$  of the Taylor series of the function  $g_0(z)$  at the point  $a$ . Note that this disc cannot contain the point  $z = 0$ , since, by virtue of the principle of analytic continuation (VI, 7.3), the sum  $h(z)$  of this series would satisfy the relation  $(h(z))^2 = z$  in  $\Delta$ , so we would have  $h(0) = 0$  and  $2h(z)h'(z) = 1$  and hence  $2h(0)h'(0) = 1$ , which is absurd. However in fact  $\Delta$  is *exactly* the disc  $|z - a| < r$  whose boundary passes through the point 0: indeed the function  $g_0$  coincides with  $g_\alpha$  in a neighbourhood of  $a$ , and  $g_\alpha$  is analytic in the latter disc, so the conclusion follows from (VII, 7.3).

This being the case, if for example  $\pi/2 < \alpha < \pi$ , the intersection  $D_0 \cap \Delta$  is *not connected* (Fig. 42) and the sum  $h(z)$  of the Taylor series of  $g_0$  at the point  $a$  coincides with  $g_0$  in the subset of  $D_0 \cap \Delta$  contained in the half-plane  $\mathcal{I}z > 0$ , but with  $-g_0$  in the subset of  $D_0 \cap \Delta$  contained in the half-plane  $\mathcal{I}z < 0$ . The point  $z = 0$  is a *singular* point of the function  $h$ , all the other points of the circle  $|z - a| = r$  being *regular*; but despite appearances,  $z = 0$  is *not an isolated singularity* in the sense defined in (2.1); it is a new kind of singular point which is sometimes called a *branch point*.

## 9. The logarithmic function

(9.1) In studying the function  $f: z \rightarrow z^2$  we discovered the existence of “maximal” open sets such as the half-plane  $H: \Re z > 0$ , for which  $f$  is injective in  $H$  but no longer injective in any larger open set. We shall show similarly the existence of such maximal open sets for the exponential function, which will enable us to define the “determinations” of its “inverse function”.

(9.2) Put  $z = x + iy$ ,  $w = s + it = e^z$  ( $x, y, s, t$  real); thus (VI, 8.5.1)

$$(9.2.1) \quad s = e^x \cos y, \quad t = e^x \sin y;$$

since  $e^x > 0$ , we deduce from this

$$(9.2.2) \quad e^x = \sqrt{s^2 + t^2}.$$

When  $w$  is given in  $\mathbf{C}$ , this equation has just one solution

$$(9.2.3) \quad x = \log \sqrt{s^2 + t^2} = \log |w|$$

provided  $w \neq 0$ ; once  $x$  is thus determined,  $y$  is a solution of the equations

$$\cos y = \frac{s}{|w|}, \quad \sin y = \frac{t}{|w|}$$

which always possess infinitely many solutions, among which precisely one,  $y_0$ , satisfies the inequalities  $-\pi < y_0 \leq \pi$ , all others having the form  $y_0 + 2k\pi$  with  $k \in \mathbf{Z}$ . If we put

$y_0 = \arg w$  (a number again called, by an abuse of language, the argument or amplitude of  $w$ ), we have, for  $w \neq 0$ ,

$$(9.2.4) \quad \arg w = 2 \arctan \frac{t}{s + |w|}$$

agreeing to take  $\arg w = \pi$  for  $t = 0, s < 0$ . Therefore, for given  $w \neq 0$  the solutions of the equation  $e^z = w$  are the complex numbers

$$(9.2.5) \quad z = \log |w| + i(\arg w + 2k\pi), \quad k \in \mathbf{Z}.$$

(9.3) These results show in particular that the function  $f: z \rightarrow e^z$  is *injective* in the *open* set  $B: -\pi < \mathcal{I}z < \pi$ . The image under the mapping  $f$  of the line  $y = y_0$  ( $-\pi < y_0 < \pi$ ) contained in  $B$  is the *open* half-line  $w = e^x e^{iy_0}$  ( $-\infty < x < +\infty$ ) (therefore not containing the point 0); the image of the segment  $x = x_0, -\pi < y < \pi$  is the circle  $|w| = e^{x_0}$  with the point  $w = -e^{x_0}$  deleted. The open set  $f(B)$  is therefore the *cut plane*  $D_0$  defined in (8.4). Applying (8.1), the function  $w \rightarrow \log |w| + i \arg w$  is *analytic* in  $D_0$  and is a bijection of  $D_0$  onto the strip  $B$ ; we denote it by  $w \rightarrow \log w$  and say that it is the *principal determination of the logarithm* in  $D_0$ . Thus

$$(9.3.1) \quad e^{\log w} = w, \quad \log(e^z) = z$$

for  $w \in D_0$  and  $z \in B$ , and

$$(9.3.2) \quad \log w = \log |w| + i \arg w \quad \text{for } w \in D_0.$$

Every other function  $g$  continuous in  $D_0$  and satisfying  $e^{g(w)} = w$  has the form  $g_k(w) = \log w + 2k\pi i$  with  $k \in \mathbf{Z}$ , since the difference  $g(w) - \log w$  must be an integral multiple of  $2\pi i$  from (9.2.5), so must be constant since it is continuous and  $D_0$  is connected (0, 5.9). The functions  $g_k$  are called *the determinations of the logarithm* in  $D_0$ ; we see here that there are *infinitely many* of them; a determination of the logarithm is completely fixed by knowledge of its value at one real point  $x > 0$ , since

$$g_k(x) = \log x + 2k\pi i.$$

At a point  $-s_0 < 0$  the function  $\log w$  has *no limit*, since as  $t$  tends to 0 in  $\mathbf{R}$  through values  $> 0$ ,  $\arg(-s_0 + it)$  tends to  $\pi$ , and  $\arg(-s_0 - it)$  tends to  $-\pi$ ; as  $w$  tends to 0 (remaining in  $D_0$ )  $\log |w|$  tends to  $+\infty$ , hence so does  $|\log w|$ . The set  $D_0$  is thus a *maximal domain* for the “inverse functions” of  $e^z$ .

Note that the function  $\log \sqrt{s^2 + t^2}$ , defined in  $\mathbf{C} - \{0\}$ , is a solution of Laplace's equation (VII, 9.5.1) in this open set; but there exists *no analytic function* in  $\mathbf{C} - \{0\}$  which has a real part  $\log \sqrt{s^2 + t^2}$ , for by virtue of the Cauchy conditions such a function would be of the form  $\log w + C$  ( $C$  constant) in  $D_0$ , and it has been seen that this function *cannot be continued* to a continuous function in  $\mathbf{C} - \{0\}$ .

(9.4) It follows from (8.1) that in  $D_0$

$$(9.4.1) \quad \frac{d}{dw} (\log w) = \frac{1}{w}$$

from which it follows that the Taylor series of the function  $\log w$  at the point  $w = 1$  is

$$(9.4.2) \quad \log(1 + z) = z - \frac{z^2}{2} + \cdots + \frac{(-1)^{n-1}}{n} z^n + \cdots$$

the same reasoning as in (8.7) showing that the disc of convergence of this power series is  $|z| < 1$ .

If  $w \in D_0$ , then also  $1/w \in D_0$  since  $\arg(1/w) = \arg(\bar{w}) = -\arg(w)$ ; therefore in  $D_0$

$$(9.4.3) \quad \log(1/w) = -\log w.$$

On the other hand, the relation

$$(9.4.4) \quad \arg(w_1 w_2) = \arg w_1 + \arg w_2 \quad \text{for } w_1, w_2 \text{ in } D_0$$

is true only if

$$(9.4.5) \quad -\pi < \arg w_1 + \arg w_2 < \pi.$$

Therefore when this last relation is satisfied

$$(9.4.6) \quad \log(w_1 w_2) = \log w_1 + \log w_2.$$

On the other hand, if  $\arg(w_1) + \arg(w_2) > \pi$  (resp.  $\arg(w_1) + \arg(w_2) < -\pi$ )

$$\log(w_1 w_2) = \log w_1 + \log w_2 - 2i\pi$$

(resp.  $\log(w_1 w_2) = \log w_1 + \log w_2 + 2i\pi$ ); if  $\arg(w_1) + \arg(w_2) = \pm\pi$ ,  $\log(w_1 w_2)$  is not defined since  $w_1 w_2$  is real and  $< 0$ .

(9.5) Instead of considering the determinations of the logarithm in  $D_0$ , we can consider them in the cut plane  $D_\alpha = e^{i\alpha} D_0$  for every  $\alpha$  satisfying  $0 \leq \alpha < 2\pi$ ; they are the functions

$$g_{\alpha,k}(w) = i\alpha + g_k(e^{-i\alpha}w) = i(\alpha + 2k\pi) + \log(e^{-i\alpha}w).$$

The Taylor series of the function  $g_{\alpha,k}$  at the point  $w_0 = re^{i\alpha}$  ( $r > 0$ ) is thus

$$(9.5.1) \quad g_{\alpha,k}(w_0 + z) = \log r + i(\alpha + 2k\pi) + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \left(\frac{z}{w_0}\right)^n$$

and its disc of convergence is  $|z| < r = w_0$ , the only singular point on its circle of convergence being the point  $z = -w_0$  (again called a “branch point”). In particular, the Taylor series of  $\log w$  at a point  $w_0 = re^{i\alpha}$  with  $-\pi/2 \leq \alpha \leq \pi/2$  is equal to  $\log(w_0 + z)$  at every point of the disc of convergence  $\Delta$ . On the other hand, for  $\pi/2 < \alpha < \pi$  (resp.  $-\pi < \alpha < -\pi/2$ ) the sum of this series is  $\log(w_0 + z)$  in the subset of  $D_0 \cap \Delta$  contained in the half-plane  $\mathcal{J}w > 0$  (resp.  $\mathcal{J}w < 0$ ), and  $\log(w_0 + z) + 2i\pi$  (resp.  $\log(w_0 + z) - 2i\pi$ ) in the subset of  $D_0 \cap \Delta$  contained in the half-plane  $\mathcal{J}w < 0$  (resp.  $\mathcal{J}w > 0$ ).

(9.6) The definition of the principal determination of the logarithm enables us to define, for every complex number  $\lambda$ , the “principal determination” of the “ $\lambda^{\text{th}}$  power” by the formula

$$(9.6.1) \quad z^\lambda = e^{\lambda \log z},$$

for every  $z \in D_0$ . With this definition, for  $z \in D_0$ ,

$$(9.6.2) \quad z^{\lambda+\mu} = z^\lambda z^\mu \quad \text{and} \quad (z^\lambda)^\mu = z^{\lambda\mu} \quad (\text{if } z^\lambda \in D_0)$$

for any complex numbers  $\lambda$  and  $\mu$  and in particular  $(e^z)^{z'} = e^{zz'}$  for  $z \in \mathbf{B}$ ,  $z' \in \mathbf{C}$ . Since

$z^1 = z$  by virtue of (9.3.1), the function  $z^n$  coincides with the  $n^{\text{th}}$  power in the usual sense for every positive or negative integer  $n$ . Also in  $D_0$

$$(9.6.3) \quad \frac{d}{dz} (z^\lambda) = \lambda z^{\lambda-1}$$

for by virtue of (9.4.1), we have

$$\frac{d}{dz} (e^{\lambda \log z}) = \frac{\lambda}{z} e^{\lambda \log z} = \frac{\lambda}{z} z^\lambda = \lambda z^{\lambda-1}$$

taking into account (9.6.2). The Taylor series of  $z^\lambda$  at the point 1 is thus

$$(9.6.4) \quad (1+z)^\lambda = 1 + \binom{\lambda}{1} z + \binom{\lambda}{2} z^2 + \cdots + \binom{\lambda}{n} z^n + \cdots$$

We see at once that *except when  $\lambda$  is an integer* (positive or negative) the function  $z^\lambda$  has no limit at the real points  $-x < 0$  and that  $D_0$  is therefore a *maximal* domain for the function  $z^\lambda$ . On the contrary, for  $n$  integer  $\geq 0$ ,  $z^n$  is an entire function and for  $n$  integer  $< 0$ ,  $z^n$  has just one pole of order  $|n|$  at the point 0. The reasoning of (8.7) therefore shows that, *except when  $\lambda$  is an integer  $\geq 0$* , the disc of convergence of the series (9.6.4) is  $|z| < 1$  and  $-1$  is the only singular point on the circle of convergence.

The formula

$$(9.6.5) \quad (z_1 z_2)^\lambda = z_1^\lambda z_2^\lambda$$

is valid only if  $-\pi < \arg z_1 + \arg z_2 < \pi$  (cf. (9.4.4)); otherwise, the second member is multiplied by  $e^{2i\lambda\pi}$  or  $e^{-2i\lambda\pi}$ . Furthermore, if we replace  $\log z$  in (9.6.1) by another “determination” of the logarithm in  $D_0$ ,  $\log z + 2k\pi i$  (with  $k \in \mathbf{Z}$ ), we obtain other determinations of the “ $\lambda^{\text{th}}$  power” in  $D_0$ , the functions  $e^{2k\lambda\pi i} z^\lambda$ . If  $\lambda$  is not a *rational number*, all these functions are *distinct*. If on the other hand  $\lambda = p/q$ , where  $p$  and  $q$  are relatively prime integers ( $q > 0$ ), there are exactly  $q$  distinct “determinations”, which can again be written  $\omega^h z^{p/q}$ , where  $\omega = e^{2i\pi/q}$  is the “ $q^{\text{th}}$  root of unity” of smallest argument  $> 0$  and  $h$  takes all integer values from 0 to  $q-1$ . In particular, for  $p/q = \frac{1}{2}$ , we again obtain the two functions  $g_0$  and  $-g_0$  considered in (8.4). There are of course corresponding “determinations” of the  $\lambda^{\text{th}}$  power in the open sets  $D_\alpha$  for  $0 \leq \alpha < 2\pi$ .

*Example (9.6.6)* As an example of an effective calculation determine the complex number  $i^i$ . By definition, it is the number  $e^{i \log i}$ , and  $\log i = i(\pi/2)$ , hence  $i^i = e^{-\frac{1}{2}\pi}$ .

(9.7) Functions are often met possessing several branch points; we do not try to formulate a general theory, but confine ourselves to considering a typical example, that of a product of the form

$$(9.7.1) \quad z^{\lambda_0} (z - a_1)^{\lambda_1} \cdots (z - a_r)^{\lambda_r}$$

where the  $a_j$  ( $1 \leq j \leq r$ ) are distinct complex numbers  $\neq 0$  and the  $\lambda_j$  are any complex numbers. To give a meaning to such a function its domain of definition must be determined precisely. If

$$a_j = \rho_j e^{i\alpha_j} \quad (0 \leq \alpha_j \leq 2\pi),$$

we take for example the plane “cut” along the half-lines  $L_0$  and  $a_j + L_{\pi + \alpha_j}$  (with the notations of (8.4)) (Fig. 43); then fix at one point of this open set  $D$  the “determination”

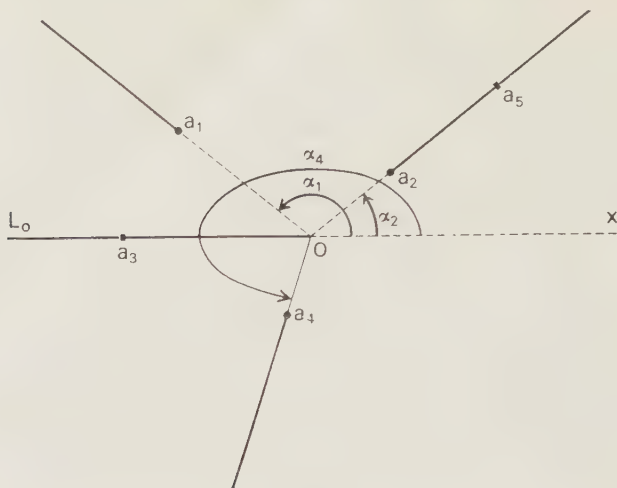


FIGURE 43

chosen for each of the factors of the product (9.7.1); this product is then a well-defined function in  $D$  and the function (9.7.1) is by definition this function. Some of the factors  $(z - a_j)^{\lambda_j}$  could of course be replaced by  $\log(z - a_j)$ , or even by powers  $(\log(z - a_j))^{\lambda_j}$  etc.

**Remark (9.8)** Let  $f$  be a function analytic in the “cut plane”  $D_0$  (8.4), and suppose that there exists a function  $g$ , continuous in  $\mathbf{C} - \{0\}$  and coinciding with  $f$  in  $D_0$  (which can also be expressed by saying that at each point  $-x < 0$  of the negative real half-axis,  $f(z)$  has a limit as  $z$  tends to  $-x$  while remaining in  $D_0$  and moreover this limit is a continuous function of  $-x$  for  $x > 0$ ). Then  $g$  is analytic in  $\mathbf{C} - \{0\}$  (i.e. 0 is a regular point or an isolated singular point for  $f$ , but not a branch point).

This follows from the following more general proposition:

(9.8.1) Let  $P_+$  and  $P_-$  be the open half-planes  $\mathcal{I}z > 0$ ,  $\mathcal{I}z < 0$ ,  $D$  an open set in  $\mathbf{C}$ ,  $f$  a continuous complex function in  $D$ . Suppose furthermore that  $f$  is analytic in each of the two open sets  $D \cap P_+$  and  $D \cap P_-$ ; then  $f$  is analytic in  $D$ .

Everything reduces to showing that  $f$  is analytic at a real point  $x_0 \in D$ ; consider a square  $Q: |x - x_0| \leq a, |y| \leq a$  of centre  $x_0$  contained in  $D$ ; if  $\gamma$  is the “perimeter of the square” as defined in (6.6), for every  $z$  interior to  $Q \cap P_+$  or to  $Q \cap P_-$ , we will show that

$$(9.8.2) \quad 2\pi i f(z) = \int_{\gamma} \frac{f(u) du}{u - z}.$$

Then the second member is a function analytic in the whole interior of the square  $Q$  (7.2); since it coincides with  $f$  in  $Q \cap P_+$  and  $Q \cap P_-$  and since  $f$  is continuous, the two members of (9.8.2) are equal in the whole interior of  $Q$ , hence the conclusion. It remains to prove

(9.8.2). Suppose for example  $\mathcal{J}z > 0$ ; then Cauchy's formula shows on the one hand that, for  $0 < h < \mathcal{J}z$

$$2\pi if(z) = \int_{\gamma'_h} \frac{f(u) du}{u - z} \quad \text{and} \quad 0 = \int_{\gamma''_h} \frac{f(u) du}{u - z}$$

$\gamma'_h$  being the "perimeter of the rectangle" with vertices  $x_0 - a + ih$ ,  $x_0 + a + ih$ ,  $x_0 + a + ia$ ,  $x_0 - a + ia$ ,  $\gamma''_h$  the perimeter of the symmetric rectangle with respect to the real axis (Fig. 44). By adding these two formulae, it is seen that the difference of the two members of (9.8.2) is equal to

$$\int_{-a}^a \left( \frac{f(x_0 + ih + t)}{x_0 + ih + t - z} - \frac{f(x_0 - ih + t)}{x_0 - ih + t - z} \right) dt \\ - i \int_{-h}^h \left( \frac{f(x_0 + a + it)}{x_0 + a + it - z} - \frac{f(x_0 - a + it - z)}{x_0 - a + it - z} \right) dt.$$

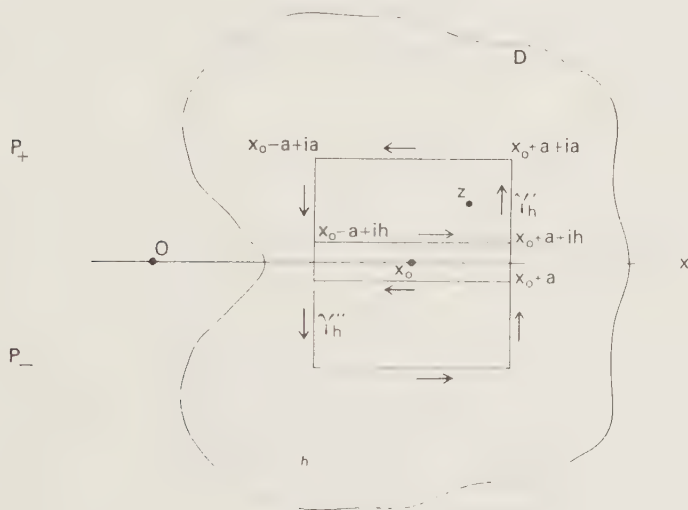


FIGURE 44

By virtue of the uniform continuity of the function  $f$  in  $Q$  (0, 5.6) and the theorem of the mean, it follows immediately that the absolute value of this sum is arbitrarily small with  $h$ , hence (9.8.2). Similar reasoning applies to  $\mathcal{J}z < 0$ .

(9.8.3) Consider in particular a function  $F(w)$  analytic in a half-plane  $S$ :  $\Re w < a$  and such that  $F(w + 2i\pi) = F(w)$  in  $S$ . Then, if  $\Delta$  is the disc  $|z| < e^a$ , there exists a function  $G(z)$  analytic in  $\Delta - \{0\}$  such that  $F(w) = G(e^w)$ ; indeed, the function  $F(\log z)$  is analytic in  $\Delta \cap D_0$  and since by hypothesis  $F(\log r - i\pi) = F(\log r + i\pi)$  for  $0 < r < e^a$ , it is continued to a continuous function  $G(z)$  in  $\Delta - \{0\}$ , to which the preceding is applied.

(9.8.4) Similarly, consider a function  $F(z)$  analytic in a disc  $\Delta$ :  $|z| < a$  and such that, for an integer  $m > 0$ ,  $F(e^{2\pi i/m} z) = F(z)$ . Then there exists a function  $G(z)$  analytic in the disc  $|z| < a^m$  and such that  $F(z) = G(z^m)$  for  $|z| < a$ . One reasons as in (9.8.3) considering the function  $F(z^{1/m})$  in  $\Delta \cap D_0$ .

(9.9) We have seen (VI, 8.7.6) that for every complex square matrix  $A$ , the exponential  $e^A$  is an *invertible* matrix. Conversely, we shall see that for *each complex invertible matrix*  $B$ , there exists at least one matrix  $A$  such that  $e^A = B$  (since  $e^{A+2n\pi i I} = e^A$ , there exist infinitely many matrices with this property). Taking into account (VI, 8.7.8), assume that  $B$  has Jordan canonical form; then by using (VI, 8.7.10), reduce to the case where  $B$  is a Jordan matrix  $\lambda I + N$ , with

$$N = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ . & . & . & . & . \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

Since  $B$  is invertible, by hypothesis  $\lambda \neq 0$ , so there exists  $\mu \in \mathbf{C}$  such that  $e^\mu = \lambda$ , and hence  $e^{\mu I} = \lambda I$ . Since  $B = \lambda I(I + \lambda^{-1}N)$ , it is sufficient to find a matrix  $C$  such that  $e^C = I + \lambda^{-1}N$ , since  $I$  commutes with  $C$  (cf. (VI, 8.7.5)). Note that  $N^n = 0$ ; it will be seen that the matrix

$$C = \lambda^{-1}N - \frac{\lambda^{-2}}{2} N^2 + \dots + (-1)^n \frac{\lambda^{-(n-1)}}{n-1} N^{n-1}$$

obtained by replacing  $z$  by  $\lambda^{-1}N$  in the  $n-1$  first terms of the power series for  $\log(1+z)$  (9.4.2) answers the question. It is clear that  $C^n = 0$  so

$$e^C = I + \frac{1}{1!}C + \frac{1}{2!}C^2 + \dots + \frac{1}{(n-1)!}C^{n-1}.$$

Since by definition  $e^{\log(1+z)} = 1+z$  for  $|z| < 1$ , the relation  $e^C = I + \lambda^{-1}N$  follows from the theorem of substitution of one power series into another power series (VI, 4.4) which proves that the coefficients of the power series obtained by substituting  $\log(1+z)$  for  $z$  in the series  $e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!}$ , are equal to 0 except for the constant term and the coefficient of  $z$ , which are equal to 1.

## 10. Application to the calculation of integrals

(10.1) As a first application we prove the formulae (4.6.4) of Chap. IV:

$$(10.1.1) \quad \begin{cases} \int_0^{+\infty} x^{\lambda-1} e^{ix} dx = e^{1/2\lambda\pi i} \Gamma(\lambda) \\ \int_0^{+\infty} x^{\lambda-1} e^{-ix} dx = e^{-1/2\lambda\pi i} \Gamma(\lambda) \end{cases}$$

for every real number  $\lambda$  satisfying  $0 < \lambda < 1$ . To prove for example the first of these formulae, consider the function  $f(z) = z^{\lambda-1} e^{iz}$  in the plane cut along the negative real axis  $L_0$ , where we take the “determination” defined by (9.6.1) for the  $(\lambda-1)^{\text{th}}$

power, which is thus equal to  $x^{\lambda-1}$  (in the real sense) for  $x$  real and  $> 0$ . Apply Cauchy's theorem (VII, 5.2) to this analytic function, taking for  $\gamma$  the loop defined as the juxtaposition  $\gamma_1 \vee \gamma_2 \vee \gamma_3 \vee \gamma_4$  of the four paths

$$\gamma_1(t) = t \quad \text{for } r \leq t \leq R$$

$$\gamma_2(t) = R e^{it} \quad \text{for } 0 \leq t \leq \frac{\pi}{2}$$

$$\gamma_3(t) = -it \quad \text{for } -R \leq t \leq -r$$

$$\gamma_4(t) = r e^{i(\frac{1}{2}\pi - t)} \quad \text{for } 0 \leq t \leq \frac{\pi}{2}$$

where  $0 < r < R$  (Fig. 45). Thus

$$(10.1.2) \quad \int_{\gamma_1} f(z) dz + \int_{\gamma_2} f(z) dz + \int_{\gamma_3} f(z) dz + \int_{\gamma_4} f(z) dz = 0$$

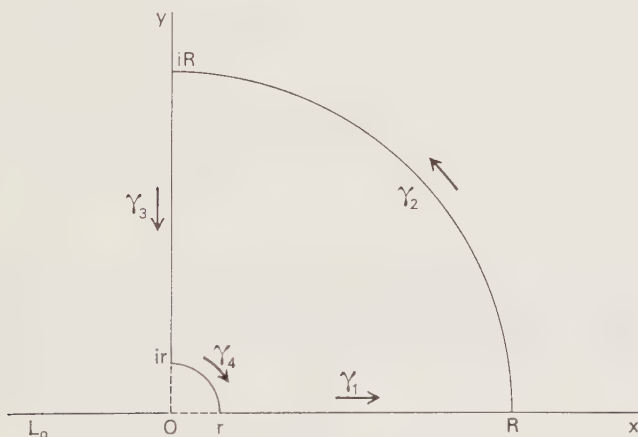


FIGURE 45

the plane cut along  $L_0$  being simply connected. By definition and the choice of the determination of  $f$ , we have

$$(10.1.3) \quad \int_{\gamma_1} f(z) dz = \int_r^R x^{\lambda-1} e^{tx} dx$$

and for  $z = iy$  with  $y > 0$  we have by definition

$$z^{\lambda-1} = e^{(\lambda-1)\log z} = e^{(\lambda-1)(\log y + i(\pi/2))} = e^{(\lambda-1)i(\pi/2)} y^{\lambda-1};$$

hence

$$(10.1.4) \quad \int_{\gamma_3} f(z) dz = -e^{1/2\lambda\pi i} \int_r^R t^{\lambda-1} e^{-t} dt.$$

Note that by definition, as  $r$  tends to 0 and  $R$  tends to  $+\infty$ , the second member of (10.1.4) tends to the second member of the first formula (10.1.1) with its sign changed. Thus everything reduces to showing that in these conditions the integrals along  $\gamma_2$  and  $\gamma_4$  tend to 0. Now

$$\left| \int_{\gamma_2} f(z) dz \right| \leq \int_0^{\pi/2} R^\lambda e^{-R \sin t} dt$$

and since  $0 < \lambda < 1$ , the second member tends to 0 with  $1/R$  by virtue of Jordan's lemma (5.3.2); similarly

$$\left| \int_{\gamma_4} f(z) dz \right| \leq r^\lambda \int_0^{\pi/2} e^{-r \sin t} dt \leq \frac{\pi r^\lambda}{2}$$

which tends to 0 with  $r$ . The second formula (10.1.1) is similarly proved, integrating this time along the closed path deduced from  $\gamma$  by symmetry with respect to the real axis.

(10.2) We now give a general method for calculating integrals of the form

$$(10.2.1) \quad \int_0^{+\infty} x^{\lambda-1} Q(x) dx$$

where  $\lambda$  is a real number  $> 0$ ,  $Q$  a rational function (with complex coefficients) possessing no pole which is real and  $\geq 0$  and such that  $Q(0) \neq 0$  and  $\lim_{x \rightarrow +\infty} x^\lambda |Q(x)| = 0$  (which is equivalent to saying that if  $Q = F/G$ , where  $F$  and  $G$  are two polynomials,  $\deg F < \deg G - \lambda$ ).

This time consider the function

$$f(z) = (-z)^{\lambda-1} Q(z)$$

in the plane  $D_{2\pi}$  cut along the positive real axis  $L_{2\pi}$ ; by definition, we then take  $(-z)^{\lambda-1} = e^{(\lambda-1)\log(-z)}$  in this open set (which has a meaning, since then  $-z \in D_0$  and the principal determination of the logarithm is precisely defined in  $D_0$ ). The function  $f$  is meromorphic in  $D_{2\pi}$ ; then consider the loop  $\gamma$  (Fig. 46), which is the juxtaposition  $\gamma_1 \vee \gamma_2 \vee \gamma_3 \vee \gamma_4$  of the four paths

$$\begin{aligned} \gamma_1(t) &= e^{i\alpha} t & \text{for } r \leq t \leq R \\ \gamma_2(t) &= R e^{it} & \text{for } \alpha \leq t \leq 2\pi - \alpha \\ \gamma_3(t) &= -e^{-i\alpha} t & \text{for } -R \leq t \leq -r \\ \gamma_4(t) &= r e^{i(2\pi-t)} & \text{for } \alpha \leq t \leq 2\pi - \alpha. \end{aligned}$$

Since  $\gamma$  is contained in  $D_{2\pi}$ , by the residue theorem (4.3),

$$(10.2.2) \quad \int_{\gamma_1} f(z) dz + \int_{\gamma_2} f(z) dz + \int_{\gamma_3} f(z) dz + \int_{\gamma_4} f(z) dz = 2\pi i \sum_k \operatorname{Res}_{a_k} f$$

the sum being taken over the poles of  $f$  with index  $\neq 0$  with respect to the closed path  $\gamma$  (it follows immediately that every point of  $D_{2\pi}$  has index with respect to  $\gamma$  equal to 0 or 1; cf. (VI, 2.2.3)). Recalling the above definition of  $(-z)^{\lambda-1}$ , note first that when  $\alpha$  tends to 0 in (10.2.2) leaving  $r$  and  $R$  fixed, the integrals along  $\gamma_1$  and  $\gamma_3$  tend respectively to

$$e^{-(\lambda-1)i\pi} \int_r^R t^{\lambda-1} Q(t) dt \quad \text{and} \quad -e^{(\lambda-1)i\pi} \int_r^R t^{\lambda-1} Q(t) dt,$$

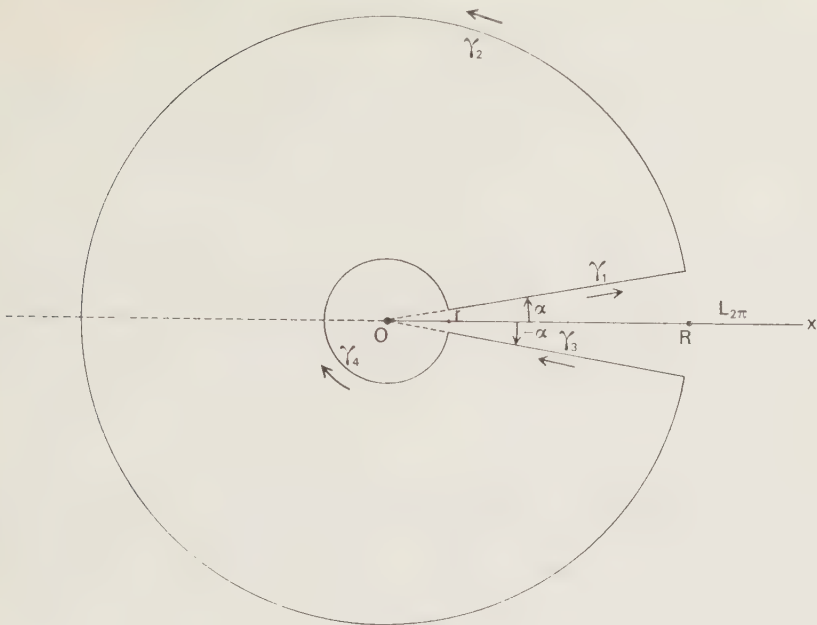


FIGURE 46

On the other hand

$$\left| \int_{\gamma_2} f(z) dz \right| \leq \int_0^{2\pi} R^\lambda |Q(Re^{it})| dt, \quad \left| \int_{\gamma_4} f(z) dz \right| \leq \int_0^{2\pi} r^\lambda |Q(re^{it})| dt$$

and because of the hypotheses made, the second members of these inequalities tend to 0 as  $R$  tend to  $+\infty$  and  $r$  tends to 0.

Therefore, in the limit, taking into account the fact that  $\sin((\lambda - 1)\pi) = -\sin \lambda\pi$ ,

$$(10.2.3) \quad \int_0^{+\infty} x^{\lambda-1} Q(x) dx = \frac{\pi}{\sin \lambda\pi} \sum_k \text{Res}_{a_k} f$$

where the sum is taken over *all the poles* of the rational function  $Q(z)$ .

For example take  $Q(z) = 1/(1+z)$  and  $0 < \lambda < 1$ . There is one simple pole,  $z = -1$ , and  $f$  has the residue 1 at this point (as  $1^{\lambda-1} = 1$  by definition of the “determination”  $z^{\lambda-1}$ ). Hence the formula

$$(10.2.4) \quad \int_0^{+\infty} \frac{x^{\lambda-1}}{1+x} dx = \frac{\pi}{\sin \lambda\pi} \quad \text{for } 0 < \lambda < 1.$$

## 11. Application to infinite products

(11.1) The idea of an “infinite product” of complex numbers is similar in conception to that of a series, multiplication replacing addition. Given a sequence  $(a_n)$  of complex

numbers, we form from it the sequence of "partial products"

$$p_1 = a_1, \quad p_2 = a_1 a_2, \quad \dots, \quad p_n = a_1 a_2 \dots a_n, \quad \dots$$

and say that the infinite product of general factor  $a_n$  is convergent if the sequence  $(p_n)$  is convergent in  $\mathbf{C}$ ; its limit  $p$  is called the infinite product of the sequence  $(a_n)$  and denoted

$$(11.1.1) \quad p = \prod_{n=1}^{\infty} a_n.$$

The particular properties of the number 0 relative to multiplication entail for infinite products certain peculiarities which have no counterpart for series: if at least one of the  $a_n$  is zero,  $p_n = 0$  for all sufficiently large  $n$  and the sequence  $(p_n)$  converges to 0 whatever the other terms of the sequence  $(a_n)$ . Let us consider only sequences  $(a_n)$  no term of which is zero (unless express mention otherwise is made), and moreover consider the infinite product to be convergent only if the number  $p = \lim_{n \rightarrow \infty} p_n$  is non-zero (it is then said that the product converges in the strict sense).

(11.2) With the preceding convention we can write for every  $n \geq 2$ ,  $a_n = p_n/p_{n-1}$ ; thus if the infinite product converges in the strict sense (hence to  $p \neq 0$  by convention), then  $\lim_{n \rightarrow \infty} a_n = 1$ . For this reason we usually write  $a_n = 1 + u_n$ , and if the infinite product is convergent in the strict sense, then  $\lim_{n \rightarrow \infty} u_n = 0$ ; of course this necessary criterion for convergence is not sufficient.

More generally, for every integer  $k \geq 0$ , for  $n \geq 2$

$$a_n a_{n+1} \dots a_{n+k} = p_{n+k}/p_{n-1}$$

and hence, if the infinite product is convergent in the strict sense, there exists a number  $n_0$  such that for  $n \geq n_0$ , for every  $k \geq 0$

$$(11.2.1) \quad |a_n a_{n+1} \dots a_{n+k} - 1| \leq \frac{1}{2}.$$

From this inequality, we deduce that, for  $n \geq n_0$  and for every  $k \geq 0$ , with the notations of (9.2.4)

$$(11.2.2) \quad |\arg(a_n a_{n+1} \dots a_{n+k})| \leq \frac{\pi}{4}$$

and

$$(11.2.3) \quad \arg(a_n a_{n+1} \dots a_{n+k}) = \arg a_n + \arg a_{n+1} + \dots + \arg a_{n+k}.$$

Indeed, relation (11.2.2) is an immediate consequence of (11.2.1); (11.2.3) is proved by induction on  $k$ , the relation being evident for  $k = 0$ : since

$$|\arg(a_n a_{n+1} \dots a_{n+k-1})| \leq \frac{\pi}{4} \quad \text{and} \quad |\arg(a_{n+k})| \leq \frac{\pi}{4},$$

$$|\arg(a_n a_{n+1} \dots a_{n+k-1}) + \arg(a_{n+k})| \leq \frac{\pi}{2},$$

and the relation (11.2.3) is then deduced from the induction hypothesis and from (9.4.4).

(11.3) For every integer  $m \geq 1$  we can write, for  $n > m$

$$p_n = p_m(a_{m+1} \cdots a_n)$$

and hence (since  $p_m \neq 0$ ), for the sequence  $(p_n)$  to converge, it is necessary and sufficient that the sequence  $(a_m a_{m+1} \cdots a_n)_{n \geq m}$  be convergent and then

$$\lim_{n \rightarrow \infty} p_n = p_{m-1} \lim_{n \rightarrow \infty} (a_m a_{m+1} \cdots a_n).$$

In other words the convergence in the strict sense of an infinite product *does not depend on the initial factors of the product* (just as for series), and one can add, subtract or change a finite number of factors without altering the strict convergence or non-convergence of a given product. Thus in order to study the convergent infinite products (in the strict sense), we may confine ourselves, by virtue of (11.2.1), to products such that  $1 + u_n = a_n$  satisfies  $|u_n| \leq \frac{1}{2}$  for every  $n$ . By virtue of (11.2.3) and (9.4.6),

$$(11.3.1) \quad \log p_n = \log a_1 + \cdots + \log a_n + 2hi\pi$$

on the understanding that the logarithms are the “principal determinations” defined in (9.3) and the integer  $h$  being constant for  $n \geq n_0$ . Taking into account the continuity of the log and exp functions, it is seen that the infinite product  $\prod_{n=1}^{\infty} a_n$  is convergent in the strict sense if, and only if, the series  $\sum_{n=1}^{\infty} \log a_n$  is convergent, and

$$(11.3.2) \quad \prod_{n=1}^{\infty} a_n = \exp \left( \sum_{n=1}^{\infty} \log a_n \right).$$

(11.4) If the series of general term  $u_n$  is absolutely convergent (and  $u_n \neq -1$  for all  $n$ ), the infinite product  $\prod_{n=1}^{\infty} (1 + u_n)$  is convergent in the strict sense.

It is sufficient to show that the series of general term  $\log(1 + u_n)$  is convergent; we shall in fact show that it is *absolutely convergent*. We may confine ourselves to the case where  $|u_n| \leq \frac{1}{2}$  for every  $n$ . Since the second derivative  $-1/(1+z)^2$  of the logarithm is bounded in absolute value by 4 in the disc  $|z| \leq \frac{1}{2}$ , Taylor’s formula (VI, 6.6) thus gives, for every  $n$ ,

$$(11.4.1) \quad |\log(1 + u_n)| \leq |u_n| + 2|u_n|^2 \leq 2|u_n|$$

hence our assertion.

We agree to say that when the series of general term  $u_n$  is absolutely convergent, the infinite product  $\prod_{n=1}^{\infty} (1 + u_n)$  is *absolutely convergent*. Note carefully that it may happen that the series  $\sum_{n=1}^{\infty} u_n$  is *convergent* (but not absolutely convergent) *without* the infinite product  $\prod_{n=1}^{\infty} (1 + u_n)$  being convergent, and vice versa (problem 28).

(11.5) Let  $(u_n)$  be a sequence of functions analytic in an open set  $D \subset \mathbf{C}$ , and suppose that for

every closed disc  $\Delta \subset D$ , the series of general term  $u_n(z)$  converges normally (V, 2.6) in  $\Delta$ . Then the infinite product  $f(z) = \prod_{n=1}^{\infty} (1 + u_n(z))^{\dagger}$  is a function analytic in  $D$ ; its zeros in  $D$  are those of each of the factors  $1 + u_n(z)$ , the order  $\omega(a; f)$  at a zero being the sum of the orders  $\omega(a; 1 + u_n)$  (zero except for a finite number of values of  $n$ ). Furthermore, for every closed disc  $\Delta \subset D$  not containing any of the zeros of  $f$ , the series of general term  $u'_n(z)/(1 + u_n(z))$  is uniformly convergent in  $\Delta$  and we have

$$(11.5.1) \quad \frac{f'(z)}{f(z)} = \sum_{n=1}^{\infty} \frac{u'_n(z)}{1 + u_n(z)} \quad \text{for every } z \in \Delta.$$

Because of the hypothesis, for every closed disc  $\Delta \subset D$  there is a convergent series  $(\alpha_n)$  of numbers  $> 0$  such that  $|u_n(z)| \leq \alpha_n$  for every  $z \in \Delta$ . If  $n_0$  is chosen so that  $\alpha_n \leq \frac{1}{2}$  for  $n \geq n_0$ , we therefore see that the series of general term  $\log(1 + u_n(z))$  (where  $n \geq n_0$ ) is normally convergent in  $\Delta$  by virtue of the inequality (11.4.1) which gives here  $|\log(1 + u_n(z))| \leq 2\alpha_n$  for every  $z \in \Delta$  and all  $n \geq n_0$ . The infinite product  $\prod_{n=n_0}^{\infty} (1 + u_n(z)) = \exp \left( \sum_{n=n_0}^{\infty} \log(1 + u_n(z)) \right)$  is thus convergent (in the strict sense, so  $\neq 0$ ) for every  $z \in \Delta$ ; being the uniform limit of a sequence of functions analytic in  $\Delta$  by virtue of (V, 2.6), it is analytic in  $\Delta$  by the Weierstrass convergence theorem (VII, 10.5).

The same is thus the case for  $\prod_{n=1}^{\infty} (1 + u_n(z))$ , which by definition is obtained by multiplying by the finite product  $\prod_{n=1}^{n_0-1} (1 + u_n(z))$ . If it is further supposed that  $1 + u_n(z) \neq 0$  in  $\Delta$  for every integer  $n$ , the terms  $u'_n(z)/(1 + u_n(z))$  are functions analytic in  $\Delta$ , and for  $n \geq n_0$  they are the derivatives of the functions  $\log(1 + u_n(z))$  which are well defined and analytic in  $\Delta$ . The uniform convergence of this series in  $\Delta$  follows from (VII, 10.1). Moreover, if we put  $g_N(z) = \prod_{n=1}^N (1 + u_n(z))$  for every integer  $N$ , the sequence  $(g_N(z))$  converges to  $f(z)$  in  $\Delta$ , hence the sequence  $(g'_N(z)/g_N(z))$  converges to  $f'(z)/f(z)$  in  $\Delta$  by virtue of (VII, 10.1), which completes the proof of (11.5).

Note that under conditions of (11.5), the sequence of partial products  $\prod_{n=1}^N (1 + u_n(z))$  converges uniformly to  $f(z)$  in every closed disc  $\Delta \subset D$ , since with the same notations

$$\prod_{n=1}^N (1 + u_n(z)) = \prod_{n=1}^{n_0-1} (1 + u_n(z)) \cdot \prod_{n=n_0}^N (1 + u_n(z))$$

so that the first factor is bounded in  $\Delta$ .

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$\dagger$  Contrary to the convention made in (11.1), we admit the case where for certain  $z \in D$ , we have  $u_n(z) = -1$  for certain values of  $n$ , and say that the infinite product converges to 0 at these points.

## PROBLEMS

1. Let  $f$  be an analytic function in an annulus  $S: 0 < r < |z| < r'$ , and let

$$f(z) = \sum_{n=1}^{\infty} \frac{a_{-n}}{z^n} + 1 + \sum_{n=1}^{\infty} a_n z^n$$

be its Laurent series. If  $r < \rho < 1$ , show that

$$2 - a_n - a_{-n} = \frac{2}{\pi} \int_0^{2\pi} f(e^{i\theta}) \sin^2 \frac{n\theta}{2} d\theta$$

$$2\rho^n + a_{-n} + a_n \rho^{2n} = \frac{2\rho^n}{\pi} \int_0^{2\pi} f(\rho e^{i\theta}) \cos^2 \frac{n\theta}{2} d\theta.$$

Deduce from this that if  $\Re f(z) \geq 0$  in  $S$ , then  $\Re a_n \leq 2/(1 - \rho^n)$  for  $n \geq 1$ , and considering the functions  $f(e^{i\alpha}z)$  and  $f(\rho/z)$ , conclude that

$$|a_n| \leq \frac{2}{1 - \rho^n} \quad \text{and} \quad |a_{-n}| \leq \frac{2\rho^n}{1 - \rho^n} \quad \text{for } n \geq 1.$$

2. Consider the power series

$$f(z) = 1 + z^2 + z^4 + z^8 + \cdots + z^{2^n} + \cdots$$

whose radius of convergence is 1. Show that

$$f(z) = z^2 + f(z^2) = z^2 + z^4 + f(z^4) = \cdots$$

and deduce from this that all the points of the circle  $|z| = 1$  are *singular* for the function  $f$  (use problem 17(b)) of Chap. VI).

3. Let  $D \subset \mathbf{C}$  be a connected open set containing the exterior  $|z| > R$  of a disc. Let  $f$  be an analytic function in  $D$  such that

$$\lim_{z \rightarrow 0} f(1/z) = c.$$

Show that for every loop  $\gamma$  contained in  $|z| > R$ ,

$$\int_{\gamma} \frac{f(z) dz}{z - x} = 2\pi i j(0; \gamma) c + 2\pi i (j(x; \gamma) - j(0; \gamma)) f(x)$$

for every  $x \in D$  not in the image of the loop. (Observe that for any  $R' > R$ ,  $\gamma$  is homotopic in  $|z| > R$  to a loop contained in the exterior of the disc  $|z| > R' > R$ .)

4. Let  $c_0 + c_1 z + \cdots + c_n z^n + \cdots$  be a power series of finite radius of convergence  $R > 0$ .

(a) Suppose that the sum of this series is the restriction to  $|z| < R$  of a function  $f$  meromorphic in an open set  $D$  containing the closed disc  $|z| \leq R$ . If  $f$  has only one pole  $z_0$  of order  $m$  on the circle  $|z| = R$ , show that for some constant  $A$ ,

$$c_n = A n^{m-1} z_0^{-n} + o(R'^{-n}) \quad \text{for some } R' > R.$$

(Consider the singular part  $u(z)$  of  $f$  at the point  $z_0$  and the difference  $f(z) - u(z)$ .)

(b) More generally, if  $z_1, \dots, z_r$  are the distinct poles of greatest order  $m$  of  $f$  on the circle  $|z| = R$ ,

$$c_n = n^{m-1} \left( \sum_{j=1}^r A_j z_j^{-n} \right) + o(n^{m-2} R^{-n}) = O(n^{m-1} R^{-n})$$

where the  $\Lambda_j$  are constants  $\neq 0$ . Deduce from this that it is not possible to have infinitely many sequences  $n_k, n_k + 1, \dots, n_k + r$  of  $r + 1$  consecutive integers such that  $c_n = 0$  for the integers of each of these sequences ("gaps of length  $r + 1$ " in the series). (Use the expression of the Vandermonde determinant.) If  $r = 1$ , we have  $\lim_{n \rightarrow \infty} c_n/c_{n+1} = z_1$ .

5. Let  $f(z) = c_0 + c_1 z + \dots + c_n z^n + \dots$  be a power series of radius of convergence  $R = 1$ ; if  $c_n = o(n^{\alpha-1})$  with  $\alpha > 0$ , show that when  $r$  tends to 1 while remaining  $< 1$ , we have  $\lim_{r \rightarrow 1} (1-r)^\alpha f(re^{i\theta}) = 0$  uniformly for  $0 \leq \theta \leq 2\pi$ . (Observe that  $\left| \binom{-\alpha}{n} \right| \sim an^{\alpha-1}$ .) Deduce from this in particular that if  $\lim_{n \rightarrow \infty} c_n = 0$ , there cannot be a singular point of  $f$  on  $|z| = 1$  which is a pole.

6. Let  $f(z) = a_0 + a_1 z + \dots + a_n z^n + \dots$  be a power series with *real* coefficients and of radius of convergence  $R$ ; suppose that on the circle  $|z| = R$ , the only singular points of  $f$  are two poles  $\operatorname{Re}^{i\alpha}$  and  $\operatorname{Re}^{-i\alpha}$  with  $0 < \alpha < \pi$ . Show that if  $V_n$  is the variation number of the sequence  $(a_j)_{0 \leq j \leq n}$ , then

$$\lim_{n \rightarrow \infty} \frac{V_n}{n} = \frac{\alpha}{\pi}.$$

7. Prove Liouville's theorem by applying the residue theorem to the integral

$$\int_\gamma \frac{f(z) dz}{(z-a)(z-b)}$$

with  $a \neq b$ , where  $\gamma$  is the loop  $t \rightarrow \operatorname{Re}^{it}$ .

8. Calculate by the method of residues an integral of the form

$$\int_0^{2\pi} R(\cos \theta, \sin \theta) d\theta,$$

where  $R(u, v)$  is a rational function whose denominator does not vanish for any value  $u = \cos \theta, v = \sin \theta$  with  $0 \leq \theta \leq 2\pi$ . Examples:

$$\int_0^{2\pi} \frac{d\theta}{(a + b \cos \theta)^2} = \frac{2\pi a}{(a^2 - b^2)^{3/2}} \quad \text{for } a > b > 0;$$

$$\int_0^{2\pi} \left( \frac{\sin n\theta/2}{\sin \theta/2} \right)^2 d\theta = 2\pi n.$$

9. Prove that

$$\begin{aligned} \int_0^{+\infty} \frac{\sin x}{x} dx &= \frac{\pi}{2}, & \int_0^{+\infty} \frac{\sin^3 x}{x^3} dx &= \frac{3\pi}{8} \\ \int_0^{+\infty} e^{a \cos bx} \sin(a \sin bx) \frac{dx}{x} &= \frac{\pi}{2} (e^a - 1) \quad (a > 0, b > 0) \end{aligned}$$

(integrate along the loop of Fig. 47).

10. Show that

$$\int_0^{2\pi} e^{\cos \theta} \cos(n\theta - \sin \theta) d\theta = \frac{2\pi}{n!} \quad (n \text{ integer } \geq 0).$$



FIGURE 47

11. (a) Show that if  $\lambda > 0$ , and  $a$  is real, then

$$\int_{-\infty}^{+\infty} e^{-\lambda x^2} \cos(2\lambda ax) dx = \sqrt{\frac{\pi}{\lambda}} e^{-\lambda a^2}$$

(integrate  $e^{-\lambda z^2}$  along a rectangle with vertices  $\pm R$  and  $\pm R + ia$ ).

- (b) Show that for  $a > 0$ ,  $b > 0$ ,

$$\int_{-\infty}^{+\infty} e^{-ia^2t^2 - 2ibt} dt = \frac{\sqrt{\pi}}{a} e^{-(\pi i/4) + i(b^2/a^2)}$$

hence

$$\int_{-\infty}^{+\infty} e^{-ia^2t^2} \cos 2bt dt = \frac{\sqrt{\pi}}{2a} e^{-(\pi i/4) + i(b^2/a^2)}.$$

12. Show that for  $a$  real

$$\int_0^{+\infty} \frac{\sin ax}{\sinh x} dx = \frac{\pi}{2} \tanh \frac{\pi a}{2}, \quad \int_0^{+\infty} \frac{x \cos ax}{\sinh x} dx = \frac{\pi^2 e^{\pi a}}{(e^{\pi a} + 1)^2}$$

(integrate along a rectangle with vertices  $\pm R$  and  $\pm R + 2\pi i$ , avoiding the points 0 and  $2\pi i$  by small semicircles).

13. Calculate the integrals (where  $t > 0$ ,  $c > 0$ )

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{e^{zt} dz}{z^{n+1}}, \quad \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{t^2 dz}{z^{n+1}}.$$

14. Let  $f$  be an analytic function in a disc  $D: |z| < R$ . Let  $z_0 = r_0 e^{i\theta_0}$  be a point  $\neq 0$  such that  $r_0 < R$ , and put  $f(z_0) = \rho_0 e^{i\omega_0}$ ; for  $\theta$  near  $\theta_0$  in  $\mathbb{R}$  and  $z = r_0 e^{i\theta}$ , we can write  $f(z) = \rho e^{i\omega}$  with  $\rho > 0$  and  $\omega$  in the interval  $[\omega_0 - \pi, \omega_0 + \pi]$ . Show that

$$(*) \quad \frac{d\omega}{d\theta} = \Re \left( z \frac{f'(z)}{f(z)} \right).$$

The example of  $f(z) = z^2$  shows that we can have

$$\Re \left( z \frac{f'(z)}{f(z)} \right) \geq 0$$

in  $D$  without  $f$  being injective. Show that if  $f$  is injective in  $D$  and if moreover

$$\Re\left(z \frac{f'(z)}{f(z)}\right) \geq 0$$

for  $z \neq 0$  in  $D$ , then we necessarily have  $f(0) = 0$  and  $f(D)$  is *starlike* with respect to 0 (apply (6.1) to the loop  $t \rightarrow re^{it}$  for  $0 < r < R$ ).

15. Show that if  $\alpha, \beta$  are two real numbers  $\neq 0$ , the number of zeros of the polynomial

$$z^{2n} + \alpha z^{2n-1} + \beta^2$$

such that  $\Re z > 0$  is equal to  $n$  if  $n$  is even; if  $n$  is odd, the number of zeros is equal to  $n - 1$  if  $\alpha > 0$ , to  $n + 1$  if  $\alpha < 0$ . (Apply (6.1) to the loop formed by the semicircle  $t \rightarrow Re^{it}$  ( $-\pi/2 \leq t \leq \pi/2$ ) and its diameter

$$t \rightarrow -it \quad (-R \leq t \leq R)$$

for  $R$  large; use formula (\*) of problem 14.)

16. Show that for  $n$  integer  $> 1$ , the equation  $\sin z - z = 0$  has two roots in the strip  $2n\pi < \Re z < (2n + 1)\pi$ , and has no root in the strip  $(2n - 1)\pi < \Re z < 2n\pi$ . Furthermore, there exists a number  $k > 0$  such that for  $n$  sufficiently large, the equation has one and only one root  $x + iy$  satisfying the conditions

$$\left| x - \left( 2n\pi + \frac{\pi}{2} \right) \right| \leq k \frac{\log n}{n}, \quad |y - \log(4n\pi)| \leq k \frac{\log n}{n}.$$

(Use the method of (6.2) applied to the perimeter of a rectangle  $2n\pi \leq \Re z \leq (2n + 1)\pi$ ,  $\alpha \leq \Im z \leq \beta$ , when  $\alpha$  and  $\beta$  are suitable numbers  $> 0$ ; then apply Newton's method.)

17. (a) Consider the equation

$$z - \frac{\pi}{2} - w \sin z = 0.$$

Show that for every  $r$  such that  $0 < r < \pi/2$ , and for  $|w| < 2r_1(e^r + e^{-r})$  this equation has one and only one root  $z = h(w)$  such that  $|z - (\pi/2)| < r$ . If  $\rho = 1.19 \dots$  is the root  $> 0$  of the equation

$$e^{2\rho} = \frac{\rho + 1}{\rho - 1}$$

deduce that  $h(w)$  is analytic in the disc  $|w| < (\rho^2 - 1)^{1/2} = 0.6627 \dots$

(b) If  $f(z) = (z - (\pi/2))/\sin z$  and if  $z_0$  is a root of the equation  $f'(z) = 0$ , show that  $h(w)$  cannot be analytic in a disc of centre 0 containing the point  $w_0 = f(z_0)$ . Show that there exists such a root  $z_0$  such that  $|w_0| = (\rho^2 - 1)^{1/2}$ , and deduce from this that the radius of convergence of the Taylor series of  $h(w)$  at the point  $w = 0$  is  $(\rho^2 - 1)^{1/2}$ .

18. Let  $f$  be an analytic function in a disc  $|z| < R$ . For every  $r$  such that  $0 < r < R$ , show that if  $f(z) \neq 0$  for  $|z| = r$ , the number of zeros of  $f(z)$  belonging to the disc  $|z| < r$  is at most equal to the least upper bound of  $\Re\left(z \frac{f'(z)}{f(z)}\right)$  on  $|z| = r$ .

19. Show that a trigonometric polynomial

$$a_m \cos m\theta + b_m \sin m\theta + a_{m+1} \cos(m+1)\theta + b_{m+1} \sin(m+1)\theta + \dots + a_n \cos n\theta + b_n \sin n\theta$$

with real coefficients  $a_i, b_j$ , has at least  $2m$  distinct roots in the interval  $0 \leq \theta \leq 2\pi$ . (Apply (6.1) to the polynomial

$$(a_m - ib_m)z^m + (a_{m+1} - ib_{m+1})z^{m+1} + \cdots + (a_n - ib_n)z^n.)$$

20. Let

$$a_0 + a_1 \cos \theta + a_2 \cos 2\theta + \cdots + a_n \cos n\theta$$

be a trigonometric polynomial where  $0 < a_0 < a_1 < \cdots < a_n$ . Show that this polynomial has  $2n$  distinct roots in the interval  $0 \leq \theta \leq 2\pi$  (use the same method as in problem 19, using problem 1(d) of Chap. II).

21. Let  $\lambda$  be real and  $> 1$ . Show that the equation

$$\lambda - z - e^{-z} = 0$$

has only one root (necessarily real) such that  $\Re z \geq 0$ .

22. Let  $f(t)$  be a real function twice continuously differentiable in the interval  $0 \leq t \leq 1$ . Consider the entire function

$$F(z) = \int_0^1 f(t) \sin zt \, dt.$$

(a) Show that if  $|f(1)| > f(0)$ ,  $F$  has infinitely many real zeros and a finite number of non-real zeros.

(b) Show that if  $|f(1)| < f(0)$ ,  $F$  has infinitely many non-real zeros and a finite number of real zeros. (Compare the zeros of  $F$  to those of

$$f(0) - f(1) \cos z$$

by means of two integrations by parts.)

23. Show that the entire function

$$\sum_{n=0}^{\infty} \frac{z^{n^3}}{(n^3)!}$$

has exactly  $N^3$  zeros in the disc  $|z| < N^3$ , for  $N$  sufficiently large (compare the power series to its term  $z^{N^3}/(N^3)!$ ).

24. (a) Show that the equation  $ze^{-z} = w$  admits an analytic solution  $z = h(w)$  in the neighbourhood of  $w = 0$ , having for its Taylor series at the point  $w = 0$

$$h(w) = w + \frac{2w^2}{2!} + \cdots + \frac{n^{n-1}w^n}{n!} + \cdots.$$

Moreover, for every complex constant  $\alpha$ , we have

$$e^{\alpha h(w)} = 1 + \sum_{n=1}^{\infty} \frac{\alpha(\alpha + n)^{n-1}}{n!} w^n$$

$$\frac{e^{\alpha h(w)}}{1 - h(w)} = \sum_{n=0}^{\infty} \frac{(\alpha + n)^n}{n!} w^n$$

these series having radius of convergence  $1/e$ . (Use (7.3.2).)

(b) Show that the series

$$1 + \sum_{n=1}^{\infty} \frac{\alpha(\alpha + n)^{n-1}}{n!} x^n e^{-nx}$$

is normally convergent for  $x \geq 0$ ; its sum is equal to  $e^{\alpha x}$  for  $0 \leq x \leq 1$ , but to  $e^{\alpha x'}$  for  $x \geq 1$ , where  $x'$  is then the real number lying between 0 and 1 such that  $x' e^{x'} = x e^x$ .

25. Calculate the integrals

$$\int_0^1 \frac{x^{1-p}(1-x)^p}{(1+x)^3} dx, \quad \int_0^1 \frac{x^{1-p}(1-x)^p}{1+x^2} dx \quad (-1 < p < 2)$$

(integrate a suitable determination of the function along the loop of Fig. 48).

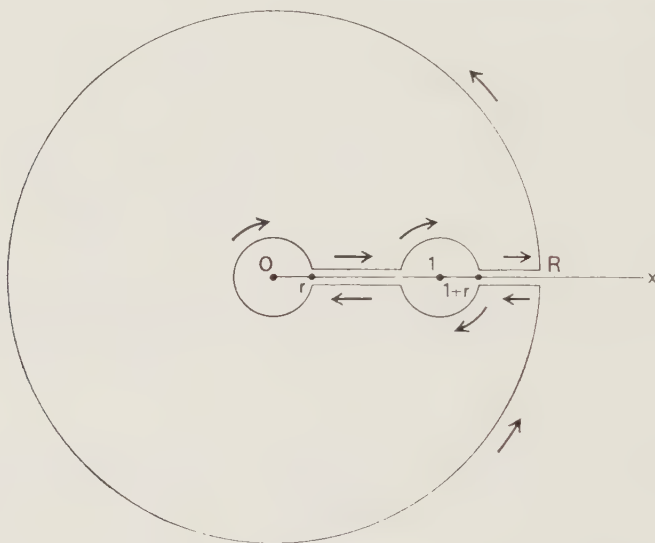


FIGURE 48

26. Calculate the integrals

$$\int_0^{+\infty} \frac{\log^2 x \, dx}{x^2 + a^2}, \quad \int_0^{+\infty} \frac{\log x \, dx}{(x+1)^2 \sqrt{x}}$$

(use the loop of Fig. 47).

27. For  $a > 0$ ,  $n$  integer  $> 0$ , calculate the integrals

$$\int_0^{+\infty} \frac{dx}{(x^2 + a^2)((\log x)^2 + (2n+1)^2 \pi^2)}, \quad \int_0^{+\infty} \frac{dx}{(x^2 + a^2)((\log x)^2 + 4n^2 \pi^2)}$$

(Consider the functions

$$\frac{1}{z^2 + a^2} \left( \frac{1}{\log z - (2n+1)i\pi} + \frac{1}{\log z - (2n-1)i\pi} + \cdots + \frac{1}{\log z + (2n-1)i\pi} \right)$$

and

$$\frac{1}{z^2 + a^2} \left( \frac{1}{\log z - 2ni\pi} + \frac{1}{\log z - (2n-2)i\pi} + \cdots + \frac{1}{\log z + (2n-2)i\pi} \right)$$

for a suitable determination of  $\log z$  and the loops of Fig. 47 or 48.)

28. (a) The infinite product  $\prod_{n=1}^{\infty} \left( 1 + \frac{(-1)^n}{\sqrt{n}} \right)$  converges to 0 although the series of general term  $(-1)^n/\sqrt{n}$  is convergent.

(b) The infinite product  $\prod_{n=1}^{\infty} (1 + u_n)$ , where  $u_{2n-1} = -1/\sqrt{n}$ ,  $u_{2n} = (1/\sqrt{n}) + (1/n)$  converges in the strict sense although the series of general term  $u_n$  is not convergent.

29. Let  $a_1, \dots, a_r$  be integers  $> 0$ . Denote by  $A_n$  the number of solutions  $(x_1, x_2, \dots, x_r)$  of the equation

$$a_1x_1 + a_2x_2 + \cdots + a_rx_r = n$$

formed of integers  $\geq 0$ .

(a) Show that the power series  $\sum_{n=0}^{\infty} A_n z^n$  is convergent in the disc  $|z| < 1$  and has for sum the rational function

$$(*) \quad \frac{1}{(1 - z^{a_1})(1 - z^{a_2}) \cdots (1 - z^{a_r})}.$$

(b) Deduce from this that if the greatest common divisor of the numbers  $a_1, \dots, a_r$  is 1, then

$$A_n \sim \frac{1}{(r-1)! a_1 a_2 \cdots a_r} n^{r-1}$$

(use problem 4a)).

(c) Show that the number  $A_n$  of solutions  $(x, y, z)$  in integers  $\geq 0$  of the equation  $x + 2y + 3z = n$  is the integer nearest to  $(n+3)^2/12$  (decompose the corresponding rational function  $(*)$  into partial fractions).

30. The infinite products  $\prod_{n=1}^{\infty} (1 + z^n)$  and  $\prod_{n=1}^{\infty} (1 - z^{2n-1})$  are convergent in the strict sense for  $|z| < 1$ . Show that

$$\prod_{n=1}^{\infty} (1 + z^n) \cdot \prod_{n=1}^{\infty} (1 - z^{2n-1}) = 1.$$

(If  $F(z)$  is the function of the first member, observe that  $F(z^2) = F(z)$ .)

31. For  $|q| < 1$ , the infinite product  $F(z) = \prod_{n=1}^{\infty} (1 - q^n z)$  is convergent in the strict sense for every  $z \in \mathbf{C}$  and is therefore an entire function. Calculate the coefficients of its development into a power series

$$F(z) = a_0 + a_1 z + \cdots + a_n z^n + \cdots$$

by using the functional equation  $F(z) = (1 - qz)F(qz)$ . Calculate similarly the coefficients of the development into a power series

$$\frac{1}{F(z)} = b_0 + b_1z + \cdots + b_nz^n + \cdots$$

Deduce from the calculation of the  $a_n$  that, for  $t$  real tending to  $+\infty$ , we have

$$F(-t) \sim A \int_1^{+\infty} q^{x(x+1)/2} t^x dx$$

where  $A$  is a constant  $\neq 0$ , and evaluate the integral by Laplace's method.

32. For which values of the numbers  $\alpha > 0, \beta > 0$  is the integral  $\int_0^{+\infty} \frac{\sin(x^\alpha)}{x^\beta} dx$  convergent, and what then is its value?

33. Let  $\omega_1 < \omega_2 < \cdots < \omega_r$  be real numbers,  $A_j$  ( $1 \leq j \leq r$ ) complex numbers  $\neq 0$ . Consider the function  $F(z) = \sum_{j=1}^r A_j \exp(\omega_j z)$ .

(a) Show that there exists a number  $c > 0$  such that all the zeros of  $F(z)$  are contained in the strip  $|\Re z| < c$ .

(b) Let  $\alpha$  be a real number. Show that if  $N(t)$  is the number of the zeros of  $F(z)$  such that  $\alpha < \Re z < \alpha + t$ , then

$$\left| N(t) - \frac{\omega_r - \omega_1}{2\pi} t \right| \leq r - 1.$$

(use the method of (6.1)).

# Application of analytic functions to approximation problems

## 1. Method of steepest descent

In Chap. IV we studied functions of a real variable  $t$  which have the form

$$(1.1) \quad I(t) = \int_a^b g(x) e^{th(x)} dx$$

in the neighbourhood of  $t = +\infty$ , in the cases where  $g$  is a real function and  $h$  a function which takes, either *real* values (IV, 2), or *purely imaginary* values (IV, 4). We shall see that with supplementary hypotheses on  $g$  and  $h$ , we can approach the study of  $I(t)$  when  $g$  and  $h$  are functions with any *complex* values. We treat only the case where the integral (1.1) arises by changes of variable (VII, 2.1.1) from an integral along a path  $L$ :

$$(1.2) \quad I(t) = \int_L g(z) e^{th(z)} dz$$

where  $L$  is contained in an open set  $D \subset \mathbf{C}$  and  $g$  and  $h$  are *analytic* in  $D$  ( $g$  and  $h$  no longer have the same meanings as in (1.1)). We suppose furthermore that  $D$  is *simply connected*; then, for every path  $L'$  in  $D$ , with *the same initial and terminal points* as  $L$ , by (VII, 5.2)

$$(1.3) \quad I(t) = \int_{L'} g(z) e^{th(z)} dz.$$

The idea of the *method of steepest descent* is to profit from this “independence” of the path  $L$  of given endpoints and to so *choose* it that Laplace’s method (slightly modified to take into account the complex values of  $h$ ) is applicable. To be precise,  $|e^{th(z)}| = e^{t\Re(h(z))}$  and a “favourable” choice of  $L'$  will have the property that  $\Re h(z)$  attains its greatest value at *one point only* of  $L'$ . By eventually dividing the path  $L'$  into the juxtaposition of two other paths, one can suppose that the point where  $\Re h(z)$  is a maximum is the *initial point* of  $L'$  (ignoring its sign), and by translation to confine oneself to the case where the initial point is the point  $z = 0$ .

$$(1.4) \quad \text{I. Hypotheses on } g \text{ and } h.$$

Since  $h$  and  $g$  are assumed analytic at the point  $z = 0$ , we have in the neighbourhood of this point (VI, 6.7)

$$(1.4.1) \quad g(z) \sim Az^n, \quad h(z) = a + cz^m + O(z^{m+1})$$

where  $n$  is an integer  $\geq 0$ ,  $m$  an integer  $\geq 1$ ,  $a, c$  and  $A$  complex numbers with  $A \neq 0$ ,  $c = \rho e^{i\alpha} \neq 0$  (it is evidently possible to limit the discussion to the case where  $h$  is not constant). The argument of the number  $cz^m$  here plays an essential part, for it determines the sign of its real part and hence that of  $\Re(h(z) - a)$  in the neighbourhood of  $z = 0$ . To be precise, let  $S_0$  be the "angular sector" formed by the points  $z = ue^{i\theta}$  with  $u \geq 0$  and

$$(1.4.2) \quad -\frac{\pi}{2m} \leq \theta - \frac{\pi - \alpha}{m} \leq \frac{\pi}{2m}.$$

Since  $cz^m = \rho u^m e^{i(\alpha + m\theta)}$ , we have  $\pi/2 \leq m\theta + \alpha \leq 3\pi/2$  in  $S_0$  and hence  $\Re(cz^m) \leq 0$  in this sector. The same is true in the other  $m-1$  angular sectors  $S_k = e^{2ki\pi/m} S_0$  ( $1 \leq k \leq m-1$ ), whereas in the  $m$  angular sectors  $S'_k = e^{(2k+1)i\pi/m} S_0$  ( $0 \leq k \leq m-1$ ), we have  $\Re(cz^m) \geq 0$  (Fig. 49). If  $\theta$  is given one of the values

$$(1.4.3) \quad \omega = \frac{(2k+1)\pi - \alpha}{m}$$

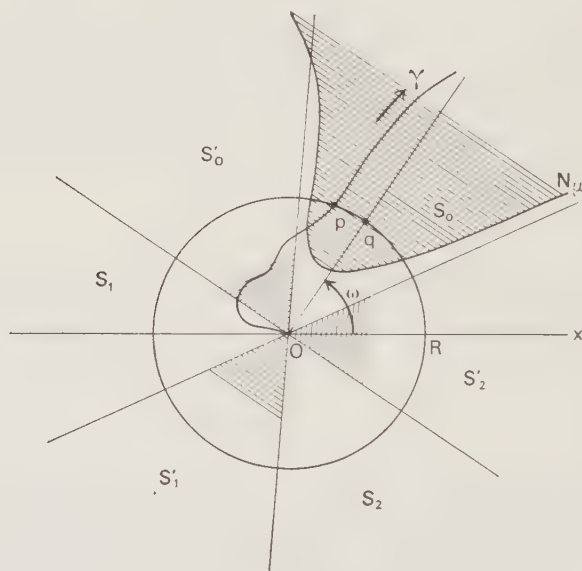


FIGURE 49

for an integer  $k$  satisfying  $0 \leq k \leq m-1$

$$(1.4.4) \quad h(ze^{i\omega}) = a - \rho z^m F(z)$$

where  $F$  is analytic at the point  $z = 0$  and  $F(0) = 1$ . The function  $z(F(z))^{1/m}$  (where the determination of the  $1/m^{\text{th}}$  power is the principal determination defined in (VIII,

9.6)) is thus analytic in an open disc  $\Delta_0: |z| < R_0$ , and  $R_0$  can be assumed sufficiently small for this mapping to be a *bijection* of  $\Delta_0$  onto an open simply connected set  $U$  containing 0 (VIII, 8.1). The inverse bijection  $w \rightarrow G(w)$  of  $U$  onto  $\Delta_0$  is moreover analytic in  $U$  and such that  $G(0) = 0$ ,  $G'(0) = 1$  (VIII, 8.1). Choose  $r > 0$  such that the disc  $V: |w| < r$  is contained in  $U$ , and  $R$  such that  $0 < R < R_0$  and such that the image under  $z \rightarrow z(F(z))^{1/m}$  of the disc  $\Delta: |z| < R$  is contained in the disc  $V$ . If for  $0 \leq s \leq R$

$$s(F(s))^{1/m} = P(s) + iQ(s) \quad (P \text{ and } Q \text{ real}),$$

it can be supposed further that  $R$  is chosen sufficiently small so that

$$|Q(R)| \leq |P(R)| \tan \frac{\pi}{4m}$$

(since  $P'(0) = 1$  and  $Q'(0) = 0$ ). In the following reasoning it is assumed that the numbers  $R$  and  $r$  satisfy the preceding conditions.

(1.5) II. *Hypotheses on the path.* It is convenient to consider a *path without endpoints* (VII, 10.3):

$$(1.5.1) \quad \gamma: \mathbf{R}_+ = [0, +\infty[ \rightarrow D,$$

the case of an ordinary path being a particular case. Assume that this path satisfies the following conditions for some integer  $k$  such that  $0 \leq k \leq m - 1$  (Fig. 49):

1. *There exists a number  $s_0 > 0$  such that*

$$\gamma(0) = 0, \quad |\gamma(s)| \leq R \quad \text{for } 0 \leq s \leq s_0, \quad |\gamma(s_0)| = R,$$

*the point  $p = \gamma(s_0)$  being in the sector  $S_k$ .*

2. *There exists a number  $\mu > 0$  such that, if  $N_\mu$  is the set of points  $z$  for which  $\Re(h(z)) \leq \Re(a) - \mu$ , we have  $\gamma(s) \in N_\mu$  for  $s \geq s_0$ , and such that the arc  $\lambda$  of the circle joining the point  $p$  to the point  $q = Re^{i\omega}$  is also contained in  $N_\mu$ .*

Then we have:

(1.6) *Suppose for some integer  $k$  the hypotheses (1.4) and (1.5) are satisfied and suppose further that the improper integral  $\int_0^{+\infty} |g(\gamma(s))| e^{\Re h(\gamma(s))} |\gamma'(s)| ds$  is convergent. Then the principal part (in the sense of (IV, 7.6)) of  $I(t)$  is given by*

$$(1.6.1) \quad \int_\gamma g(z) e^{th(z)} dz \sim \frac{A}{m} \Gamma\left(\frac{n+1}{m}\right) (\rho t)^{-(n+1)/m} e^{at + (n+1)i\omega}$$

Evidently  $e^{at}$  can be put as a factor in the integral; suppose for simplicity that  $a = 0$ . The same reasoning as in (IV, 2.3) then shows that

$$(1.6.2) \quad \int_{s_0}^{+\infty} |g(\gamma(s))| e^{t\Re h(\gamma(s))} |\gamma'(s)| ds \leq B e^{-(t-1)\mu}$$

where  $B = \int_0^{+\infty} |g(\gamma(s))| e^{\Re h(\gamma(s))} |\gamma'(s)| ds$ . Since the second member of (1.6.2) is negligible compared to the absolute value of the second member of (1.6.1), it is sufficient to show that this second member is the principal part of

$$\int_{\gamma_0} g(z) e^{th(z)} dz$$

where  $\gamma_0: [0, s_0] \rightarrow \Delta$  is the *restriction* of the path  $\gamma$ . By virtue of Cauchy's theorem we also have, with the notations of (VII, 3.2)

$$(1.6.3) \quad \int_{\gamma_0} g(z) e^{t h(z)} dz = \int_0^q g(z) e^{t h(z)} dz + \int_{\lambda} g(z) e^{t h(z)} dz.$$

By the hypothesis (1.5, 2) on the arc  $\lambda$ , the second integral of the second member of (1.6.3) is majorized in absolute value by the second member of (1.6.2), where the constant B is replaced by the constant

$$R \frac{\pi}{2m} \sup_{|z| \leq R} |g(z) e^{h(z)}|.$$

It thus remains to evaluate the first integral of the second member of (1.6.3). Now, with the notations of (1.4),

$$\int_0^q g(z) e^{t h(z)} dz = \int_{\beta} g_1(w) e^{-\rho t w^m} dw$$

where  $g_1(w) = e^{t\omega} g(G(w) e^{t\omega}) G'(w)$ , and  $\beta: [0, R] \rightarrow V$  is the path

$$s \rightarrow s(F(s))^{1/m} = P(s) + iQ(s)$$

with initial point 0 and terminal point  $w_0$ . Applying Cauchy's theorem in U

$$(1.6.4) \quad \int_{\beta} g_1(w) e^{-\rho t w^m} dw = \int_0^r g_1(w) e^{-\rho t w^m} dw + \int_{\sigma} g_1(w) e^{-\rho t w^m} dw$$

where  $\sigma$  is the segment with initial point  $w_0$  and terminal point  $r$  (Fig. 50). Because of the choice of R and  $r$ , we have  $\Re w^m \geq (\sqrt{2}/2) |w_0|^m$  at every point on  $\sigma$ , hence

$$(1.6.5) \quad \left| \int_{\sigma} g_1(w) e^{-\rho t w^m} dw \right| \leq M e^{-k t}$$

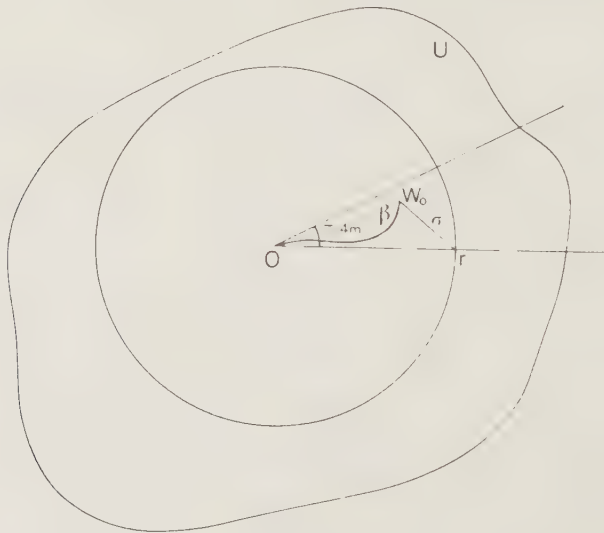


FIGURE 50

with

$$M = \sup_{|w| \leq r} |g_1(w)| \quad \text{and} \quad k = \rho \frac{\sqrt{2}}{2} |w_0|^m.$$

It remains to evaluate the principal part of

$$(1.6.6) \quad \int_0^r g_1(w) e^{-\rho t w^m} dw$$

and since  $g_1(w) \sim A w^n e^{(n+1)i\omega}$  in the neighbourhood of 0, we have reduced the problem to the case treated by Laplace's method (IV, 2.3) (considering separately the real and imaginary part of (1.6.6)). Q.E.D.

(1.7) Let us return to the integral (1.2) initially proposed. The whole difficulty of the "method of steepest descent" consists of the *choice* of the path  $L'$ , which must be made in such a way that (1.6) can be applied to the integral (1.3), or the latter can be decomposed into two others to which (1.6) is applicable. There are two possibilities:

1. There exists a path  $L'$  in  $D$  joining the endpoints of  $L$  and such that  $\Re(h(z))$  attains in  $L'$  its greatest value at one of the endpoints. The "normal" case will then be that where  $m = 1, n = 0$  in (1.4.1).
2. There is no path  $L'$  contained in  $D$  with the preceding property. In this case (1.6) can be applied only if the point  $z_0$  of  $L'$  where  $\Re h(z)$  attains its greatest value is *such that*  $m \geq 2$ , in other words satisfies the equation

$$(1.7.1) \quad h'(z_0) = 0.$$

Indeed, in the contrary case ( $m = 1$ ), the principal parts of the two integrals into which (1.3) has been decomposed, obtained by application of (1.6), would be *opposite*, so *no information* would be obtained about (1.3). This comes from the fact that for  $m = 1$  there is *only one sector*  $S_k$ . In the case where  $z_0$  satisfies (1.7.1), it is necessary, for the same reason, that the application of (1.6) to the two partial integrals be made for *different* sectors  $S_k$  (in terms of pictures, the path  $L'$  must "arrive" at the point  $z_0$  and "depart" from  $z_0$  through different sectors  $S_k$ ).

The "normal" case will then be that where  $n = 0$  and  $m = 2$ , i.e.:

$$(1.7.2) \quad h''(z_0) \neq 0.$$

(1.8) To determine  $L'$  it is usually necessary to study the curves  $\Re(h(z)) = u$  for real values of the parameter  $u$  ("level curves" of the surface  $\Sigma$  of equation

$$x_3 = \Re(h(x_1 + ix_2))$$

in the space  $\mathbf{R}^3$ ). Case 1 of (1.7) is that where, denoting by  $u_1$  and  $u_2 \geq u_1$  the values of  $\Re h(z)$  at the endpoints of  $L$ , the open set of  $z \in D$  such that  $u_1 < \Re(h(z)) < u_2$  is *connected* (Fig. 51). If this is not the case, we must begin by determining the solutions  $z_0$  of the equation (1.7.1). The "normal" case where in addition the condition (1.7.2) is satisfied, is that where the surface  $\Sigma$  presents a "saddle" at the point  $z_0$  (Fig. 52); the level curve

$$\Re(h(z)) = \Re(h(z_0))$$

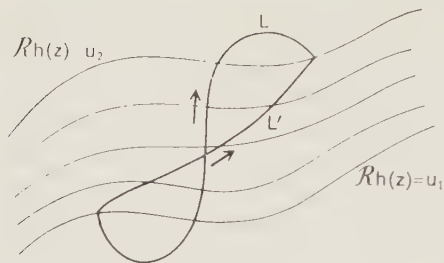


FIGURE 51

line  $L_+$ , is entirely situated in the set of points  $z$  such that  $\Re(h(z)) \geq \Re(h(z_0))$ , and the other, the *thalweg line*  $L_-$ , is entirely situated in the set of points  $z$  such that  $\Re(h(z)) \leq \Re(h(z_0))$  (the two “valleys” joining at the saddle). The result of the method of steepest descent can also be summarized by saying that the integral (1.3) can be replaced by the integral taken along a small segment of the tangent to the thalweg line  $L_-$  at the saddle  $z_0$  and that in this latter integral the functions  $g$  and  $h$  be replaced by their developments (1.4.1). This presupposes of course that it has been possible to replace the initial path  $L$  by a path  $L'$  passing through the saddle and satisfying the conditions (1.5). Taking into account the fact that the integral (1.3) has been decomposed into two others, its principal part is then given (for the “normal” case  $n = 0, m = 2$ ) by the formula (which generalizes (IV, 2.5.1))

$$(1.8.1) \quad \pm \sqrt{\frac{2\pi}{t|h''(z_0)|}} e^{i(\pi-\alpha)/2} g(z_0) e^{th(z_0)}$$

where

$$(1.8.2) \quad h''(z_0) = |h''(z_0)| e^{i\alpha}.$$

The sign of (1.8.1) depends on the path along the thalweg line  $L_-$ , which must be chosen so that its direction is from the “valley” containing the initial point of  $L'$  to that containing the terminal point.

*Remarks (1.9)* If there are several solutions of the equation (1.7.1), one can be certain in advance that at most one solution is suitable (i.e. one that permits the joining of the endpoints of  $L$  by a path  $L'$  satisfying all the conditions (1.5) and passing through  $z_0$ ), unless  $\Re(h(z))$  has the same value at two of these “saddles” (otherwise there would be two non-equivalent principal parts, which is absurd). The choice of this “saddle” may be a difficult matter.

(1.10) When  $L$  is a “path without endpoints” (VII, 10.3) in the integral (1.2), we cannot simply replace  $L$  by another “path without endpoints”  $L'$  in  $D$  without some

then has two “branches” at the point  $z_0$ , whose tangents are the lines bounding the two sectors  $S_0$  and  $S_1$ . The curve given by the equation

$$\mathcal{I}(h(z)) = \mathcal{I}(h(z_0))$$

also presents in the neighbourhood of  $z_0$  two branches whose tangents are the bisectors of the angle made by the two lines tangent to the level curve  $\Re(h(z)) = \Re(h(z_0))$ . These two “branches” are lines of steepest slope of the surface  $\Sigma$ , of which one, the ridge

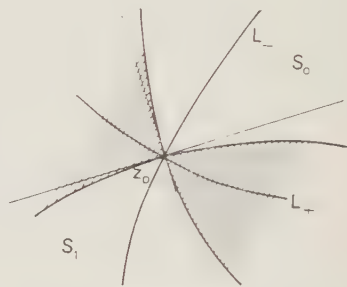


FIGURE 52

justification, and this *is not* a direct consequence of Cauchy's theorem. Usually one must pass to the limit from paths in the usual sense to which Cauchy's theorem can be applied (see below, examples (2.1) and (2.2)).

(1.11) The saddle method can be extended to the case where  $g$  and  $h$  have *isolated singularities* in  $D$  (VIII, 2.1); it is then necessary to add to the second member of (1.3) terms arising from the *residues* of  $g(z)e^{th(z)}$  at these points (VIII, 4.3).

(1.12) The asymptotic study of more general integrals of the form  $\int_L e^{h(t, z)} dz$  can often be reduced to the method of steepest descent, or to an application of similar arguments. One attempts to "distort" the path  $L$  so as to make it pass through the "saddles" of the surface of equation

$$x_3 = \Re(h(t, x_1 + ix_2)),$$

which may now depend on  $t$ ; a change of variable sometimes enables a reduction to the case where these "saddles" are fixed (cf. (2.2)).

## 2. Examples of application of the method of steepest descent

(2.1) Let us take

$$(2.1.1) \quad I(t) = \int_0^{+\infty} e^{t((1+i)x - x^3)} dx$$

which is absolutely convergent (IV, 9.1) for every  $t > 0$ , since

$$|e^{t((1+i)x - x^3)}| = e^{t(x - x^3)} \ll e^{-tx^3/2}.$$

The path  $L$  without endpoints is here the positive real axis  $\mathbf{R}_+$ ; observe first that we can replace  $L$  by any half-line  $L': s \rightarrow e^{i\varphi}s$  ( $0 \leq s < +\infty$ ) provided  $0 \leq \varphi < \pi/6$ . It is sufficient to apply Cauchy's theorem to the loop, which is the juxtaposition of the three paths (Fig. 53)

$$\begin{aligned} \gamma_1: s &\rightarrow s, & 0 \leq s \leq R \\ \gamma_2: s &\rightarrow Re^{is}, & 0 \leq s \leq \varphi \\ \gamma_3: s &\rightarrow e^{i\varphi}(R - s), & 0 \leq s \leq R. \end{aligned}$$

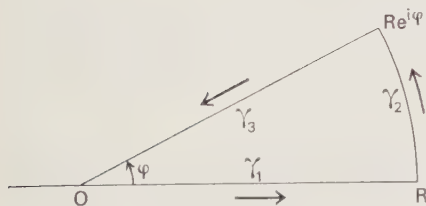


FIGURE 53

The integral along this loop is 0, since  $e^{t((1+i)z - z^3)}$  is an entire function. On the other hand for  $0 \leq s \leq \varphi$

$$|\exp((1+i)Re^{is} - R^3e^{3is})| = \exp(R(\cos s - \sin s) - R^3 \cos 3s)$$

and it follows immediately that as soon as  $R$  is sufficiently large,

$$R \cdot \exp(R(\cos s - \sin s) - R^3 \cos 3s) \leq e^{-kR^3}$$

with  $k = \frac{1}{2} \cos 3\varphi$ , for  $0 \leq s \leq \varphi$ , hence  $\int_{\gamma_2} e^{t((1+i)z - z^3)} dz$  tends to 0 as  $R$  tends to  $+\infty$ , which proves our assertion.

To apply the method of steepest descent here,  $g(z) = 1$ ,  $h(z) = (1 + i)z - z^3$  and hence  $h'(z) = 1 + i - 3z^2$ ,  $h''(z) = -6z$ ; there are two "saddles"  $\pm z_0$  with

$$(2.1.2) \quad z_0 = 2^{1/4} 3^{-1/2} e^{i\pi/8}.$$

By virtue of the preceding, to calculate (2.1.1) replace  $L = \mathbf{R}_+$  by the half-line  $L': s \rightarrow se^{i\pi/8}$  ( $0 \leq s < +\infty$ ) passing through the saddle  $z_0$ . Here we have the "normal" conditions  $n = 0$ ,  $m = 2$ , and  $\alpha = 9\pi/8$ , so  $\omega = -\pi/16$ . Furthermore, along  $L'$

$$h(z) = h(se^{i\pi/8}) = e^{3\pi i/8} (2^{1/2} s - s^3)$$

and it is immediately verified that  $\Re(h(se^{i\pi/8}))$  attains its maximum at the single point  $s_0 = 2^{1/4} 3^{-1/2}$  for  $0 \leq s < +\infty$ . The conditions of (1.5) are satisfied by  $L'$ ; moreover, the Taylor development of  $h$  in the neighbourhood of  $z_0$  is  $h(z_0 + u) = h(z_0) - 3z_0 u^2 + o(u^3)$ , and the thalweg line passing through  $z_0$  has its tangent at  $z_0$  parallel to the real axis. By virtue of the rule given in (1.8), the principal part of the integral (2.1.1) is thus equal to that of  $e^{ih(z_0)} J(t)$ , with

$$(2.1.3) \quad J(t) = \int_{-r}^{+r} e^{-3tz_0 x^2} dx = 2t^{-1/2} \int_0^{r\sqrt{i}} f(x) dx$$

where  $r > 0$  and where

$$(2.1.4) \quad f(z) = \exp(-(2^{1/8} 3^{1/4} e^{i\pi/16} z)^2).$$

The same reasoning as in the beginning proves that  $\int_L f(z) dz = \int_{L'} f(z) dz$ , where  $L'$  is the half-line  $u \rightarrow e^{-i\pi/16} u$  ( $0 \leq u < +\infty$ ), which gives finally

$$(2.1.5) \quad I(t) \sim e^{-\pi i/16} 2^{-1/8} 3^{-1/4} \pi^{1/2} t^{-1/2} \exp(2^{7/4} 3^{-3/2} e^{3i\pi/8} t).$$

(2.2) *Airy integrals.* Let  $L$  be a path without endpoints

$$s \rightarrow r(s) e^{i\varphi(s)}$$

defined for  $-\infty < s < +\infty$ , with the following conditions:

1.  $\lim_{s \rightarrow +\infty} r(s) = +\infty$ , and in the neighbourhood of  $+\infty$

$$\frac{2\pi}{3} \leq \varphi(s) \leq \frac{2\pi}{3} + \frac{\pi}{6}.$$

2.  $\lim_{s \rightarrow -\infty} r(s) = +\infty$ , and in the neighbourhood of  $-\infty$

$$-\frac{2\pi}{3} - \frac{\pi}{6} \leq \varphi(s) \leq -\frac{2\pi}{3}$$

(Fig. 54). We shall first show that for every real  $t > 0$ , the *Airy integral* (cf. VII, 10.3)

$$(2.2.1) \quad \text{Ai}(t^2) = \frac{1}{2i\pi} \int_L \exp(t^2 z - \frac{1}{3} z^3) dz$$

exists and is *independent* of the path  $L$  provided that this path satisfies the preceding conditions. To see this, consider the path without endpoints  $L_0$  which is the juxtaposition of the two half-lines

$$\begin{aligned} s &\rightarrow se^{2i\pi/3} & (0 \leq s < +\infty) \\ s &\rightarrow -se^{-2i\pi/3} & (-\infty < s \leq 0). \end{aligned}$$

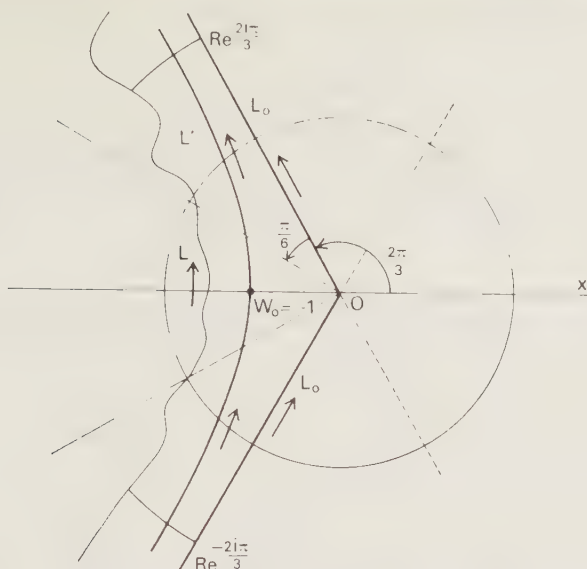


FIGURE 54

Since  $|\exp(as - \frac{1}{3}s^3)| \leq \exp(-\frac{1}{6}s^3)$  for every complex constant  $a$  in the neighbourhood of  $+\infty$ , it is clear that the integral  $\int_{L_0} \exp(t^2z - \frac{1}{3}z^3) dz$  exists. Applying Cauchy's theorem in the usual way, our assertion is a consequence of the fact that, for  $R > 0$  and

$$\frac{2\pi}{3} \leq \theta \leq \frac{2\pi}{3} + \frac{\pi}{6},$$

we have

$$|\exp(t^2 R e^{i\theta} - \frac{1}{3} R^3 e^{3i\theta})| \leq \exp\left(t^2 R \cos \frac{5\pi}{6}\right)$$

and as  $R$  tends to  $+\infty$ ,  $\operatorname{Re} \exp(t^2 R \cos 5\pi/6)$  tends to 0 (cf. VIII, 5.3.2).

The method of steepest descent will be applied to the Airy integral as  $t$  tends to  $+\infty$ . Making the change of variable  $z = tw$

$$(2.2.2) \quad \operatorname{Ai}(t^2) = \frac{t}{2i\pi} \int_{t^{-1}L} \exp(t^3(w - \frac{1}{3}w^3)) dw$$

where  $t^{-1}L$  clearly satisfies conditions (1) and (2) above. This last integral is of the type (1.2) with  $g(w) = 1$ ,  $h(w) = w - \frac{1}{3}w^3$ ; there are two saddles,  $w = \pm 1$ , roots of  $h'(w) = 1 - w^2 = 0$ . Since  $h''(w) = -2w$ , this is the "normal" situation of (1.7.2). The method can be applied taking the saddle  $w_0 = -1$ . If  $w = x + iy$  ( $x, y$  real)

$$\mathcal{J}(h(w)) = y(1 - x^2 + \frac{1}{3}y^2)$$

and it is immediately verified that the ridge line passing through  $w_0$  is the real axis, the thalweg line is the branch of the hyperbola

$$L': 1 - x^2 + \frac{1}{3}y^2 = 0$$

corresponding to  $x < 0$ , the two half-lines composing  $L_0$  being the asymptotes. The integral (2.2.2) can thus be calculated by taking for  $t^{-1}L$  the *thalweg line* itself (be careful to note that this is rather exceptional in applications of the saddle method). The rule given in (1.8) consists then in replacing  $L'$  by the path

$$u \rightarrow -1 + iu \quad (-r \leq u \leq r)$$

for an  $r > 0$ , and  $h(-1 + iu)$  by the first two terms of its Taylor development  $-\frac{2}{3} - u^2$ ; the integral

$$\frac{t e^{-2t^{3/3}}}{2\pi} \int_{-r}^r e^{-t^3 u^2} du$$

is obtained, hence finally the principal part

$$(2.2.3) \quad \text{Ai}(t^2) \sim \frac{e^{-2t^{3/3}}}{2\sqrt{\pi t}}.$$

*Remark (2.3)* The formula (2.2.3) is also exact when  $t$  is replaced by  $te^{i\omega}$  ( $t$  again tending to  $+\infty$ ) provided that  $|\omega| < \pi/6$ ; this is shown by an argument almost identical to the preceding one.

### 3. Eulerian developments

(3.1) Let  $f$  be a function *meromorphic* in  $\mathbf{C}$  (VIII, 3.5),  $(a_n)_{n \geq 1}$  the sequence (finite or infinite) of its poles, enumerated so that

$$|a_n| \leq |a_{n+1}|;$$

if the sequence  $(a_n)$  is infinite, we have  $\lim_{n \rightarrow \infty} |a_n| = +\infty$ . The following method, due to Cauchy, enables us in some important cases to express  $f(z)$  by means of a development in a convergent series for every  $z$  distinct from the poles  $a_n$ , which generalizes the "development in partial fractions" of the rational functions.

Consider an increasing sequence  $(r_\nu)_{\nu \geq 0}$  of real numbers tending to  $+\infty$  and distinct from the absolute values  $|a_n|$ ; denote by  $\Gamma_\nu$  the circles described once in the positive sense

$$\Gamma_\nu: \theta \rightarrow r_\nu e^{i\theta} \quad (0 \leq \theta \leq 2\pi).$$

The residue theorem (VIII, 4.3) gives for each  $\nu$ ,

$$(3.1.1) \quad \sum_{|a_k| < r_\nu} \text{Res}_{a_k} f = \frac{1}{2\pi i} \int_{\Gamma_\nu} f(z) dz = \frac{1}{2\pi} \int_0^{2\pi} F_\nu(\theta) d\theta$$

where

$$(3.1.2) \quad F_\nu(\theta) = r_\nu f(r_\nu e^{i\theta}) e^{i\theta}.$$

If as  $\nu$  tends to  $+\infty$ , the integral  $(1/2\pi) \int_0^{2\pi} F_\nu(\theta) d\theta$  has a limit  $A$ , it is thus seen that the series with general term

$$\sum_{r_\nu < |a_k| < r_{\nu+1}} \text{Res}_{a_k} f$$

is convergent, and

$$(3.1.3) \quad A = \sum_{\nu=0}^{\infty} \left( \sum_{r_\nu < |a_k| < r_{\nu+1}} \text{Res}_{a_k} f \right).$$

In particular:

(3.2) *The notations being those of (3.1), suppose that the function  $f$  is odd and that there exists a number  $M > 0$  such that  $|f(r_\nu e^{i\theta})| \leq M$  for every integer  $\nu$  and every  $\theta$ . Then for every  $x \in \mathbf{C}$  distinct from the poles  $a_n$*

$$(3.2.1) \quad -f(x) = \sum_{\nu=0}^{\infty} \left( \sum_{r_\nu < |a_k| < r_{\nu+1}} \text{Res}_{a_k} \left( \frac{f(z)}{z-x} \right) \right)$$

where the series of the second member is convergent.

Apply (3.1) to the meromorphic function  $g(z) = f(z)/(z-x)$  which has the poles  $a_n$  ( $n \geq 1$ ) and the simple pole  $x$ , of residue  $f(x)$ . Since the function  $f$  is odd

$$\begin{aligned} \int_0^{2\pi} F_\nu(\theta) d\theta &= \int_0^{2\pi} \frac{r_\nu f(r_\nu e^{i\theta}) d\theta}{r_\nu e^{i\theta} - x} \\ &= \int_0^\pi r_\nu f(r_\nu e^{i\theta}) e^{i\theta} \left[ \frac{1}{r_\nu e^{i\theta} - x} - \frac{1}{r_\nu e^{i\theta} + x} \right] d\theta \\ &= 2x \int_0^\pi \frac{r_\nu f(r_\nu e^{i\theta}) e^{i\theta} d\theta}{r_\nu^2 e^{2i\theta} - x^2} \end{aligned}$$

and for every  $\nu$  such that  $|x| < r_\nu$

$$\left| \int_0^{2\pi} F_\nu(\theta) d\theta \right| \leq 2\pi M \frac{r_\nu |x|}{r_\nu^2 - |x|^2}$$

where the second member tends to 0 as  $\nu$  tends to  $+\infty$ , hence the result.

(3.3) Let us apply (3.2) to the meromorphic function  $f(z) = \cot z$ , which is odd and has simple poles at the points  $n\pi$  ( $n \in \mathbf{Z}$ ). Take here  $r_\nu = (2\nu + 1)\pi/2$ . Because of the periodicity of  $f$ , it follows from (VIII, 6.6.4) that the conditions of (3.2) are fulfilled: indeed, the function  $f(z)$  is continuous in the bounded closed set  $R_0$  defined by

$$0 \leq \Re z \leq \pi, \quad |z| > \frac{\pi}{4}, \quad |z - \pi| \geq \frac{\pi}{4}, \quad |\Im z| \leq 1$$

(Fig. 55). If  $M_0$  is the least upper bound of  $|f(z)|$  in  $R_0$ , we also have  $|f(z + k\pi)| \leq M_0$  in  $R_0 + k\pi$  for every integer  $k \in \mathbf{Z}$ . It is concluded from (VIII, 6.6.4) that there exists a number  $M > 0$  such that the relation  $|z - n\pi| \geq \pi/4$  for every integer  $n \in \mathbf{Z}$  implies that  $|\cot z| \leq M$ . Since  $|z - n\pi| \geq \pi/4$  for every  $z$  satisfying  $|z| = (2\nu + 1)\pi/2$ , whatever the integers  $n$  and  $\nu$ , the conclusion is deduced.

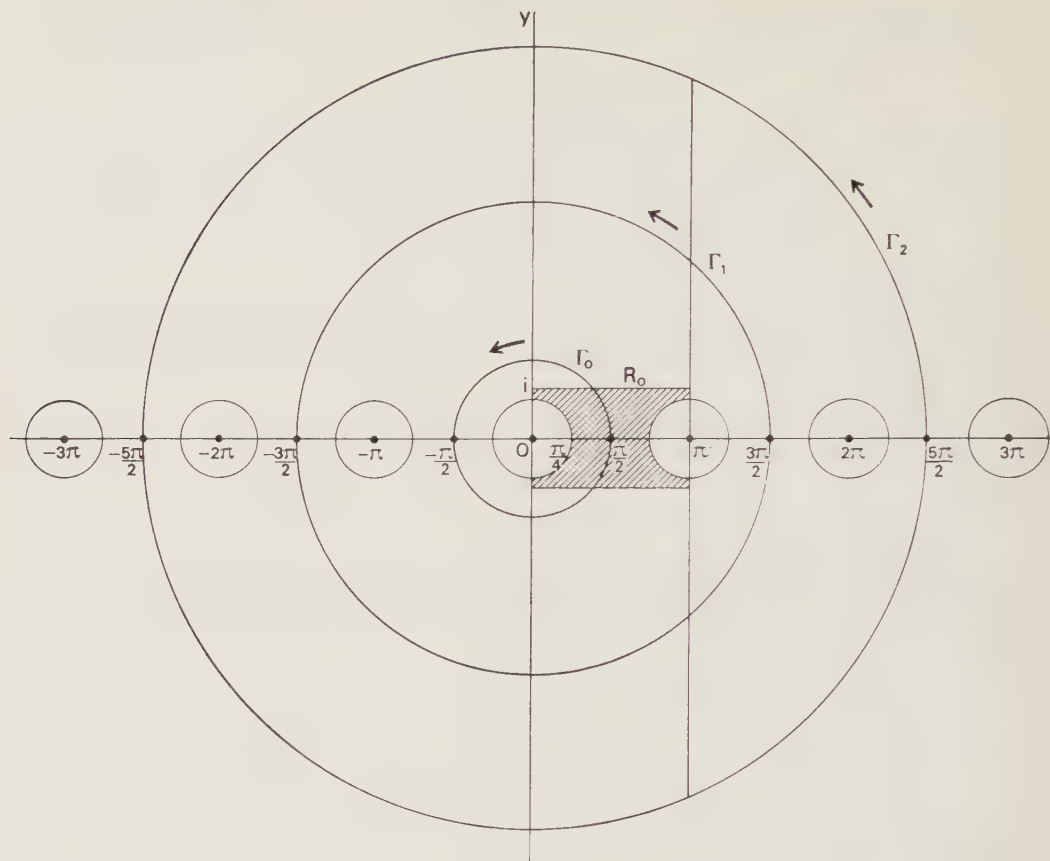


FIGURE 55

The residue of  $\cot z/(z - x)$  at each of the points  $n\pi$  is equal to  $1/(n\pi - x)$  by virtue of the formula (VIII, 4.4.4) applied to the function  $\cos z/((z - x) \sin z)$ . Hence the *Eulerian development* of  $\cot z$

$$(3.3.1) \quad \cot z = \frac{1}{z} + 2z \sum_{n=1}^{\infty} \frac{1}{z^2 - n^2\pi^2}.$$

(3.4) From this we obtain the *Eulerian development* of  $\sin z$  in an infinite product (VIII, 11):

$$(3.4.1) \quad \sin z = z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2\pi^2}\right)$$

where the product is *uniformly convergent* (VIII, 11.6) in every closed disc  $|z| \leq R$ . The uniform convergence of the product follows from (VIII, 11.5) and from the fact that the series of general term  $z^2/n^2\pi^2$  is *normally convergent* in the disc  $|z| \leq R$ . Since the

logarithmic derivative of  $1 - (z^2/n^2\pi^2)$  is  $2z/(z^2 - n^2\pi^2)$ , it follows from (VIII, 11.5) that if  $h(z)$  is the entire function equal to the second member of (3.4.1)

$$\frac{h'(z)}{h(z)} = \cot z$$

for  $z$  distinct from the numbers  $n\pi$ . This immediately implies that  $h(z)/\sin z$  has a zero derivative for  $z \neq n\pi$ , hence is *constant*. On the other hand, since

$$\lim_{z \rightarrow 0} \frac{h(z)}{z} = \lim_{z \rightarrow 0} \frac{\sin z}{z} = 1,$$

the formula (3.4.1) is proved.

(3.5) It is not of course possible to have uniform convergence for the series (3.3.1) in the neighbourhood of a pole  $n\pi$ . For every pair of numbers such that  $0 < r < R$ , the series converges *normally* in the closed set  $E$  defined by  $|z| \leq R$ ,  $|z - n\pi| \geq r$  for every  $n$ . For  $n\pi > R$  we have

$$\left| \frac{1}{z^2 - n^2\pi^2} \right| \leq \frac{1}{n^2\pi^2 - R^2} \quad \text{for } z \in E;$$

since for  $n\pi \leq R$ , each of the functions  $1/(z^2 - n^2\pi^2)$  is analytic in  $E$ , there exists  $\alpha_n$  such that  $|1/(z^2 - n^2\pi^2)| \leq \alpha_n$  for these values of  $n$ , which proves our assertion.

#### 4. Gamma function in the complex domain

Euler's formula (IV, 3.5.1) can also be written for every  $x$  *real* and  $> 0$

$$\begin{aligned} (4.1) \quad \frac{1}{\Gamma(x)} &= \lim_{n \rightarrow \infty} \frac{x(x+1) \dots (x+n)}{(n+1)^x n!} \\ &= \lim_{n \rightarrow \infty} x \prod_{k=1}^n \left(1 + \frac{x}{k}\right) \exp\left(x \log \frac{k}{k+1}\right) \end{aligned}$$

in other words,  $1/\Gamma(x)$  can be represented as an *infinite product*. Now, each factor of this product is the value for real  $x$  of the *entire function*

$$(4.2) \quad 1 + u_n(z) = \left(1 + \frac{z}{n}\right) \exp\left(z \log \frac{n}{n+1}\right)$$

defined for every  $z \in \mathbf{C}$ . It will be seen that the infinite product  $\prod_{n=1}^{\infty} (1 + u_n(z))$  is *uniformly convergent* in each disc  $|z| < R$  (VIII, 11.6); it is sufficient to prove that the series of general term  $u_n(z)$  is *normally convergent* in this disc (VIII, 11.5). Now, Taylor's formula (VI, 6.6.6) applied to the function

$$\log\left(1 + \frac{z}{n}\right) + z \log \frac{n}{n+1}$$

analytic for  $|z/n| < 1$ , shows that for  $|z/n| \leq \frac{1}{2}$

$$\left| \log \left( 1 + \frac{z}{n} \right) + z \log \frac{n}{n+1} \right| \leq 2 \left| \frac{z^2}{n^2} \right|.$$

Applying Taylor's formula to the function  $e^z$ , we obtain for  $|z/n| \leq \frac{1}{2}$

$$|u_n(z)| = \left| \exp \left( \log \left( 1 + \frac{z}{n} \right) + z \log \frac{n}{n+1} \right) - 1 \right| \leq 2 e^{1/2} \left| \frac{z^2}{n^2} \right|$$

which proves our assertion.

Thus for every complex number  $z$ , we define

$$(4.3) \quad \frac{1}{\Gamma(z)} = \lim_{n \rightarrow \infty} \frac{z(z+1) \dots (z+n)}{(n+1)^z n!} = \lim_{n \rightarrow \infty} \frac{z(z+1) \dots (z+n)}{n^z \cdot n!}.$$

An equivalent form of this definition is

$$(4.4) \quad \frac{1}{\Gamma(z)} = z e^{\gamma z} \prod_{n=1}^{\infty} \left( 1 + \frac{z}{n} \right) e^{-z/n}$$

(Weierstrass formula), where  $\gamma$  is Euler's constant (III, 11.12.2). Indeed, we have

$$\begin{aligned} \prod_{k=1}^n \exp \left( z \log \frac{k}{k+1} + \frac{z}{k} \right) \\ = \exp \left( z \left( \log \frac{1}{2} + \log \frac{2}{3} + \dots + \log \frac{n}{n+1} + 1 + \frac{1}{2} + \dots + \frac{1}{n} \right) \right) \\ = \exp \left( z \left( 1 + \frac{1}{2} + \dots + \frac{1}{n} - \log(n+1) \right) \right) \end{aligned}$$

which by definition tends to  $e^{\gamma z}$  as  $n$  tends to  $+\infty$ . The formula (4.4) thus follows from the definition of an infinite product. Moreover, as above, for  $|z/n| < 1$

$$\left| \log \left( 1 + \frac{z}{n} \right) - \frac{z}{n} \right| \leq 2 \left| \frac{z^2}{n^2} \right|$$

and so, if  $1 + v_n(z) = (1 + z/n)e^{-z/n}$ , the series of general term  $v_n(z)$  is also *normally convergent* in every disc  $|z| \leq R$ .

(4.5) The function  $1/\Gamma(z)$  defined by (4.3) or (4.4) is thus an *entire function*, having *simple zeros* at the points  $z = -n$  ( $n$  integer  $\geq 0$ ) and satisfying  $\Gamma(\bar{z}) = \overline{\Gamma(z)}$ . Its reciprocal  $\Gamma(z)$  is thus *meromorphic* in  $\mathbf{C}$  and has simple poles at the points  $-n$ . It follows directly from (4.3), or the principle of analytic continuation (VI, 7.3), that the function  $\Gamma$  thus continued again satisfies the functional equation

$$(4.5.1) \quad \Gamma(z+1) = z\Gamma(z).$$

Also from Weierstrass's formula (4.4) and from (VIII, 11.5.1), for every  $z$  distinct from the points  $-n$

$$(4.5.2) \quad \frac{\Gamma'(z)}{\Gamma(z)} = -\gamma - \frac{1}{z} + \sum_{n=1}^{\infty} \frac{z}{n(z+n)}$$

the series being normally convergent in every closed disc not containing any of the poles  $-n$ . In particular, since

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left( \frac{1}{n} - \frac{1}{n+1} \right) = 1,$$

for  $z = 1$ ,

$$(4.5.3) \quad \Gamma'(1) = -\gamma.$$

(4.6) *For every  $z \in \mathbf{C}$*

$$(4.6.1) \quad \frac{1}{\Gamma(z)\Gamma(1-z)} = \frac{1}{\pi} \sin \pi z$$

(Euler).

It follows from (4.3) that the first member of (4.6.1) is the limit as  $n$  tends to  $+\infty$  of

$$\frac{z(z+1)\dots(z+n)(1-z)(2-z)\dots(n+1-z)}{n^{z+1-z}(n!)^2}$$

$$= z \frac{n+1-z}{n} \left(1 - \frac{z^2}{1^2}\right) \left(1 - \frac{z^2}{2^2}\right) \dots \left(1 - \frac{z^2}{n^2}\right)$$

and the conclusion follows from the Eulerian development (3.4.1) of  $\sin z$ . In particular, for every *real*  $t$

$$(4.6.2) \quad |\Gamma(it)| = \sqrt{\frac{\pi}{t \sinh \pi t}}$$

and

$$(4.6.3) \quad |\Gamma(\tfrac{1}{2} + it)| = \sqrt{\frac{\pi}{\cosh \pi t}}.$$

These follow at once from the Euler formula (4.6.1), noting that from (4.5.1)

$$\Gamma(1 - it) = -it\Gamma(-it) = -it\overline{\Gamma(it)}$$

and

$$\Gamma(1 - (\tfrac{1}{2} + it)) = \Gamma(\tfrac{1}{2} - it) = \overline{\Gamma(\tfrac{1}{2} + it)}.$$

(4.7) *For every integer  $p > 1$*

$$(4.7.1) \quad \Gamma\left(\frac{z}{p}\right)\Gamma\left(\frac{z+1}{p}\right)\dots\Gamma\left(\frac{z+p-1}{p}\right) = (2\pi)^{(p-1)/2} p^{1/2-z}\Gamma(z)$$

(Legendre–Gauss formula).

Since

$$\Gamma\left(\frac{z+p}{p}\right) = \frac{z}{p} \Gamma\left(\frac{z}{p}\right),$$

it is sufficient to calculate the expression

$$f(z) = \Gamma\left(\frac{z+1}{p}\right)\Gamma\left(\frac{z+2}{p}\right)\dots\Gamma\left(\frac{z+p}{p}\right).$$

It follows from (4.3) that  $1/f(z)$  is the limit of the expression

$$(4.7.2) \quad \frac{\left(\frac{z+1}{p}\right)\left(\frac{z+1}{p}+1\right)\dots\left(\frac{z+1}{p}+n\right)\left(\frac{z+2}{p}\right)\dots\left(\frac{z+2}{p}+n\right)\dots\left(\frac{z+p}{p}\right)\left(\frac{z+p}{p}+1\right)\dots\left(\frac{z+p}{p}+n\right)}{(n!)^p n^{(z+1)/p+(z+2)/p+\dots+(z+p)/p}}$$

as  $n$  tends to  $+\infty$ . This expression can also be written

$$\frac{(z+1)(z+2)\dots(z+(n+1)p)}{(n!)^p p^{(n+1)p} n^{z+(p+1)/2}}$$

and in particular

$$(4.7.3) \quad \frac{1}{f(0)} = \lim_{n \rightarrow \infty} \frac{((n+1)p)!}{(n!)^p p^{(n+1)p} n^{(p+1)/2}}.$$

Therefore  $f(0)/f(z)$  is the limit of the expression

$$\frac{(z+1)(z+2)\dots(z+(n+1)p)}{n^z ((n+1)p)!} = \frac{1}{z} \left( \frac{(n+1)p}{n} \right)^z \frac{z(z+1)(z+2)\dots(z+(n+1)p)}{((n+1)p)^z ((n+1)p)!}$$

and since  $((n+1)p/n)^z$  tends to  $p^z$ , by (4.3)

$$(4.7.4) \quad \frac{f(0)}{f(z)} = \frac{p^z}{z} \frac{1}{\Gamma(z)}.$$

It remains to evaluate  $f(0)$  with the help of (4.7.3), which is an immediate application of Stirling's formula (IV, 3.8.2).

(4.8) Note finally an expression for  $1/\Gamma(z)$  as an integral along a path without endpoints, valid for every  $z \in \mathbf{C}$ . Let  $L$  be a path without endpoints  $t \rightarrow r(t)e^{i\varphi(t)}$  defined for  $-\infty < t < +\infty$ , contained in  $D_0$  (the plane cut along the negative real axis) and satisfying the following conditions:

1.  $\lim_{t \rightarrow +\infty} r(t) = +\infty$ , and in the neighbourhood of  $+\infty$ ,  $(\pi/2) + \delta \leq \varphi(t) < \pi$ .
2.  $\lim_{t \rightarrow -\infty} r(t) = +\infty$ , and in the neighbourhood of  $-\infty$ ,  $-\pi < \varphi(t) \leq (-\pi/2) - \delta$

for some  $\delta > 0$  (Fig. 56). It will be seen that for every  $z \in \mathbf{C}$

$$(4.8.1) \quad \frac{1}{\Gamma(z)} = \frac{1}{2\pi i} \int_L u^{-z} e^u du$$

where of course the integral of the second member exists under the conditions imposed on  $L$  (*Hankel's integral*); the principal determination has been taken for  $u^{-z}$  (VIII, 9.6). The calculation is very like the analogous one for the Airy integral (2.2) and a mere outline is given. Consider the particular path without endpoints  $H_{\alpha, r}$  (Fig. 56); by applica-

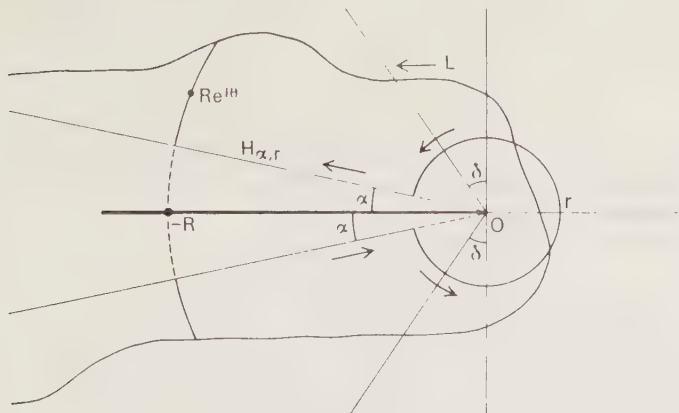


FIGURE 56

tion of Cauchy's theorem in the open simply connected domain  $D_0$ , the integral in (4.8.1) is defined and equal to the same integral taken along  $H_{\alpha, r}$ , using the fact that for  $u = R e^{i\theta}$  and  $(\pi/2) + \delta \leq \theta \leq (3\pi/2) - \delta$ , one has the majorization

$$|u^{-z} e^u| \leq R^\beta e^{-R \sin \delta},$$

where  $\beta$  is independent of  $u$ . Letting  $\alpha$  tend to 0, the integral along  $H_{\alpha, r}$  is equal to

$$\int_\gamma u^{-z} e^u du - e^{-i\pi z} \int_r^{+\infty} t^{-z} e^{-t} dt + e^{i\pi z} \int_r^{+\infty} t^{-z} e^{-t} dt$$

where  $\gamma$  is the path  $t \rightarrow re^{it}$ ,  $-\pi \leq t \leq \pi$ . Assume now that  $\Re(1 - z) > 0$ . Then the integral along  $\gamma$  tends to 0 with  $r$  and  $\int_r^{+\infty} t^{-z} e^{-t} dt$  tends to  $\Gamma(1 - z)$  (VII, 10.4.1); thus (4.6.1) the value of  $\int_L u^{-z} e^u du$  is

$$2i \sin \pi z \Gamma(1 - z) = \frac{2i\pi}{\Gamma(z)}.$$

But the second member of (4.8.1) is an *entire* function of  $z$  (VII, 10.4); the equality (4.8.1) is thus valid for *every*  $z \in \mathbf{C}$  by the principle of analytic continuation.

## 5. Bernoulli numbers and polynomials

(5.1) The function  $1/(e^z - 1)$  is meromorphic in  $\mathbf{C}$  and has for poles the roots  $2n\pi i$  of the equation  $e^z - 1 = 0$ , which are evidently simple poles since the derivative of  $e^z - 1$  does not vanish anywhere. Furthermore

$$(5.1.1) \quad \frac{1}{e^z - 1} = -\frac{1}{2} + \frac{1}{2} \frac{e^z + 1}{e^z - 1}$$

and  $(e^z + 1)/(e^z - 1)$  is an *odd* function, so has only terms of odd exponent in its Laurent development in the neighbourhood of  $z = 0$ . The residue at the point  $z = 0$

being 1, the Laurent development of  $1/(e^z - 1)$  (which converges for  $|z| < 2\pi$  by virtue of (VII, 7.3)) is therefore of the form

$$(5.1.2) \quad \frac{1}{e^z - 1} = \frac{1}{z} - \frac{1}{2} + \sum_{n=1}^{\infty} (-1)^{n-1} \frac{B_n}{(2n)!} z^{2n-1}$$

and by definition the coefficient  $B_n$  is called the  $n^{\text{th}}$  *Bernoulli number*. They are *rational* numbers, for

$$\frac{1}{e^z - 1} = \frac{1}{z(1 - u(z))}$$

with

$$u(z) = \frac{1 + z - e^z}{z} = -\frac{z}{2!} - \frac{z^2}{3!} - \dots - \frac{z^n}{(n+1)!} - \dots$$

and the Taylor development of  $z/(e^z - 1)$  is therefore obtained by replacing  $z$  by  $u(z)$  in the power series

$$\frac{1}{1 - z} = 1 + z + z^2 + \dots + z^n + \dots$$

(VI, 4.4), hence our assertion. The coefficient of  $z^n$  in the development of  $z/(e^z - 1)$  is the sum of the coefficients of  $z^n$  in the expressions  $(u(z))^k$  for  $k \leq n$ .

For the first few values of  $n$ , calculation gives

$$(5.1.3) \quad \begin{array}{llll} B_1 = \frac{1}{6}, & B_2 = \frac{1}{30}, & B_3 = \frac{1}{42}, & B_4 = \frac{1}{30}, \\ B_5 = \frac{5}{66}, & B_6 = \frac{691}{2730}, & B_7 = \frac{7}{6}, & B_8 = \frac{3617}{510}. \end{array}$$

The numerators of these numbers play an important role in mathematics, from the theory of algebraic numbers to differential topology. It will be proved (6.3) that they are all  $> 0$ , and a principal part of  $B_n$  as  $n$  tends to  $+\infty$  will be obtained.

(5.2) For every complex number  $x$ , the function  $e^{zx}/(e^z - 1)$  has the same poles as  $1/(e^z - 1)$ . Its Laurent series in the neighbourhood of  $z = 0$  is therefore obtained by multiplying the development (5.1.2) by that of  $e^{zx} = \sum_{n=0}^{\infty} \frac{x^n z^n}{n!}$ . This gives, for  $|z| < 2\pi$  and for every  $x \in \mathbf{C}$

$$(5.2.1) \quad \frac{e^{zx}}{e^z - 1} = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{\varphi_n(x)}{n!} z^{n-1}$$

where, for each integer  $n \geq 1$ ,  $\varphi_n(x)$  is a *polynomial*

$$(5.2.2) \quad \varphi_n(x) = x^n - \frac{n}{2} x^{n-1} + \binom{n}{2} B_1 x^{n-2} - \binom{n}{4} B_2 x^{n-4} + \binom{n}{6} B_3 x^{n-6} - \dots$$

the sum being continued up to the last exponent  $\geq 0$ . From the identity

$$\frac{e^{z(x+1)}}{e^z - 1} = e^{zx} + \frac{e^{zx}}{e^z - 1}$$

and by substituting  $x + 1$  for  $x$  in (5.2.1) and using (VI, 3.4), the relation

$$(5.2.3) \quad \varphi_n(x+1) - \varphi_n(x) = nx^{n-1}$$

is deduced.

Similarly, from the identity

$$\frac{e^{z(1-x)}}{e^z - 1} = -\frac{e^{-zx}}{e^{-z} - 1},$$

substituting  $1 - x$  for  $x$  in (5.2.1), the relation

$$(5.2.4) \quad \varphi_n(1-x) = (-1)^n \varphi_n(x)$$

is deduced.

Comparing (5.1.2) with (5.2.1) for  $x = 0$

$$\varphi_{2k+1}(0) = 0 \quad \text{and} \quad \varphi_{2k}(0) = (-1)^{k+1} B_k.$$

The polynomials  $\varphi_n(x)$  are called *Bernoulli polynomials*. The expression (5.2.2) for these polynomials gives, when substituting 1 for  $x$  and taking (5.2.3) into account

$$(5.2.5) \quad 0 = 1 - n + \binom{2n}{2} B_1 - \binom{2n}{4} B_2 + \cdots + (-1)^n \binom{2n}{2n-2} B_{n-1}$$

hence a recurrence relation enabling one to calculate the Bernoulli numbers.

(5.3) From the formula (5.2.1), one has

$$(5.3.1) \quad \frac{\varphi_n(x)}{n!} = \text{Res}_0 \frac{e^{zx}}{z^n(e^z - 1)} \quad \text{for } n \geq 1$$

and hence also, by the residue theorem,

$$(5.3.2) \quad \frac{\varphi_n(x)}{n!} = \frac{1}{2\pi i} \int_{\gamma} \frac{e^{zx} dz}{z^n(e^z - 1)}$$

for every loop  $\gamma: t \rightarrow re^{it}$  ( $0 \leq t \leq 2\pi$ ) where  $0 < r < 2\pi$ . Differentiating with respect to  $x$  the integral of the second member of (5.3.2) and applying the Leibniz formula

$$(5.3.3) \quad \varphi'_n(x) = n\varphi_{n-1}(x) \quad \text{for } n \geq 2.$$

## 6. Trigonometric developments of Bernoulli polynomials

Put  $f_k(z, x) = e^{zx}/(z^k(e^z - 1))$  for  $k \geq 1$ ; for each  $x \in \mathbf{C}$  this is a meromorphic function of  $z$  in  $\mathbf{C}$ , having a pole of order  $k + 1$  at the point 0 and a simple pole at each of the points  $2n\pi i$  for  $n$  integer such that  $|n| \geq 1$ . For every integer  $\nu \geq 1$ , put  $r_\nu = (2\nu - 1)\pi$ . We are going to apply the residue theorem to  $f_k(z, x)$  and the loops

$$\Gamma_\nu: \theta \rightarrow r_\nu e^{i\theta} \quad (0 \leq \theta \leq 2\pi).$$

Taking into account the formula (5.3.1) and (VIII, 4.4.4)

$$(6.1) \quad \frac{\varphi_k(x)}{k!} + \sum_{n=1}^{\nu-1} \left( \frac{e^{2n\pi i x}}{(2n\pi i)^k} + \frac{e^{-2n\pi i x}}{(-2n\pi i)^k} \right) = \frac{1}{2\pi} \int_0^{2\pi} r_\nu f_k(r_\nu e^{i\theta}, x) e^{i\theta} d\theta.$$

From this we shall deduce the following proposition:

(6.2) Suppose  $x$  real and such that  $0 \leq x \leq 1$ . Then, for every  $k \geq 1$

$$(6.2.1) \quad \varphi_{2k}(x) = (-1)^{k+1} 2(2k)! \sum_{n=1}^{\infty} \frac{\cos 2n\pi x}{(2n\pi)^{2k}}$$

$$(6.2.2) \quad \varphi_{2k+1}(x) = (-1)^{k+1} 2(2k+1)! \sum_{n=1}^{\infty} \frac{\sin 2n\pi x}{(2n\pi)^{2k+1}}$$

the series being normally convergent for  $0 \leq x \leq 1$ . Also, for  $0 < x < 1$ ,

$$(6.2.3) \quad x - \frac{1}{2} = \varphi_1(x) = - \sum_{n=1}^{\infty} \frac{\sin 2n\pi x}{n\pi}$$

where the series of the second member is uniformly convergent in every interval  $[\alpha, 1 - \alpha]$  ( $0 < \alpha < \frac{1}{2}$ ), and where the partial sums  $\sum_{n=1}^N \frac{\sin 2n\pi x}{n\pi}$  are in absolute value uniformly bounded by a number  $A$  depending neither on  $N$  nor on  $x$  (for  $0 \leq x \leq 1$ ).

The normal convergence of the series (6.2.1) and (6.2.2) being trivial for  $k \geq 1$ , to establish the first assertion it is sufficient to prove that the integral of the second member of (6.1), for all  $x$  such that  $0 \leq x \leq 1$ , and for  $k \geq 2$ , tends to 0 as  $\nu$  tends to  $+\infty$ . Now, if  $z = s + it$

$$\left| \frac{e^{zx}}{e^z - 1} \right| = \frac{e^{xs}}{|e^z - 1|} \leq \frac{1}{|e^z - 1|}$$

when  $s \leq 0$ , and

$$\left| \frac{e^{zx}}{e^z - 1} \right| = \left| \frac{e^{-s(1-x)}}{1 - e^{-z}} \right| \leq \frac{1}{|1 - e^{-z}|}$$

when  $s \geq 0$ , by virtue of the relations  $x \geq 0$  and  $1 - x \geq 0$ . The same reasoning as in (3.3) shows that there is a number  $M$  independent of  $\nu$  such that  $|1/(e^z - 1)| \leq M$  for  $|z| = r_\nu$ . Thus, for every  $\nu \geq 1$  and every  $k \geq 1$

$$(6.2.4) \quad |r_\nu f_k(r_\nu e^{i\theta}, x)| \leq \frac{M}{r_\nu^{k-1}}$$

for every  $x \in [0, 1]$  and  $\theta \in [0, 2\pi]$ , hence our conclusion (V, 3.4).

The same reasoning shows that the partial sums of the series (6.2.3) are uniformly bounded in  $[0, 1]$  by the number  $M + \frac{1}{2}$ .

We prove finally the uniform convergence of (6.2.3) to  $\varphi_1(x)$  for  $\alpha \leq x \leq 1 - \alpha$ . Given  $\varepsilon > 0$ , determine first a number  $\delta > 0$  such that  $4M\delta \leq \varepsilon$ . Then from (6.2.4) for  $k = 1$

$$\left| \int_{\frac{\pi}{2} - \delta}^{\frac{\pi}{2} + \delta} r_\nu f_1(r_\nu e^{i\theta}, x) e^{i\theta} d\theta \right| \leq \frac{\varepsilon}{2} \quad \text{and} \quad \left| \int_{\frac{3\pi}{2} - \delta}^{\frac{3\pi}{2} + \delta} r_\nu f_1(r_\nu e^{i\theta}, x) e^{i\theta} d\theta \right| \leq \frac{\varepsilon}{2}.$$

On the other hand, for  $\frac{\pi}{2} + \delta \leq \theta \leq \frac{3\pi}{2} - \delta$ , and  $\alpha \leq x \leq 1 - \alpha$ ,

$$|e^{xr_\nu e^{i\theta}}| = e^{xr_\nu \cos \theta} \leq e^{-\alpha r_\nu \sin \delta},$$

therefore, for these values of  $\theta$  and  $x$ ,  $|r_\nu f_1(r_\nu e^{i\theta}, x)| \leq M e^{-\alpha r_\nu \sin \delta}$ . Using the relation

$e^{zx}/(e^z - 1) = e^{-z(1-x)}/(1 - e^{-z})$ , one proves similarly that the same inequality occurs for  $0 \leq \theta \leq (\pi/2) - \delta$  and  $(3\pi/2) + \delta \leq \theta \leq 2\pi$ , and  $\alpha \leq x \leq 1 - \alpha$ . Thus

$$\left| \frac{1}{2\pi} \int_0^{2\pi} r_\nu f_1(r_\nu e^{i\theta}, x) e^{i\theta} d\theta \right| \leq \varepsilon + M e^{-\alpha r_\nu \sin \delta} \leq 2\varepsilon$$

as soon as  $\nu$  is sufficiently large, which completes the proof.

(6.3) The formula (6.2.1) gives in particular, for  $x = 0$ , and taking into account (5.2.2), Euler's formula for the Bernoulli numbers

$$(6.3.1) \quad B_k = \frac{2(2k)!}{(2\pi)^{2k}} \sum_{n=1}^{\infty} \frac{1}{n^{2k}}.$$

Thus these numbers are all  $> 0$ ; moreover, since

$$\sum_{n=2}^{\infty} \frac{1}{n^{2k}} \leq \int_1^{\infty} \frac{dx}{x^{2k}} = \frac{1}{2k-1}$$

$\sum_{n=1}^{\infty} \frac{1}{n^{2k}}$  tends to 1 as  $k$  tends to  $+\infty$ , and Stirling's formula (IV, 3.9.2) gives for the principal part of  $B_k$ :

$$(6.3.2) \quad B_k \sim 4\sqrt{\pi} \frac{k^{2k+1/2}}{(e\pi)^{2k}}$$

a function tending very rapidly to  $+\infty$ .

(6.4) The second members of (6.2.1) and (6.2.2) are functions *continuous* in  $\mathbf{R}$  and *periodic* with period 1, which coincide respectively with  $\varphi_{2k}$  and  $\varphi_{2k+1}$  in the interval  $[0, 1]$ , and which are denoted by  $\tilde{\varphi}_{2k}$  and  $\tilde{\varphi}_{2k+1}$  ( $k \geq 1$ ). From (5.1.2) and (5.2.1)

$$(6.4.1) \quad \tilde{\varphi}_{2k}(0) = (-1)^{k+1} B_k$$

and hence, for *every*  $x \in \mathbf{R}$ , by virtue of (6.2.1)

$$(6.4.2) \quad |\tilde{\varphi}_{2k}(x)| \leq B_k.$$

On the other hand

$$(6.4.3) \quad \tilde{\varphi}_{2k+1}(0) = \tilde{\varphi}_{2k+1}(\tfrac{1}{2}) = 0$$

and taking into account (5.3.3) and (5.2.4), the theorem of the mean gives from (6.4.2), the inequality valid for *every*  $x \in \mathbf{R}$

$$(6.4.4) \quad |\tilde{\varphi}_{2k+1}(x)| \leq (k + \tfrac{1}{2}) B_k.$$

(6.5) The series of the second member of (6.2.3) also converges for *every*  $x \in \mathbf{R}$  and is *periodic* with period 1, but its sum is 0 at the points  $x = n \in \mathbf{Z}$  and equal to  $x - \frac{1}{2}$  for  $0 < x < 1$ ; this sum, denoted by  $\tilde{\varphi}_1(x)$ , is thus a function *piecewise-continuous* in  $\mathbf{R}$  but *discontinuous* at the points  $n \in \mathbf{Z}$  with the relations  $\tilde{\varphi}_1(n) = 0$ ,  $\tilde{\varphi}_1(n+) = -\frac{1}{2}$ ,  $\tilde{\varphi}_1(n-) = \frac{1}{2}$ . This implies that the series (6.2.3) cannot converge uniformly in *any* interval containing a point  $n \in \mathbf{Z}$  (V, 3.1) (cf. problem 34).

It follows from the preceding and from (5.3.3) that the functions  $\tilde{\varphi}_k$  for  $k \geq 3$  are *continuously differentiable* in  $\mathbf{R}$  and that

$$(6.5.1) \quad \tilde{\varphi}'_k(x) = k\tilde{\varphi}_{k-1}(x).$$

The function  $\tilde{\varphi}_2$  has a continuous derivative equal to  $2\tilde{\varphi}_1$  *except* at the points  $n \in \mathbf{Z}$ . Fig. 57 gives the graphs of the functions  $\tilde{\varphi}_k$  for  $k \leq 4$ .

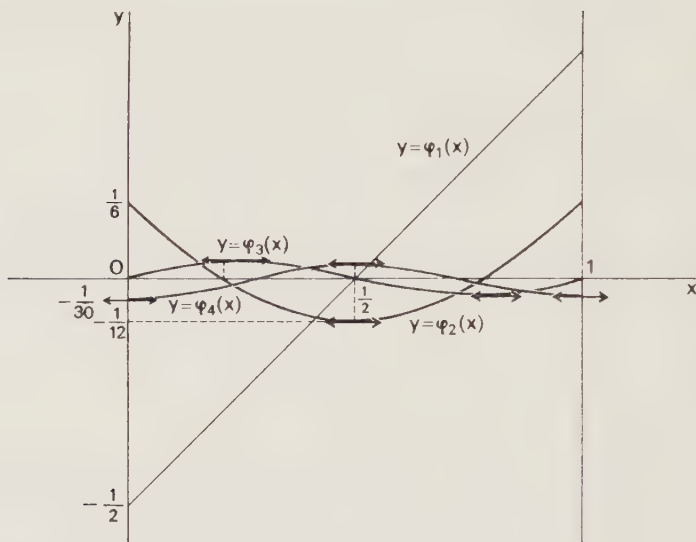


FIGURE 57

## 7. Euler-Maclaurin formula

(7.1) Let  $\alpha, \beta$  be two real numbers,  $m, n$  integers such that

$$m - 1 < \alpha \leq m < n \leq \beta < n + 1.$$

Then for every complex function  $f$  continuous in the interval  $[\alpha, \beta]$ , which is the primitive of a function  $f'$  piecewise-continuous in  $[\alpha, \beta]$

$$(7.1.1) \quad f(m) + f(m+1) + \cdots + f(n)$$

$$= \int_{\alpha}^{\beta} f(t) dt + \tilde{\varphi}_1(\alpha-)f(\alpha) - \tilde{\varphi}_1(\beta+)f(\beta) + \int_{\alpha}^{\beta} \tilde{\varphi}_1(t)f'(t) dt.$$

Decompose the interval  $[\alpha, \beta]$  into subintervals at the points  $m, m+1, \dots, n$  and calculate the integral of the piecewise-continuous function  $\tilde{\varphi}_1(x)f'(x)$  in each of these inter-

vals, taking into account the fact that  $\varphi'_1$  is constant and equal to 1. Thus if  $m \leq k < k+1 \leq n$ , the formula for integration by parts gives

$$\begin{aligned}\int_k^{k+1} \tilde{\varphi}_1(t) f'(t) dt &= \frac{1}{2}(f(k+1) + f(k)) - \int_k^{k+1} f(t) dt \\ \int_\alpha^m \tilde{\varphi}_1(t) f'(t) dt &= \frac{1}{2}f(m) - \tilde{\varphi}_1(\alpha-)f(\alpha) - \int_\alpha^m f(t) dt \\ \int_n^\beta \tilde{\varphi}_1(t) f'(t) dt &= \tilde{\varphi}_1(\beta+)f(\beta) + \frac{1}{2}f(n) - \int_n^\beta f(t) dt\end{aligned}$$

as can be seen by considering separately the cases  $m = \alpha$ ,  $m > \alpha$  (resp.  $n = \beta$ ,  $n < \beta$ ); thus obtaining (7.1.1) on adding these integrals.

In particular, for  $\alpha = m < n = \beta$

$$\begin{aligned}(7.1.2) \quad f(m) + f(m+1) + \cdots + f(n) \\ = \int_m^n f(t) dt + \frac{1}{2}(f(m) + f(n)) + \int_m^n \tilde{\varphi}_1(t) f'(t) dt.\end{aligned}$$

(7.2) Suppose now that the  $(2r+1)^{\text{th}}$  derivative of  $f$  exists in  $[\alpha, \beta]$  and is *piecewise-continuous*. Then, for  $1 \leq h \leq 2r$ , taking into account (6.5.1) and proceeding by induction on  $h$ ,

$$\begin{aligned}(7.2.1) \quad \frac{1}{h!} \int_\alpha^\beta \tilde{\varphi}_h(t) f^{(h)}(t) dt &= \frac{1}{(h+1)!} (\tilde{\varphi}_{h+1}(\beta) f^{(h)}(\beta) - \tilde{\varphi}_{h+1}(\alpha) f^{(h)}(\alpha)) \\ &\quad - \frac{1}{(h+1)!} \int_\alpha^\beta \tilde{\varphi}_{h+1}(t) f^{(h+1)}(t) dt.\end{aligned}$$

Hence, by combining these formulae member by member and by (7.1.1), the relation

$$\begin{aligned}(7.2.2) \quad f(m) + f(m+1) + \cdots + f(n) \\ = \int_\alpha^\beta f(t) dt + \sum_{h=1}^{2r+1} (-1)^h \frac{\tilde{\varphi}_h(\beta+)f^{(h-1)}(\beta) - \tilde{\varphi}_h(\alpha-)f^{(h-1)}(\alpha)}{h!} + R_{2r+1}\end{aligned}$$

with

$$(7.2.3) \quad R_{2r+1} = \frac{1}{(2r+1)!} \int_\alpha^\beta \tilde{\varphi}_{2r+1}(t) f^{(2r+1)}(t) dt.$$

This gives in particular for  $\alpha = m < n = \beta$ , by virtue of (6.4.1) and (6.4.3), the *Euler-Maclaurin summation formula*

$$\begin{aligned}(7.2.4) \quad f(m) + f(m+1) + \cdots + f(n) &= \int_m^n f(t) dt + \frac{1}{2}(f(m) + f(n)) \\ &\quad + \sum_{h=1}^r (-1)^{h-1} \frac{B_h}{(2h)!} (f^{(2h-1)}(n) - f^{(2h-1)}(m)) + R_r\end{aligned}$$

with the majorization of the remainder

$$(7.2.5) \quad |R_r| \leq \frac{(r + \frac{1}{2})B_r}{(2r+1)!} \int_m^n |f^{(2r+1)}(t)| dt$$

which comes from the formula of the mean applied to (7.2.3), taking into account the majorization (6.4.4). Using (6.3.1),

$$(7.2.6) \quad |R_r| \leq \frac{2}{(2\pi)^{2r}} \int_m^n |f^{(2r+1)}(t)| dt.$$

(7.3) Amongst other things the Euler-Maclaurin formula enables us to obtain an arbitrarily precise *asymptotic development* for a sum

$$(7.3.1) \quad G(n) = g(0) + g(1) + \cdots + g(n)$$

as  $n$  tends to  $+\infty$ , where the function  $g$  is supposed defined for  $x \geq 0$ , *real, indefinitely differentiable* with all its derivatives  $g^{(h)}$  *monotonic* and satisfying

$$(7.3.2) \quad g^{(h+1)}(x) = o(|g^{(h)}(x)|)$$

in the neighbourhood of  $+\infty$ . Suppose first that the integer  $r$  is such that

$$|g^{(2r-1)}(x)|$$

tends to  $+\infty$  with  $x$ . Then

$$(7.3.3) \quad g(1) + g(2) + \cdots + g(n) = \int_1^n g(t) dt + \frac{1}{2}g(n) + \frac{B_1}{2}g'(n) \\ + \cdots + (-1)^{r-1} \frac{B_r}{(2r)!} g^{(2r-1)}(n) + o(g^{(2r-1)}(n))$$

where each term of the second member is negligible compared to the preceding term. If on the other hand there is an integer  $q < r$  such that  $|g^{(2q-1)}(x)|$  tends to  $+\infty$ , but  $g^{(2q+1)}(x)$  tends to 0, then

$$(7.3.4) \quad g(1) + g(2) + \cdots + g(n) = \int_1^n g(t) dt + \frac{1}{2}g(n) + \sum_{h=1}^q (-1)^{h-1} \frac{B_h}{(2h)!} g^{(2h-1)}(n) \\ + C + \sum_{h=q+1}^r (-1)^{h-1} \frac{B_h}{(2h)!} g^{(2h-1)}(n) \\ + o(g^{(2r-1)}(n))$$

where  $C$  is a constant (in general  $\neq 0$ ), and each term of the second member is again negligible compared to the preceding term. One treats similarly the case where  $g(x)$  tends to 0 but the series of general term  $g(n)$  is divergent, or the case where this series is convergent ( $n$  must then tend to  $+\infty$  in (7.2.4) to have a development of the remainder of the series). It is then necessary to replace each term in the expressions obtained by an asymptotic development with respect to  $\mathcal{E}$  (III, 6).

*Examples* (7.4) Take  $g(x) = \exp(x^{1/\lambda})$  with  $\lambda > 1$ ; this is a case where all the derivatives of  $g$  tend to  $+\infty$ . Taking into account the development

$$(7.4.1) \quad \int_1^u t^{\lambda-1} e^t dt = u^{\lambda-1} e^u - (\lambda-1)u^{\lambda-2} e^u + \cdots \\ + (-1)^k (\lambda-1) \cdots (\lambda-k+1) u^{\lambda-k} e^u + o(u^{\lambda-k} e^u)$$

one obtains by (7.3.3), assuming for example  $\lambda < \frac{5}{4}$ , the development to 6 terms

$$(7.4.2) \quad \exp(1^{1/\lambda}) + \exp(2^{1/\lambda}) + \cdots + \exp(n^{1/\lambda}) \\ = \exp(n^{1/\lambda}) \left[ \lambda n^{1-\frac{1}{\lambda}} + \frac{1}{2} + \frac{1}{12\lambda} n^{\frac{1}{\lambda}-1} - \frac{1}{720\lambda^3} n^{\frac{3}{\lambda}-3} \right. \\ \left. - \lambda(\lambda-1)n^{1-\frac{2}{\lambda}} + \frac{\lambda-1}{240\lambda^3} n^{\frac{2}{\lambda}-3} + o(n^{\frac{2}{\lambda}-3}) \right]$$

(7.5) Take  $g(x) = x^\alpha$  with  $\alpha \geq 0$ . If  $q$  is an integer such that

$$2q-1 < \alpha < 2q+1,$$

we are in the case for application of (7.3.4), hence the development

$$(7.5.1) \quad 1^\alpha + 2^\alpha + \cdots + n^\alpha = \frac{n^{\alpha+1}}{\alpha+1} + \frac{1}{2} n^\alpha \\ + \sum_{h=1}^q (-1)^{h-1} \frac{B_h}{(2h)!} \alpha(\alpha-1) \cdots (\alpha-2h+2) n^{\alpha-2h+1} + C \\ + \sum_{h=q+1}^r (-1)^{h-1} \frac{B_h}{(2h)!} \alpha(\alpha-1) \cdots (\alpha-2h+2) n^{\alpha-2h+1} + o(n^{\alpha-2r+1})$$

where  $C$  is a constant. We remark here that when  $\alpha = k$  is an integer  $\geq 1$ , we have, from (5.2.3), the exact formula

$$(7.5.2) \quad 1^k + 2^k + \cdots + n^k = \frac{1}{k+1} (\varphi_{k+1}(n+1) - \varphi_{k+1}(0)).$$

(7.6) *Asymptotic development of  $\log \Gamma(z)$ .* Let us agree that in the following,  $\log(-x) = \log x + i\pi$  for  $x$  real and  $> 0$ . The expression for  $1/\Gamma(z)$  as an infinite product (4.2) proves that if  $x$  is real and  $> 0$  and  $z$  any complex number in  $D_0$  (the plane cut along the negative real axis), the series of general term

$$\log(z+n) - \log(x+n) + (z-x) \log \frac{n}{n+1}$$

is convergent and has the sum

$$\log \Gamma(x) - \log \Gamma(z) + 2ki\pi$$

where  $k$  is an integer. Put  $f(t) = \log(z+t) - \log(x+t)$  for  $t$  real and  $\geq 0$ ; apply to  $f$  the Euler-Maclaurin summation formula (7.2.4), which gives

$$(7.6.1) \quad f(0) + f(1) + \cdots + f(n) = \int_0^n f(t) dt + \frac{1}{2}(f(0) + f(n)) \\ + \sum_{h=1}^p (-1)^{h-1} \frac{B_h}{(2h)!} (f^{(2h-1)}(n) - f^{(2h-1)}(0)) + T_p(n)$$

with

$$(7.6.2.) \quad |T_p(n)| \leq \frac{2}{(2\pi)^{2p}} \int_0^n |f^{(2p+1)}(t)| dt.$$

Now,  $f(n)$  and all the derivatives  $f^{(h)}(n)$  tend to 0 with  $1/n$ ; on the other hand

$$\int_0^n \log(z+t) dt = (z+n)(\log(z+n) - 1) - z(\log z - 1)$$

from which

$$\int_0^n f(t) dt = (z-x) \log n + h(x, z) + o\left(\frac{1}{n}\right)$$

where  $h$  does not depend on  $n$ . Hence, taking into account the remark at the beginning, as  $n$  tends to  $+\infty$  ( $p$  being fixed)  $T_p(n)$  has a limit  $R_p(x, z)$  and

$$(7.6.3) \quad \log \Gamma(z) - g(z) = \log \Gamma(x) - g(x) + R_p(x, z) + 2ki\pi$$

where

$$(7.6.4) \quad g(z) = z \log z - z - \frac{1}{2} \log z + \sum_{h=1}^p \frac{(-1)^{h-1} B_h}{2h(2h-1)z^{2h-1}}.$$

Let us find an upper bound for  $R_p(x, z)$ , assuming that  $x \geq A$  and  $\omega(z) = \sup(\Re z, |\Im z|) \geq A$ . Since

$$f^{(m)}(t) = (-1)^{m-1}(m-1)! \left( \frac{1}{(z+t)^m} - \frac{1}{(x+t)^m} \right),$$

it is sufficient to majorize  $\int_0^{+\infty} \frac{dt}{|z+t|^m}$ . If  $\Re z \geq A$ , we have  $|z+t| \geq A+t$ , hence

$$\int_0^{+\infty} \frac{dt}{|z+t|^m} \leq \int_0^{+\infty} \frac{dt}{(A+t)^m} = \frac{1}{(m-1)A^{m-1}}$$

and if  $|\Im z| \geq A$ , we have  $|z+t| \geq (A^2 + (s+t)^2)^{1/2}$  where  $s = \Re z$ , hence

$$\int_0^{+\infty} \frac{dt}{|z+t|^m} \leq \int_{-\infty}^{+\infty} \frac{dt}{(A^2 + t^2)^{m/2}} = \frac{2}{A^{m-1}} \int_0^{+\infty} \frac{dt}{(1+t^2)^{m/2}}.$$

From this it may be concluded that there is a constant  $C_p$ , depending only on  $p$ , such that under the preceding conditions

$$|R_p(x, z)| \leq \frac{C_p}{A^{2p}}.$$

This being so, we know by Stirling's formula that as  $x$  tends to  $+\infty$ ,  $\log \Gamma(x) - g(x)$  tends to  $\frac{1}{2} \log 2\pi$ . Thus, when  $\omega(z)$  tends to  $+\infty$ , we have the Stirling development

$$(7.6.5) \quad \log \Gamma(z) = z \log z - z - \frac{1}{2} \log z + \frac{1}{2} \log 2\pi + \sum_{n=1}^p \frac{(-1)^{n-1} B_n}{2n(2n-1)} \frac{1}{z^{2n-1}} + 2k_z i\pi + O\left(\frac{1}{(\omega(z))^{2p}}\right)$$

where  $k_z$  is an integer depending on  $z$ .

This formula enables us to generalize the formulae given in (IV, 3.8) to the case where the parameters are complex numbers. In particular, for every complex number  $z$

$$(7.6.6) \quad z(z+1) \dots (z+n) \sim \frac{\sqrt{2\pi}}{\Gamma(z)} n^{n+z+\frac{1}{2}} e^{-n}$$

and

$$(7.6.7) \quad \frac{\Gamma(n+z)}{\Gamma(n)} \sim n^z.$$

Similarly, if  $z$  is a complex number distinct from an integer  $\geq 0$

$$(7.6.8) \quad \binom{z}{n} = \frac{(-1)^n}{\Gamma(-z)} \frac{\Gamma(n-z)}{\Gamma(n+1)} \sim \frac{(-1)^n}{\Gamma(-z)} n^{-z-1}.$$

By the same reasoning

$$(7.6.9) \quad \frac{\Gamma'(z)}{\Gamma(z)} = \log z - \frac{1}{2z} - \sum_{h=1}^p \frac{(-1)^{h-1} B_h}{2h} \frac{1}{z^{2h}} + O\left(\frac{1}{(\omega(z))^{2p+1}}\right).$$

Here the formula

$$\int_x^{x+1} \frac{\Gamma'(t)}{\Gamma(t)} dt = \log \Gamma(x+1) - \log \Gamma(x) = \log x$$

must be used to prove that the difference of the two members of (7.6.9) tends to 0 for  $x$  real.

## 8. Fourier series and approximation by trigonometric polynomials

(8.1) A *trigonometric polynomial* of a real variable  $x$  is a linear combination (with complex coefficients) of a finite number of exponentials functions  $e^{imx}$  ( $m$  positive or negative integer), in other words a function of the form

$$(8.1.1) \quad P(x) = \sum_{m=-N}^N c_m e^{imx} \quad \text{with } c_m \in \mathbf{C} \text{ for } -N \leq m \leq N.$$

For a fixed  $N$  the functions (8.1.1) are called the *trigonometric polynomials of degree  $\leq N$* .

Decomposing  $e^{imx}$  into real and imaginary parts, it amounts to the same thing to say that a trigonometric polynomial is a function of the form

$$(8.1.2) \quad x \rightarrow \sum_{m=0}^N (a_m \cos mx + b_m \sin mx)$$

where  $a_m$  and  $b_m$  are the complex numbers given by

$$(8.1.3) \quad \begin{aligned} a_m &= c_m + c_{-m}, & b_m &= i(c_m - c_{-m}) \quad \text{for } m > 0 \\ a_0 &= c_0, & b_0 &= 0. \end{aligned}$$

(8.2) A trigonometric polynomial is evidently a *continuous function periodic* with period  $2\pi$ . Moreover, the coefficients are well determined by the values of  $P(x)$  (in other terms, a trigonometric polynomial is identically zero only if its coefficients are all zero).

For the relations  $\int_0^{2\pi} e^{mit} dt = 0$  if  $m \neq 0$ ,  $\int_0^{2\pi} dt = 2\pi$ , give the formulae

$$(8.2.1) \quad c_m = \frac{1}{2\pi} \int_0^{2\pi} P(t) e^{-imt} dt \quad \text{for } -N \leq m \leq N$$

and

$$\int_0^{2\pi} P(t)e^{-imt} dt = 0 \quad \text{for } |m| > N.$$

Note further that because of the periodicity of the integrand the interval  $[0, 2\pi]$  can be replaced in these formulae by any interval  $[a, a + 2\pi]$  whatever. It follows from (8.2.1) that if the values of the trigonometric polynomial  $P(x)$  are all *real*,  $c_{-m} = \bar{c}_m$ , and hence the coefficients  $a_m$  and  $b_m$  are *real* by (8.1.3); the converse is obvious.

(8.3) We now consider more generally a complex function  $f$  *periodic* with period  $2\pi$  and *piecewise-continuous* in  $[0, 2\pi]$  (and so by the periodicity piecewise-continuous in  $\mathbf{R}$ ). Note that any complex function  $g$  piecewise-continuous in  $[0, 2\pi]$  can be considered in  $]0, 2\pi[$  to be the restriction of a piecewise-continuous periodic function  $f$  of period  $2\pi$ , by modifying its values at the points 0 and  $2\pi$  so that  $g(0) = g(2\pi)$ . Now take

$$f(x + 2k\pi) = g(x)$$

for every integer  $k \in \mathbf{Z}$  and for  $0 \leq x \leq 2\pi$ ; note that  $g$  is the restriction to  $[0, 2\pi]$  of a *continuous* periodic function only if it is continuous in  $[0, 2\pi]$  and satisfies  $g(0) = g(2\pi)$ .

This being so, with the preceding notations, the complex numbers

$$(8.3.1) \quad c_m = \frac{1}{2\pi} \int_0^{2\pi} f(t)e^{-imt} dt$$

for every integer  $m \in \mathbf{Z}$  are called the *Fourier coefficients* of  $f$  (or of  $g$ ). In general there will be infinitely many of these coefficients which are  $\neq 0$ ; note that these coefficients are *not altered* if the value of  $f$  is modified at a *finite* number of points. Again put

$$(8.3.2) \quad \begin{cases} a_m = c_m + c_{-m} &= \frac{1}{\pi} \int_0^{2\pi} f(t) \cos mt dt \\ b_m = i(c_m - c_{-m}) &= \frac{1}{\pi} \int_0^{2\pi} f(t) \sin mt dt \end{cases} \quad \text{for } m \text{ integer } > 0$$

$$(8.3.3) \quad a_0 = c_0 = \frac{1}{2\pi} \int_0^{2\pi} f(t) dt$$

so that the  $a_m$  and  $b_m$  are *real* if  $f$  takes real values. The interval  $[0, 2\pi]$  can of course be replaced in these formulae by any interval  $[a, a + 2\pi]$ .

By analogy with (8.1), one may naturally associate with  $f$  the trigonometric polynomials

$$(8.3.4) \quad P_N(x) = \sum_{m=-N}^N c_m e^{imx}$$

where for the  $c_m$  the Fourier coefficients of  $f$  are taken. The  $P_N$  are said to be the *Fourier polynomials* of  $f$ ; they are also the *partial sums* of the series

$$(8.3.5) \quad a_0 + \sum_{m=1}^{\infty} (a_m \cos mx + b_m \sin mx)$$

called the *Fourier series* of  $f$ . One could hope for this series to converge (at least simply) to  $f$ ; however examples are known of *continuous* periodic functions whose Fourier series

is *divergent* at some points (problem 30). Criteria are given below for the convergence and uniform convergence of the Fourier series of a piecewise-continuous function.

(8.4) Let  $f$  be a *piecewise-continuous* function periodic with period  $2\pi$ . For brevity  $f$  is said to be *piecewise-differentiable* if  $[0, 2\pi]$  can be decomposed into a finite number of sub-intervals  $[\alpha, \beta]$  such that in each open interval  $] \alpha, \beta[$   $f$  is *continuous* and is the *primitive* of a function  $f'$  *piecewise-continuous* in  $] \alpha, \beta[$  ((0, 4.3) and (III, 9.7)). It is further supposed that the integral  $\int_{\alpha}^{\beta} |f'(t)| dt$  is defined (it can eventually be improper (III, 9.7)).

This terminology being fixed, we have the following criterion:

(8.5) Let  $f$  be a complex function periodic with period  $2\pi$ , *piecewise-continuous* and *piecewise-differentiable* in  $\mathbf{R}$  (8.4). Then the Fourier series (8.3.5) of  $f$  converges for every  $x \in \mathbf{R}$  and has the sum  $\frac{1}{2}(f(x+) + f(x-))$ . Moreover, the convergence of this series is *uniform* in every closed interval not containing any point of discontinuity of  $f$ , and the partial sums  $P_N(x)$  are bounded in absolute value by a number  $M$  independent of  $N$  and  $x$ .

(A) Suppose first that  $f$  is *continuous* (and piecewise-differentiable). Then the integral  $\int_0^{2\pi} \left(\frac{t}{2\pi} - \frac{1}{2}\right) f'(t) dt$  is by hypothesis *absolutely convergent* and hence (III, 9.8) the formula for integration by parts can be applied:

$$(8.5.1) \quad \int_0^{2\pi} \left(\frac{t}{2\pi} - \frac{1}{2}\right) f'(t) dt = \left(\frac{t}{2\pi} - \frac{1}{2}\right) f(t) \Big|_0^{2\pi} - \frac{1}{2\pi} \int_0^{2\pi} f(t) dt.$$

Or again, since  $f$  is continuous and periodic

$$(8.5.2) \quad f(0) = \frac{1}{2\pi} \int_0^{2\pi} f(t) dt + \int_0^{2\pi} \left(\frac{t}{2\pi} - \frac{1}{2}\right) f'(t) dt.$$

Replace the function  $t \rightarrow f(t)$  in this formula by the function  $t \rightarrow f(t+x)$  (for any  $x \in \mathbf{R}$ ), which gives by the periodicity of  $f$

$$(8.5.3) \quad f(x) = \frac{1}{2\pi} \int_0^{2\pi} f(t) dt + \int_0^{2\pi} \left(\frac{t}{2\pi} - \frac{1}{2}\right) f'(t+x) dt.$$

Now, for  $0 < t < 2\pi$ , it has been seen that

$$(8.5.4) \quad \frac{t}{2\pi} - \frac{1}{2} = - \sum_{n=1}^{\infty} \frac{\sin nt}{n\pi}$$

the convergence being *uniform* in every interval  $[\alpha, 2\pi - \alpha]$  with  $0 < \alpha < \pi$  and the partial sums

$$S_N(t) = - \sum_{n=1}^N \frac{\sin nt}{n\pi}$$

being *bounded* in absolute value by a number  $A$  independent of  $N$  and of  $t$  (6.2). We shall deduce from this that for every  $x \in \mathbf{R}$

$$(8.5.5) \quad \int_0^{2\pi} \left(\frac{t}{2\pi} - \frac{1}{2}\right) f'(t+x) dt = - \sum_{n=1}^{\infty} \int_0^{2\pi} \frac{\sin nt}{n\pi} f'(t+x) dt$$

the series being *uniformly convergent* in  $\mathbf{R}$  and the partial sums

$$Q_N(x) = - \sum_{n=1}^N \int_0^{2\pi} \frac{\sin nt}{n\pi} f'(t+x) dt = \int_0^{2\pi} S_N(t) f'(t+x) dt$$

being bounded in absolute value by a number *independent of*  $N$  and  $x$ .

This last point follows at once from the theorem of the mean, which gives

$$|Q_N(x)| \leq A \int_0^{2\pi} |f'(t+x)| dt = A \int_0^{2\pi} |f'(t)| dt$$

by periodicity, taking into account the hypothesis on  $f'$ . This same hypothesis implies (0,3.4 and III, 9.6) that for each  $\varepsilon > 0$ , there exists a  $\delta > 0$  such that

$$\int_0^\delta |f'(t+x)| dt \leq \varepsilon, \quad \int_{2\pi-\delta}^{2\pi} |f'(t+x)| dt \leq \varepsilon$$

for all  $x \in \mathbf{R}$ . The number  $\delta$  being thus chosen,  $N_0$  can be found such that, for  $N \geq N_0$ , we have  $|\frac{t}{2\pi} - \frac{1}{2} - S_N(t)| \leq \varepsilon$  for all  $t$  in the interval  $[\delta, 2\pi - \delta]$ . We conclude that for  $N \geq N_0$  and for *every*  $x \in \mathbf{R}$

$$\begin{aligned} \left| \int_0^{2\pi} \left( \frac{t}{2\pi} - \frac{1}{2} \right) f'(t+x) dt - Q_N(x) \right| \\ \leq (A+1) \int_0^\delta |f'(t+x)| dt + (A+1) \int_{2\pi-\delta}^{2\pi} |f'(t+x)| dt \\ + \varepsilon \int_\delta^{2\pi-\delta} |f'(t+x)| dt \\ \leq \varepsilon \left( 2(A+1) + \int_0^{2\pi} |f'(t)| dt \right) \end{aligned}$$

which proves our assertion. But by integration by parts, for  $n \geq 1$ , by virtue of the continuity of  $f$

$$\begin{aligned} \int_0^{2\pi} \frac{\sin nt}{n\pi} f'(t+x) dt &= \frac{\sin nt}{n\pi} f(t+x) \Big|_0^{2\pi} - \frac{1}{\pi} \int_0^{2\pi} f(t+x) \cos nt dt \\ &= -\frac{1}{\pi} \int_0^{2\pi} f(t) \cos n(t-x) dt = - (a_n \cos nx + b_n \sin nx). \end{aligned}$$

Since  $(1/2\pi) \int_0^{2\pi} f(t) dt = a_0$ , we do indeed find the Fourier series of  $f$  in the second member of (8.5.3), by replacing the second integral by its value extracted from (8.5.5).

(B) Pass to the general case where  $f$  is only supposed piecewise-continuous (and piecewise-differentiable) and let  $x_1 < x_2 < \dots < x_r$  be its points of discontinuity in  $[0, 2\pi]$ . For every  $k$  such that  $1 \leq k \leq r$ , put

$$(8.5.6) \quad \lambda_k = f(x_k+) - f(x_k-)$$

(“jump” of the function  $f$  at the point  $x_k$ ). Then it is clear that the function

$$g(x) = f(x) - \sum_{k=1}^r \lambda_k \tilde{\varphi}_1\left(\frac{x - x_k}{2\pi}\right)$$

(notation of (6.4)) is continuous, periodic and piecewise-differentiable in  $\mathbf{R}$ . The result of (A) can thus be applied to it. It is immediately verified that the second member of (8.5.4) is the Fourier series of the function  $\tilde{\varphi}_1(t)$ , and the conclusion follows from the reasoning in (6.2) and from the relation

$$\tilde{\varphi}_1(0+) = -\tilde{\varphi}_1(0-) = -\frac{1}{2}.$$

(8.6) It has already been mentioned that if the function  $f$  is only supposed periodic (of period  $2\pi$ ) and *continuous* in  $\mathbf{R}$ , the Fourier series of  $f$  is not necessarily convergent at every point. It may also happen that the Fourier series is convergent, but that its partial sums are not *uniformly bounded* in  $\mathbf{R}$  (problem 30).

However we can *approximate  $f$  uniformly* in  $\mathbf{R}$  by trigonometric polynomials, in other words for continuous periodic functions there is a result analogous to the Weierstrass approximation theorem:

(8.7) *Let  $f$  be a function continuous in  $\mathbf{R}$  and periodic with period  $2\pi$ . Then, for each  $\varepsilon > 0$ , there exists a trigonometric polynomial  $P$  such that*

$$(8.7.1) \quad |f(x) - P(x)| \leq \varepsilon$$

*for every  $x \in \mathbf{R}$ .*

With the notations of (V, 4.4), consider the functions  $f_n = f * \rho_n$ , where the “regularizing” function  $\rho$  has been chosen *continuously differentiable*. It is then known that  $f_n$  is *continuously differentiable* (V, 4.6), and it follows at once from the definition (V, 4.4.4) that  $f_n$  is *periodic* of period  $2\pi$ . Lastly (V, 4.5), we can find  $n$  sufficiently large that  $|f(x) - f_n(x)| \leq \varepsilon/2$  in  $[0, 2\pi]$ , so everywhere in  $\mathbf{R}$  by periodicity. But (8.5) can now be applied to the function  $f_n$  and there is therefore a partial sum  $P$  of its Fourier series such that  $|f_n(x) - P(x)| \leq \varepsilon/2$  for every  $x \in \mathbf{R}$ . It is clear that the polynomial  $P$  has the required property.

(8.8) One often has to consider functions piecewise-continuous in  $\mathbf{R}$  and periodic with period  $T > 0$ ; if  $f$  is such a function, a function piecewise-continuous and periodic with period  $2\pi$  is obtained by putting  $g(x) = f(Tx/2\pi)$ . The *Fourier coefficients* of  $f$  are by definition the Fourier coefficients of  $g$ ; by virtue of (8.3.1), they are written

$$(8.8.1) \quad c_m = \frac{1}{T} \int_0^T f(t) \exp\left(-\frac{2\pi imt}{T}\right) dt.$$

All the preceding definitions and results can be immediately transcribed to functions of any period  $T$ .

Fourier series are of considerable interest in physical and mechanical applications, the “simple” periodic functions of the type  $e^{imt}$  being particularly easy to handle, hence the idea of approximating any periodic function (which occur in many phenomena) by these “simple” functions or a linear combination of a finite number of them. This approximate expression for the function  $f$  as a “superposition” of “simple” periodic functions, also called “harmonic analysis” of  $f$  (the functions  $e^{mit}$  for  $m > 1$  being often called the “harmonics” of  $e^t$ ), corresponds to phenomena which can often be checked experimentally.

## 9. Approximation by quadratic means and Fourier series

(9.1) The idea of the “approximation” of one function by another, introduced in Chap. V, gives rise to notions other than that of “uniform approximation”, which was studied at that time. We shall confine our attention to complex functions defined in a bounded interval  $I = [a, b]$  of  $\mathbf{R}$  and *piecewise-continuous* in  $I$ . Instead of taking for the “distance” between two such functions the number  $d(f, g)$  defined in (V, 1.3) we can choose the number

$$(9.1.1) \quad d_1(f, g) = \int_a^b |f(t) - g(t)| dt.$$

When  $f$  and  $g$  are real, this number has a simple geometrical significance: it is the “area” bounded by the two graphs of  $f$  and  $g$  (Fig. 58). Note that the value of this number is *independent* of the values of  $f$  and  $g$  at their points of discontinuity, which already shows that this is a notion of “approximation” very different from uniform approximation. But even for *continuous* functions  $f$  and  $g$ ,  $d_1(f, g)$  can be very small when  $d(f, g)$  is not: for example, for the functions  $g_n$  defined in (V, 2.3.1),  $d(0, g_n) = 1$  whereas  $d_1(0, g_n) = 1/2n$ . Graphically speaking it can be said that if  $d_1(f, g)$  is “small”, the values of  $f$  and  $g$  may be very different, although only in “small” intervals, and the “larger” the difference the “smaller” the intervals. To express this “compensation” we speak of “approximation in the mean” for the notion of approximation based on the “distance”  $d_1(f, g)$ .

(9.2) In the formula (9.1.1),  $|f(t) - g(t)|$  can be replaced by  $|f(t) - g(t)|^p$ , where  $p$  is a fixed number  $> 0$ . For various reasons we then take as the “distance” between  $f$  and  $g$  the number

$$(9.2.1) \quad d_p(f, g) = \left( \int_a^b |f(t) - g(t)|^p dt \right)^{1/p}$$

which has the property of being *homogeneous*: if  $f$  and  $g$  are multiplied by the same constant  $c$ ,  $d_p(f, g)$  is multiplied by  $|c|$ .

Note that for each of these “distances” the *piecewise-continuous* functions in  $I$  can be *arbitrarily approximated* by *continuous* functions (contrary to what happens for uniform approximation (V, 3.1)). If  $a_1 < a_2 < \dots < a_r$  are the points of discontinuity of a piecewise-continuous function  $f$ , and if  $f_n$  is the continuous function equal to  $f$  except in the intervals  $[a_k - (1/n), a_k + (1/n)]$  where it has the form  $t \rightarrow \alpha t + \beta$  (Fig. 59), it is seen that

$$|f(t) - f_n(t)| \leq 2M,$$

designating by  $M$  the least upper bound of  $|f(t)|$  in  $I$ . It follows at once that

$$d_p(f, f_n) \leq 2M(2r/n)^{1/p}$$

which is arbitrarily small with  $1/n$ .

It is even possible to take  $f_n$  such that  $f_n(a) = f_n(b) = 0$  by replacing  $f$  by a suitable affine linear function  $t \rightarrow \alpha t + \beta$  in the intervals  $[a, a + (1/n)]$ ,  $[b - (1/n), b]$ .

(9.3) Calculations with the “distances” (9.2.1) are particularly simple when  $p = 2$ ; in this case we speak of the “quadratic distance” and of “approximation in the quadratic mean”. It will

be seen that the Fourier polynomials (8.3.5) possess a remarkable *extremal* property for the quadratic distance:

(9.4) Let  $f$  be a function piecewise-continuous in  $I = [0, 2\pi]$ , and let

$$c_m = \frac{1}{2\pi} \int_0^{2\pi} f(t) e^{-imt} dt$$

be its Fourier coefficients. Among all the trigonometric polynomials  $Q(t)$  of degree  $\leq N$ , the Fourier polynomial  $P_N(t)$  of  $f$  is the only one for which the quadratic distance  $d_2(f, Q)$  attains its smallest value, and

$$(9.4.1) \quad (d_2(f, P_N))^2 = \int_0^{2\pi} |f(t)|^2 dt - 2\pi \sum_{m=-N}^N |c_m|^2.$$

For any trigonometric polynomial  $Q(t) = \sum_{m=-N}^N d_m e^{imt}$  of degree  $\leq N$ ,

$$\begin{aligned} |f(t) - Q(t)|^2 &= (f(t) - Q(t))(\overline{f(t)} - \overline{Q(t)}) \\ &= |f(t)|^2 - \sum_{m=-N}^N d_m \overline{f(t)} e^{imt} - \sum_{m=-N}^N \bar{d}_m f(t) e^{-imt} + \sum_{m=-N}^N \sum_{n=-N}^N d_m \bar{d}_n e^{i(m-n)t}. \end{aligned}$$

Integrating and taking into account the definition (8.3.1)

$$(d_2(f, Q))^2 = \int_0^{2\pi} |f(t)|^2 dt - 2\pi \sum_{m=-N}^N (\bar{c}_m d_m + c_m \bar{d}_m) + 2\pi \sum_{m=-N}^N |d_m|^2$$

which can also be written

$$\begin{aligned} (d_2(f, Q))^2 &= \int_0^{2\pi} |f(t)|^2 dt - 2\pi \sum_{m=-N}^N |c_m|^2 + 2\pi \sum_{m=-N}^N |c_m - d_m|^2 \\ &= (d_2(f, P_N))^2 + 2\pi \sum_{m=-N}^N |c_m - d_m|^2 \end{aligned}$$

and this immediately proves the proposition.

From this and (8.7) we obtain

(9.5) For each function  $f$  piecewise-continuous in  $[0, 2\pi]$ , the series  $\sum_{m=-\infty}^{+\infty} |c_m|^2$  is convergent and

$$(9.5.1) \quad \sum_{m=-\infty}^{+\infty} |c_m|^2 = \frac{1}{2\pi} \int_0^{2\pi} |f(t)|^2 dt$$

(Parseval's relation).

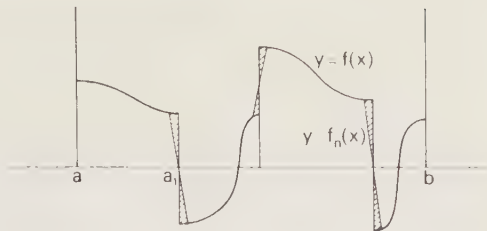


FIGURE 59

First deduce from (9.4.1) the inequality

$$(9.5.2) \quad \sum_{-N}^N |c_m|^2 \leq \frac{1}{2\pi} \int_0^{2\pi} |f(t)|^2 dt$$

for every integer  $N$ , which proves the convergence of the series of the first member of (9.5.1). On the other hand, for each  $\varepsilon > 0$ , there exists a function  $g$  continuous and periodic of period  $2\pi$  such that  $d_2(f, g) \leq \varepsilon$  (9.2). From (8.7) there is a trigonometric polynomial  $Q$  such that  $|g(x) - Q(x)| \leq \varepsilon$  in  $[0, 2\pi]$ , hence on integrating,  $d_2(g, Q) \leq \sqrt{2\pi} \varepsilon$ , and finally, by Minkowski's inequality (I, 4.6)

$$d_2(f, Q) \leq \varepsilon(1 + \sqrt{2\pi}).$$

Now if  $Q$  has degree  $\leq N$ , we have  $d_2(f, P_N) \leq d_2(f, Q)$  by (9.4.1), and this shows that the difference between the two members of (9.5.2) can be made arbitrarily small, hence (9.5.1).

When the function  $f$  is real, the formula (9.5.1) can also be written with the notations of (8.3.2)

$$(9.5.3) \quad 2a_0^2 + \sum_{m=1}^{\infty} (a_m^2 + b_m^2) = \frac{1}{\pi} \int_0^{2\pi} (f(t))^2 dt.$$

The following corollary expresses the fact that a piecewise-continuous function is essentially determined by knowledge of its Fourier coefficients:

(9.6) *If  $f$  and  $g$  are two functions piecewise-continuous in  $[0, 2\pi]$  and having the same Fourier coefficients, then  $f(t) = g(t)$  except at the points of discontinuity of  $f$  or of  $g$ .*

By considering the difference  $f - g$ , reduce at once to the case where  $g = 0$ , therefore the coefficients  $c_m$  in (9.5.1) are by hypothesis all zero. From the relation  $\int_0^{2\pi} |f(t)|^2 dt = 0$ , then deduce that  $f(t) = 0$  except at the points of discontinuity of  $f$  (I, 3.1).

## 10. Fourier coefficients and regularity properties

(10.1) It follows from (9.5) that for every function  $f$  piecewise-continuous in  $[0, 2\pi]$ , the sequence  $(c_m)_{m \in \mathbf{Z}}$  of Fourier coefficients of  $f$  is such that  $\sum_{-\infty}^{+\infty} |c_m|^2$  is convergent, and in particular  $\lim_{m \rightarrow \infty} c_m = \lim_{m \rightarrow -\infty} c_{-m} = 0$ . It will be seen that the rapidity of the convergence of the sequence of Fourier coefficients to 0 is closely bound up with the "regularity" of the function  $f$ : roughly speaking, it can be said that the more the function  $f$  is differentiable, the more the Fourier coefficients tend rapidly to 0, and conversely.

(10.2) *For every function  $f$  continuous and periodic of period  $2\pi$ , which is the primitive of a piecewise-continuous function  $f'$ ,  $\lim_{m \rightarrow \infty} m c_m = \lim_{m \rightarrow -\infty} m c_{-m} = 0$ , and more precisely, the series  $\sum_{-\infty}^{+\infty} m^2 |c_m|^2$  is convergent.*

Integrating by parts, if  $m \neq 0$ ,

$$c_m = \frac{1}{2\pi} \int_0^{2\pi} f(t) e^{-imt} dt = \frac{-1}{2\pi im} f(t) e^{-imt} \Big|_0^{2\pi} + \frac{1}{2\pi im} \int_0^{2\pi} f'(t) e^{-imt} dt$$

and hence, if  $c'_m$  is the Fourier coefficient of subscript  $m$  of  $f'$ , by virtue of the periodicity of  $f$ ,

$$(10.2.1) \quad c'_m = imc_m.$$

It is then sufficient to apply (9.5) to the function  $f'$ .

By induction:

(10.3) *If the continuous periodic function  $f$  is  $k$  times continuously differentiable, the series  $\sum_{-\infty}^{+\infty} |m^k c_m|^2$  is convergent, and in particular  $c_m = o(1/m^k)$  as  $m$  tends to  $\pm\infty$ .*

(10.4) The preceding propositions have no converses: for example, a continuous function  $f$  can be such that the series  $\sum_{-\infty}^{+\infty} |mc_m|^2$  is convergent, although  $f$  is not everywhere differentiable. This question is bound up with a more general one: given a sequence  $(c_m)_{m \in \mathbf{Z}}$  (infinite in both directions), does there exist a periodic function  $f$  piecewise-continuous, or continuous in  $\mathbf{R}$ , or  $k$  times continuously differentiable, for which  $(c_m)$  is the sequence of Fourier coefficients? No necessary and sufficient condition answering this question is known, which is not a tautology. We shall give only simple sufficient conditions partially answering the preceding question.

(10.5) *If the series  $\sum_{-\infty}^{+\infty} |c_m|$  (resp.  $\sum_{-\infty}^{+\infty} m^k |c_m|$ ) is convergent, the sequence  $(c_m)$  is the sequence of Fourier coefficients of a periodic function  $f$  continuous in  $\mathbf{R}$  (resp.  $k$  times continuously differentiable) and the Fourier series of  $f$  (resp. of each of the first  $k$  derivatives of  $f$ ) converges normally to  $f$  (resp. to the considered derivative).*

Since  $|c_m e^{mit}| = |c_m|$ , the series  $\sum_{-\infty}^{+\infty} c_m e^{mit}$  is normally convergent in  $\mathbf{R}$ , and its sum  $g(t)$  is thus a continuous periodic function (V, 3.1). In calculating the Fourier coefficient  $(1/2\pi) \int_0^{2\pi} f(t) e^{-int} dt$  of  $f$ , we can replace  $f(t)$  by the series  $\sum_{-\infty}^{+\infty} c_m e^{mit}$  and integrate term by term (V, 3.4), which gives the value  $c_n$  and so proves the proposition, when it is supposed the series  $\sum_{-\infty}^{+\infty} |c_m|$  convergent. If the series  $\sum_{-\infty}^{+\infty} m |c_m|$  is supposed convergent, so is the series  $\sum_{-\infty}^{+\infty} |c_m|$ , since  $|c_m| \leq m |c_m|$  for  $m \neq 0$ . From the preceding, the function  $g(t) = \sum_{-\infty}^{+\infty} imc_m e^{imt}$  is continuous and periodic and by virtue of (V, 3.4)

$$f(t) - f(0) = \int_0^t g(u) du,$$

therefore  $g(t) = f'(t)$ . Similar reasoning by induction on  $k$  applies when the series  $\sum_{-\infty}^{+\infty} m^k |c_m|$  is convergent.

One should be careful to observe that a "trigonometric series" (8.3.5) can be convergent for every  $x \in \mathbf{R}$  without its sum  $f$  being continuous (or even bounded). It can happen for example that  $f$  is continuous in the open interval  $]0, 2\pi[$  but tends to  $+\infty$  or  $-\infty$  at the end points and that the (improper) integrals  $\int_0^{2\pi} f(t)e^{-imt} dt$  are not convergent so that one cannot even talk about the Fourier coefficients of  $f$ .

The question posed in (10.4) can be answered completely for analytic functions:

(10.6) For the sequence  $(c_m)_{m \in \mathbf{Z}}$  to be the sequence of Fourier coefficients of the restriction to  $\mathbf{R}$  of a function of period  $2\pi$  analytic in an open strip  $|\Im z| < r$ , it is necessary and sufficient that for each  $\rho$  satisfying  $0 < \rho < r$ , we have  $c_m = O(e^{-|m|\rho})$  as  $m$  tends to  $\pm\infty$ .

Suppose that  $f$  is periodic of period  $2\pi$  and analytic in the strip  $|\Im z| < r$ . Then, for  $0 < \rho < r$ , application of Cauchy's theorem to the rectangle (Fig. 60) which is the juxtaposition of

$$t \rightarrow t \quad (0 \leq t \leq 2\pi)$$

$$t \rightarrow 2\pi + it \quad (0 \leq t \leq \rho)$$

$$t \rightarrow 2\pi + i\rho - t \quad (0 \leq t \leq 2\pi)$$

$$t \rightarrow i\rho - it \quad (0 \leq t \leq \rho)$$

gives, because of the periodicity of  $f$ , the formula

$$c_m = \frac{e^{m\rho}}{2\pi} \int_0^{2\pi} f(i\rho + t) e^{-imt} dt;$$

hence, as  $m$  tends to  $-\infty$ ,  $c_m = O(e^{m\rho})$ . It is similarly shown that as  $m$  tends to  $+\infty$ ,  $c_m = O(e^{-m\rho})$  by considering the reflection of the preceding rectangle through the real axis.

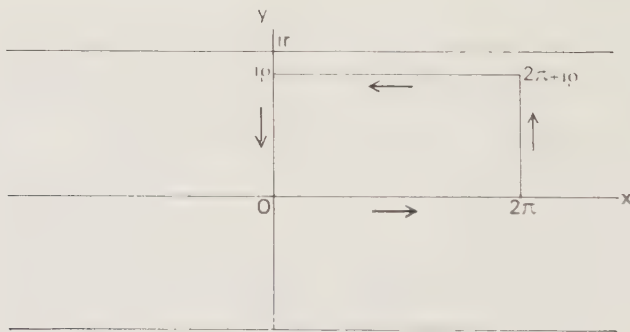


FIGURE 60

Conversely, suppose that for  $0 < \rho < r$ , we have  $c_m = O(e^{-|m|\rho})$  as  $m$  tends to  $\pm\infty$ . Then the power series

$$\sum_{m=0}^{\infty} c_m w^m \quad \text{and} \quad \sum_{m=1}^{\infty} c_{-m} w^{-m}$$

in  $w$  and  $1/w$  are respectively convergent for  $|w| < e^\rho$  and for  $|w| > e^{-\rho}$ . Hence

$$(10.6.1) \quad g(w) = \sum_{m=0}^{\infty} c_m w^m + \sum_{m=1}^{\infty} c_{-m} w^{-m}$$

is a function analytic in the annulus  $e^{-\rho} < |w| < e^\rho$ , the second member being the Laurent development (VIII, 2). Since the series  $\sum_{m=-\infty}^{+\infty} |c_m|$  is evidently convergent, the sequence  $(c_m)_{m \in \mathbf{Z}}$  is the sequence of Fourier coefficients of the function  $f(x) = g(e^{ix})$ . Moreover the second member of (10.6.1) is convergent for  $e^{-r} < |w| < e^r$  by hypothesis, so the function  $g(e^{iz})$  is analytic for  $|\Im z| < r$ .

*Remark (10.7)* Let  $f$  be a *real* function piecewise-continuous in  $\mathbf{R}$  and periodic of period  $2\pi$ , and consider its Fourier series (8.3.5) (remember that this is not necessarily convergent at every point, even if  $f$  is continuous). By (10.1) the coefficients  $a_m$  and  $b_m$  tend to 0 with  $1/m$ , therefore the power series  $\sum_{m=0}^{\infty} (a_m - ib_m)z^m$  converges for  $|z| < 1$ . If further the series  $\sum_{m=0}^{\infty} (|a_m| + |b_m|)$  is convergent, then the preceding power series converges even for  $|z| = 1$ , and its sum  $F(z)$  is then a function continuous for  $|z| \leq 1$ , analytic for  $|z| < 1$  and such that  $f(\theta) = \Re(F(e^{i\theta}))$ . However when  $f$  is only assumed continuous and periodic, such a function  $F(z)$  with the above properties does not in general exist.

## APPENDIX

### Runge phenomenon

A method very much in vogue for approximating a complex continuous function defined in a bounded closed interval  $I = [a, b]$  of  $\mathbf{R}$ , consists of considering  $n$  points  $a_1 < a_2 < \dots < a_n$  of  $I$  (usually equidistant with  $a_1 = a$ ,  $a_n = b$ ), and the “*interpolation polynomial*”  $P_n(x)$  of degree  $\leq n - 1$  determined by the conditions

$$(1) \quad P_n(a_j) = f(a_j) \quad \text{for } 1 \leq j \leq n.$$

Let

$$(2) \quad \omega_n(x) = (x - a_1)(x - a_2) \dots (x - a_n).$$

It is easily verified (VIII, 4.4.4) that the unique polynomial  $P_n(x)$  of degree  $\leq n - 1$  answering the question is given by the *Lagrange interpolation formula*

$$(3) \quad P_n(x) = \sum_{j=1}^n \frac{f(a_j) \omega_n(x)}{(x - a_j) \omega_n'(a_j)}.$$

(This solution is indeed a solution of the problem, and the difference of two solutions must be a multiple of  $\omega_n(x)$ , and so is 0 since it is a polynomial of degree  $\leq n - 1$ .)

There is a tendency (particularly in applications to Physics) to admit without proof that when  $n$  tends to  $+\infty$ ,  $P_n(x)$  converges uniformly to  $f(x)$  in  $[a, b]$ . Now this may not be true, even for functions  $f$  defined and analytic in an open set  $D \subset \mathbf{C}$  containing the interval  $I$  of the real axis (*Runge phenomenon*).

Let us take for  $\alpha > 0$

$$(4) \quad f(z) = \frac{1}{z^2 + \alpha^2}$$

a function meromorphic in  $\mathbf{C}$  with simple poles at  $\pm \alpha i$ ; let  $\gamma: t \rightarrow \operatorname{Re} t$  for  $0 \leq t \leq 2\pi$  with  $R \geq |x|$ ,  $R \geq \alpha$ ,  $R \geq |a_j|$  for  $1 \leq j \leq n$ , and consider the integral

$$\frac{1}{2\pi i} \int_{\gamma} \frac{f(z)\omega_n(x) dz}{(z-x)\omega_n(z)}$$

where  $x$  is distinct from the  $a_j$  and from  $\pm \alpha i$ . It is clear that the absolute value of this integral is majorized by  $A/R^{n+2}$ , where  $A$  is a constant. The residue of the integrand

$$g(z) = \frac{f(z)\omega_n(x)}{(z-x)\omega_n(z)}$$

at the pole  $x$  is  $f(x)$ , and at each of the poles  $a_j$  is equal to

$$\frac{f(a_j)\omega_n(x)}{(a_j-x)\omega'_n(a_j)}.$$

Lastly

$$\operatorname{Res}_{\alpha i}(g) = \frac{\omega_n(x)}{2\alpha i(\alpha i - x)\omega_n(\alpha i)}, \quad \operatorname{Res}_{-\alpha i}(g) = \frac{\omega_n(x)}{2\alpha i(\alpha i + x)\omega_n(-\alpha i)}.$$

We confine ourselves to the case where  $[a, b] = [-1, +1]$ , where  $n = 2m$  is even and where we take for the  $a_j$  the points  $\pm(2k+1)/2m$  with  $0 \leq k \leq m-1$ . Then

$$(5) \quad \omega_n(\alpha i) = \omega_n(-\alpha i) = (-1)^m \left( \alpha^2 + \frac{1}{4m^2} \right) \left( \alpha^2 + \frac{9}{4m^2} \right) \dots \left( \alpha^2 + \frac{(2m-1)^2}{4m^2} \right)$$

and application of the residue theorem gives, on letting  $R$  tend to  $+\infty$  and taking into account (3), the formula

$$(6) \quad f(x) - P_n(x) = \frac{1}{x^2 + \alpha^2} \frac{\omega_n(x)}{\omega_n(\alpha i)}$$

This formula enables us to show that when  $\alpha$  is taken sufficiently small, the difference  $f(1) - P_n(1)$  does not tend to 0 as  $n$  tends to  $+\infty$ . We have

$$(7) \quad \omega_n(1) = \frac{1}{2m} \cdot \frac{3}{2m} \cdot \frac{5}{2m} \dots \frac{4m-1}{2m} \sim \sqrt{2} \left( \frac{2}{e} \right)^n$$

by virtue of Stirling's formula. On the other hand, by (5),

$$\log |\omega_n(\alpha i)| = \sum_{k=0}^{m-1} \log \left( \alpha^2 + \frac{(2k+1)^2}{4m^2} \right)$$

and the Euler-Maclaurin formula shows easily that

$$(8) \quad |\omega_n(\alpha i)| \sim c \cdot \beta^n \quad (c \text{ constant} \neq 0)$$

with

$$(9) \quad \log \beta = \int_0^1 \log(\alpha^2 + t^2) dt = \log(1 + \alpha^2) - 2 + 2\alpha \arctan \frac{1}{\alpha}.$$

When  $\alpha$  tends to 0, this last expression tends to  $-2$ , so there exists  $\alpha > 0$  such that  $\log \beta < \log(2/e) = \log 2 - 1$ . The formulae (6), (7) and (8) then give

$$f(1) - P_n(1) \sim c' \left( \frac{2}{e\beta} \right)^n \quad (c' \text{ constant} \neq 0)$$

and this difference thus tends in absolute value to  $+\infty$ .

## PROBLEMS

1. Show, by the method of steepest descent, that as  $t$  tends to  $+\infty$

$$\int_{-\infty}^{+\infty} \left( \frac{e^{ix}}{1+x^2} \right)^t dx \sim \frac{(\pi(2-\sqrt{2})/t)^{1/2} e^{-(\sqrt{2}-1)t}}{(2\sqrt{2}-2)^t}$$

2. Show by the method of steepest descent that when the integer  $n$  tends to  $+\infty$

$$\int_{-\infty}^{+\infty} \frac{e^{ix}}{(x^3 + 3x - 2i)^n} dx \sim 2e\left(\frac{i}{4}\right)^n \sqrt{\frac{\pi}{3n}}$$

(There are two roots of (1.7.1), but only one is suitable).

3. Let  $A_n = \{1, 2, \dots, n\}$ ; consider the set  $\mathcal{C}_n \subset \mathfrak{P}(\mathfrak{P}(A_n))$  formed by the partitions of  $A_n$  into non-empty subsets, and denote by  $d_n$  the number of its elements.

- (a) Show that  $d_1 = 1$  and, setting  $d_0 = 1$ , prove the recurrence relation

$$d_{n+1} = \binom{n}{0} d_0 + \binom{n}{1} d_1 + \dots + \binom{n}{n} d_n$$

for  $n \geq 0$ . (For each  $k$ , consider the partitions of  $A_{n+1}$  of which one element is a subset of  $A_{n+1}$  having  $k+1$  elements and containing  $n+1$ .)

- (b) Deduce from (a) that

$$\exp(e^z - 1) = \sum_{n=0}^{\infty} \frac{d_n}{n!} z^n$$

for every  $z \in \mathbf{C}$ , and hence

$$d_n = \frac{n!}{2\pi i} \int_{\gamma} \exp(e^z) z^{-n-1} dz$$

where  $\gamma$  is a closed path such that  $j(0; \gamma) = 1$ .

- (c) Show that application of the method of steepest descent leads one to consider the saddle  $z = u$ , where  $u$  is the real root of the equation  $xe^x = n+1$ . Prove first that

$$\int_{\gamma} \exp(e^z) z^{-n-1} dz = \int_{u-i\infty}^{u+i\infty} \exp(e^z) z^{-n-1} dz.$$

Putting  $z = u + iy$  with  $y \in \mathbf{R}$ , show then that this integral is equivalent to

$$i \exp(e^u - u e^u \log u) \int_{-\infty}^{+\infty} \exp(e^u(e^{iy} - 1)) dy.$$

By applying the method of steepest descent to this integral, obtain finally

$$d_n \sim \frac{n!}{e\sqrt{2\pi}} \exp(e^u - u e^u \log u - \tfrac{1}{2}u)$$

and, by using an asymptotic development of  $u$ ,

$$\log d_n \sim n \log n.$$

4. The entire function

$$\sin z - az \cos z \quad (a \text{ complex} \neq 1)$$

possesses  $2n + 1$  simple zeros at 0 and  $\pm\lambda_j$  ( $1 \leq j \leq n$ ) in the open disc  $|z| < n\pi$  for  $n$  sufficiently large. Prove the formula

$$\sin z - az \cos z = (1 - a)z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{\lambda_n^2}\right)$$

(apply the same method as for the development of  $\sin z$  into an Eulerian product).

5. Let  $F(z)$  be an entire function such that

$$|F(z)| < C e^{\rho|z|}$$

where  $C$  and  $\rho$  are constants  $> 0$ . Show that

$$\frac{d}{dz} \left( \frac{F(z)}{\sin \rho z} \right) = - \sum_{n=-\infty}^{+\infty} \frac{(-1)^n \rho F(n\pi/\rho)}{(\rho z - n\pi)^2}$$

(method of no. 3). Deduce that if further  $F$  is odd

$$\frac{F(z)}{2\rho z \cos \rho z} = \sum_{n=0}^{+\infty} \frac{(-1)^n F\left(\frac{(n + \frac{1}{2})\pi}{\rho}\right)}{(n + \frac{1}{2})^2 \pi^2 - \rho^2 z^2}$$

6. Prove by the methods of no. 3 the identities

$$\frac{\pi \sin az}{2 \sin \pi z} = \sum_{n=1}^{\infty} (-1)^n \frac{n \sin na}{z^2 - n^2} \quad (-\pi \leq a \leq \pi)$$

$$\frac{\pi \cos az}{2z \sin \pi z} = \frac{1}{2z^2} + \sum_{n=1}^{\infty} (-1)^n \frac{\cos na}{z^2 - n^2}$$

$$\sum_{n=0}^{\infty} \frac{1}{a + bn^2} = -\frac{1}{2a} + \frac{\pi}{2\sqrt{ab}} \coth\left(\pi\sqrt{\frac{a}{b}}\right), \quad 0 < a, 0 < b$$

$$\sum_{n=1}^{\infty} (-1)^n \frac{n}{e^{\pi an} - e^{-\pi an}} = -\frac{1}{4\pi a} - \frac{1}{a^2} \sum_{n=1}^{\infty} (-1)^n \frac{n}{e^{n\pi/a} - e^{-n\pi/a}} \quad (a \text{ real and } \neq 0).$$

7. Prove the formula

$$\prod_{n=1}^{\infty} \left(1 + \frac{z(1-z)}{n(n+1)}\right) = \frac{\sin \pi z}{\pi z(1-z)}.$$

8. Let  $a_1, \dots, a_k, b_1, \dots, b_k$  be  $2k$  complex numbers distinct from the integers  $\geq 0$ . Show that for the infinite product

$$\prod_{n=1}^{\infty} \frac{(n - a_1)(n - a_2) \dots (n - a_k)}{(n - b_1)(n - b_2) \dots (n - b_k)}$$

to be absolutely convergent, it is necessary and sufficient that

$$a_1 + \dots + a_k = b_1 + \dots + b_k.$$

When this is the case, show that the product is equal to

$$\prod_{j=1}^k \frac{\Gamma(1 - b_j)}{\Gamma(1 - a_j)}$$

9. Let  $a, b$  be two complex numbers such that  $\Re a > 0, \Re b > 0, \Re(a + b) > 1$ . Show that

$$\int_{-\infty}^{+\infty} \frac{dt}{(1 + it)^a (1 - it)^b} = \frac{\pi}{2^{a+b-2}} \frac{\Gamma(a + b - 1)}{\Gamma(a)\Gamma(b)}.$$

10. Let  $f$  be a function analytic in a strip  $\alpha < \Re z < \beta$ , and let  $m, n$  be two integers such that  $\alpha < m < n < \beta$ . Putting  $z = s + it$  ( $s, t$  real) write

$$q(s, t) = (1/2i)(f(s + it) - f(s - it))$$

(a) Suppose that

$$\lim_{t=\pm\infty} e^{-2\pi|t|} f(s + it) = 0$$

uniformly for  $\alpha < s < \beta$ . Show that

$$f(m) + f(m+1) + \cdots + f(n)$$

$$= \int_m^n f(s) ds + \frac{1}{2}(f(m) + f(n)) + 2 \int_0^{+\infty} \frac{q(n, t) - q(m, t)}{e^{2\pi t} - 1} dt.$$

(Apply the residue theorem to the function  $\pi f(z) \cot \pi z$ , integrating along a rectangle with vertices  $m \pm iN$ ,  $n \pm iN$  and avoiding the points  $m$  and  $n$  by small semicircles; then let  $N$  tend to  $+\infty$ .)

(b) Let  $k$  be an integer  $> 0$ , and suppose that

$$\lim_{t=\pm\infty} e^{-2(k+1)\pi|t|} f(s + it) = 0$$

uniformly for  $\alpha < s < \beta$ ; suppose moreover that  $\alpha < 0$ . Show that

$$\begin{aligned} f(0) + f(1) + \cdots + f(n-1) &= \frac{1}{2}(f(0) - f(n)) + \int_0^n f(s) ds \\ &+ 2 \sum_{h=1}^k \left( \int_0^n f(s) \cos 2\pi hs ds \right) + 2 \int_0^{+\infty} \frac{(q(n, t) - q(0, t)) e^{-2k\pi t}}{e^{2\pi t} - 1} dt \end{aligned}$$

(same method, writing

$$\frac{e^{-2k\pi iz}}{e^{2\pi iz} - 1} = \frac{1}{e^{2\pi iz} - 1} - e^{-2\pi iz} - e^{-4\pi iz} - \cdots - e^{-2k\pi iz}).$$

11. Deduce from problem 10(a) that

$$\log n! = (n + \frac{1}{2}) \log n - n + 1 - 2 \int_0^{+\infty} \frac{\arctan t}{e^{2\pi t} - 1} dt + 2 \int_0^{+\infty} \frac{\arctan(t/n)}{e^{2\pi t} - 1} dt$$

With the help of Stirling's formula deduce that

$$\int_0^{+\infty} \frac{\arctan t}{e^{2\pi t} - 1} dt = \frac{1 - \log \sqrt{2\pi}}{2}.$$

For Euler's constant  $\gamma$ , prove similarly the formula

$$\gamma = \frac{1}{2} + 2 \int_0^{+\infty} \frac{t}{1 + t^2} \frac{dt}{e^{2\pi t} - 1}.$$

12. Let  $p$  and  $n$  be two relatively prime integers  $> 1$ .

(a) Show that if  $p$  and  $n$  are odd

$$\sum_{k=0}^{n-1} e^{(p\pi i/n)k^2} = 0$$

(replace  $k$  by  $n - k$ ).

(b) Suppose that  $p$  and  $n$  are not both odd. Using problem 10(b) show that

$$\sum_{k=0}^{n-1} e^{(p\pi i/n)k^2} = \int_0^n F_0(s) ds + 2 \sum_{h=1}^{p-1} \int_0^n F_h(s) ds + i \int_0^{+\infty} e^{-(p\pi i/n)t^2} \left( \frac{e^{2\pi t} - e^{-(4p-2)\pi t}}{e^{2\pi t} - 1} \right) dt$$

where  $F_h(z) = e^{(p\pi i/n)z^2} \cos 2h\pi z$ .

(c) With the help of Cauchy's theorem show that

$$\int_n^{+\infty} F_h(s) ds = i \int_0^{+\infty} F_h(n + it) dt = \frac{i}{2} \int_0^{+\infty} e^{-(p\pi i/n)t^2 - 2p\pi t} (e^{2h\pi t} + e^{-2h\pi t}) dt$$

Using problem 11(b) of Chap. VIII, deduce that

$$\begin{aligned} \int_0^n F_0(s) ds + 2 \sum_{h=1}^{p-1} \int_0^n F_h(s) ds \\ = e^{\pi i/4} \sqrt{\frac{n}{p}} \left( \frac{1}{2} + \sum_{h=1}^{p-1} e^{-(n\pi i/p)h^2} \right) + i \int_0^{+\infty} e^{-(p\pi i/n)t^2} \frac{(e^{-(4p-2)\pi t} - 1)}{e^{2\pi t} - 1} dt. \end{aligned}$$

(d) Conclude that

$$\sum_{k=0}^{n-1} e^{(p\pi i/n)k^2} = e^{\pi i/4} \sqrt{\frac{n}{p}} \sum_{h=0}^{p-1} e^{-(n\pi i/p)h^2}$$

In particular, for  $p = 2$  and  $n$  odd, we have the expression for *Gauss sums*

$$\sum_{h=0}^{n-1} e^{(2\pi i/n)h^2} = \frac{1-i}{i-1} n \sqrt{n}.$$

13. Using the expression for Euler's constant  $\gamma$  given in Chap. IV, problem 13, and the formula  $1/(z+m) = \int_0^{+\infty} e^{-(z+m)t} dt$ , deduce from the formula (4.5.2) the expression

$$\frac{\Gamma'(z)}{\Gamma(z)} = \int_0^{+\infty} \left( \frac{e^{-t}}{t} - \frac{e^{-zt}}{1-e^{-t}} \right) dt$$

valid for  $\Re z > 0$ .

14. Deduce from the Stirling development that, for  $z = s + it$ , as  $t$  tends to  $+\infty$

$$\Gamma(s + it) = e^{i(\pi/2)(s-(1/2))} e^{-(\pi/2)t} t^{s-(1/2)} e^{it(\log t - 1)} (1 + u(s, t))$$

where  $u(s, t)$  tends to 0 with  $1/t$ , uniformly for  $s$  varying in every bounded interval of  $\mathbf{R}$ .

15. Let  $t$  be a fixed real number  $\neq 0$ ; show that as  $s$  tends to  $+\infty$

$$|\Gamma(-s + it)| \sim \sqrt{\frac{\pi}{2}} s^{-s-(1/2)} e^s (\sinh^2 \pi t + \sin^2 \pi s)^{-1/2}$$

(use (4.6.1)).

16. Starting from the integral expression for Euler's constant

$$\gamma = \int_0^{+\infty} \left( \frac{1}{1+t} - e^{-t} \right) \frac{dt}{t}$$

(V, problem 13), show, by using Cauchy's theorem, that

$$\gamma = \int_0^{+\infty} \left( \frac{1}{1+t^2} - \cos t \right) \frac{dt}{t} = \int_0^1 \frac{1 - \cos t}{t} dt - \int_1^{+\infty} \frac{\cos t}{t} dt.$$

17. Let  $b$  be a number  $> 0$ ; designate by  $R_n$  the rectangle defined by

$$-n - \frac{1}{2} < \Re z < -n + \frac{1}{2}, \quad |\Im z| < b.$$

For every complex number  $w$ , show that, for  $n$  sufficiently large, the equation  $1/\Gamma(z) = w$  has exactly one zero in  $R_n$  (use Rouché's theorem and (4.6.1)).

18. Put

$$\frac{1}{\Gamma(1-z)} = \sum_{n=0}^{\infty} \frac{c_n}{n!} z^n$$

the power series converging in the whole of  $\mathbf{C}$ .

(a) By using Hankel's integral (4.8.1) and the change of variable  $u = e^v$ , show that

$$(*) \quad c_n = \mathcal{J} \int_{\gamma} v^n \exp(e^v) dv$$

where  $\gamma$  is the juxtaposition of the two paths

$$\begin{aligned} \gamma_1: t &\rightarrow \pi + it, & 0 \leq t \leq \beta \\ \gamma_2: t &\rightarrow t + i\beta, & \pi \leq t < +\infty. \end{aligned}$$

(b) Application of the method of steepest descent to the integral which occurs in (\*) leads to a consideration of the roots of the equation

$$(**) \quad ze^z = -n \quad (n \text{ integer } \geq 1).$$

By studying the real and imaginary parts of  $ze^z$  for  $z = x + iy$  such that  $0 < y < \pi$ , show that in the strip  $0 < y < \pi$  the equation (\*\*) has exactly one root  $\rho(n) = \alpha(n) + i\beta(n)$ . By majorizing  $|ze^z|$ , show that

$$\log n - \log \log n < \alpha(n) < \log n$$

as soon as  $n$  is sufficiently large. Putting  $z = (1+w) \log(-n)$  in (\*\*) (with the convention of (7.6) for the logarithms of negative real numbers), show that the Lagrange inversion formula can be applied to the equation giving  $w$ , and deduce the asymptotic development

$$\begin{aligned} \alpha(n) &= \log n - \log \log n + \frac{\log \log n}{\log n} + O\left(\left(\frac{\log \log n}{\log n}\right)^2\right) \\ \beta(n) &= \pi i \left(1 - \frac{1}{\log n} - \frac{\log \log n}{(\log n)^2} + O\left(\frac{1}{(\log n)^2}\right)\right). \end{aligned}$$

(c) To apply the method of steepest descent, choose  $\beta = \beta(n)$ , so that the path  $\gamma$  passes through  $\rho(n)$  as soon as  $n$  is sufficiently large. To evaluate the integral along  $\gamma_2$ , partition the interval of integration  $\pi \leq t < +\infty$  at the points

$$\alpha(n) \pm n^{-2/5},$$

and use evaluations of the derivatives of  $f(z) = e^z + n \log z$  on  $\gamma_2$ , up to the third derivative. Conclude from this the principal part of  $c_n$

$$c_n \sim \sqrt{\frac{2 \log n}{\pi n}} \mathcal{J}(\rho(n)^n e^{-n/\rho(n)}).$$

19. Let  $x_n$  be the root of the equation  $\Gamma'(x) = 0$  belonging to the interval  $]-n, -n+1[$ . Show that

$$x_n = -n + \frac{1}{\log n} + O\left(\frac{1}{(\log n)^2}\right)$$

(use formulae (4.5.2) and (4.6.1)). Deduce that

$$\Gamma(x_n) \sim \frac{(-1)^n}{\sqrt{2\pi}} n^{-n-(1/2)} e^{n+1} \log n.$$

20. Let  $g(z)$  be a function analytic in a half-plane  $\Re z > c_0$ . Suppose further that there exists a number  $\alpha$  satisfying  $0 < \alpha < \pi$ , for which the following condition is satisfied: for each  $c > c_0$  and each  $\varepsilon > 0$ , there exists  $t_0 > 0$  such that, for  $t \geq t_0$  and  $-\pi/2 \leq \theta \leq \pi/2$

$$|g(c + t e^{i\theta})| < e^{(\alpha + \varepsilon)t}.$$

(a) Let  $m$  be an integer  $> c_0$  (positive or negative), and let  $c$  be real such that  $m - 1 < c < m$ . Show that there exists a number  $r_0 > 0$  such that, for  $x$  real and satisfying  $-r_0 < x \leq 0$ ,

$$(*) \quad \sum_{n=m}^{\infty} g(n)x^n = - \int_{c-i\infty}^{c+i\infty} \frac{g(x)x^z dz}{e^{2\pi iz} - 1}$$

(with the notations of Chap. VII, problem 20).

(Use the residue theorem applied to a loop which is a semicircle of centre the point  $c$  and of radius  $n + \frac{1}{2} - c$  ( $n$  tending to  $+\infty$ ).)

(b) Show that the second member of  $(*)$  is a function of the complex variable  $x = re^{i\theta}$ , which is analytic in the open set  $S$  defined by

$$r > 0, \quad \alpha < \theta < 2\pi - \alpha.$$

The power series  $f(x) = \sum_{n=0}^{\infty} g(n)x^n$ , which has radius of convergence  $R \geq e^{-\alpha}$ , can be continued to a function  $F(x)$  analytic in the open union of the disc of convergence of  $f$  and of  $S$ . If  $-(k+1) < c_0 < -k$ , where  $k$  is an integer  $> 0$ , the asymptotic development

$$F(x) = -\frac{g(-1)}{x} - \frac{g(-2)}{x^2} - \dots - \frac{g(-k)}{x^k} + o(|x|^{c_0})$$

is valid in every "sector"  $\alpha + \varepsilon \leq \theta \leq 2\pi - \alpha - \varepsilon$  ( $\varepsilon > 0$  arbitrary).

21. Consider the entire function

$$f(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(1 + \alpha n)}$$

( $0 < \alpha < 2$ ). Deduce from problem 20 that, for every  $k > 0$ , there is an asymptotic development

$$f(z) = -\frac{1}{z\Gamma(1-\alpha)} - \frac{1}{z^2\Gamma(1-2\alpha)} - \dots - \frac{1}{z^k\Gamma(1-k\alpha)} + O(|z|^{-c})$$

with  $k < c < k+1$ , valid in every sector  $(\pi\alpha/2) + \varepsilon \leq \theta \leq 2\pi - (\pi\alpha/2) - \varepsilon$ .

22. Let  $\beta$  be a real number  $> 2$ . Consider the entire function

$$E_{\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{(\log(n+\beta))^n}.$$

(a) Show that for each  $\varepsilon > 0$ , there is an asymptotic development

$$E_{\beta}(z) = -\frac{\log(\beta-1)}{z} + o\left(\frac{1}{z}\right)$$

valid in the sector  $\varepsilon \leq \theta \leq 2\pi - \varepsilon$ .

(b) If  $\gamma > \beta > 2$ , show that the entire function

$$f(z) = \exp(-E_\beta(z)) - \exp(-E_\gamma(z))$$

which is not identically zero, is such that, for every  $\theta$

$$\lim_{r \rightarrow +\infty} f(re^{i\theta}) = 0.$$

Why does this result not contradict Liouville's theorem?

23. Let  $f, g$  be two functions  $> 0$ , indefinitely differentiable for  $x > 0$  and put  $h(x) = f(x)e^{tg(x)}$ .

(a) Suppose that in the neighbourhood of  $+\infty$ ,  $h(x) = o(1)$ , the integral

$$\int_1^{+\infty} f(t)e^{tg(t)} dt$$

is convergent and the integral  $\int_1^{+\infty} |h'(t)| dt$  is convergent. Show that the series  $\sum_{n=1}^{\infty} f(n)e^{tg(n)}$  is convergent (use the Euler-Maclaurin summation formula).

(b) Suppose that  $f$  and  $g$  satisfy one of the three hypotheses of Chap. III, problem 16, and furthermore that  $\int_1^x |h''(t)| dt = o(|h(x)|)$ . Show that then

$$\sum_{k=1}^n f(k)e^{tg(k)} \sim \int_1^n f(t)e^{tg(t)} dt$$

(same method).

24. Show with the help of problem 23 that the series

$$\sum_{n=2}^{\infty} \frac{n^{-a} - 1}{\log \log n}$$

is convergent for every real number  $a \neq 0$ , without being absolutely convergent. Deduce that the power series

$$\sum_{n=2}^{\infty} \frac{n^{-a} - 1}{\log \log n} z^n$$

which has for disc of convergence  $|z| < 1$ , is *uniformly convergent* for  $|z| \leq 1$ , but *not absolutely convergent* for  $|z| = 1$  (use Abel's partial summation).

25. Show that the integral  $\int_1^{+\infty} e^{t\Gamma(t)} dt$  is convergent, but that for every  $\theta$  of the form  $2\pi p/q$  ( $p, q$  integers), the partial sums  $\sum_{k=1}^n e^{t\theta\Gamma(k)}$  tend to  $+\infty$  with  $n$ .

26. Show that in the neighbourhood of  $+\infty$

$$\frac{1}{\sin \frac{\pi}{n}} + \frac{1}{\sin \frac{2\pi}{n}} + \cdots + \frac{1}{\sin \frac{(n-1)\pi}{n}} = \frac{2n}{\pi} \log n + \frac{2n}{\pi} \left( \gamma - \log \frac{\pi}{2} \right) + o(1)$$

(apply the Euler-Maclaurin formula to the sum

$$\sum_{k=1}^{n-1} \left( \frac{1}{\sin \frac{k\pi}{n}} - \frac{1}{k\pi} - \frac{1}{(n-k)\pi} \right)).$$

27. Show that, for  $x > 0$

$$\int_x^{x+1} \log \Gamma(t) dt = x(\log x - 1) + \frac{1}{2} \log 2$$

(*Raabe's integral*). One can show that the first member is of the form  $x(\log x - 1) + C$  where  $C$  is a constant, by using the functional equation of  $\Gamma$ , then, either apply Stirling's formula, or pass to the limit in the Legendre-Gauss formula to find  $C$ .

28. The Fourier coefficients  $c_m$  are still defined for a function  $f$  piecewise-continuous in the open interval  $]0, 2\pi[$ , and such that the improper integral  $\int_0^{2\pi} |f(t)| dt$  is convergent. Suppose in particular that  $f(x)$  is the sum of a trigonometric series  $a_0 + \sum_{m=1}^{\infty} (a_m \cos mx + b_m \sin mx)$  which converges *uniformly* to  $f(x)$  in every interval  $[\alpha, 2\pi - \alpha]$  ( $\alpha > 0$ ). If further the integral  $\int_0^{2\pi} |f(t)| dt$  is convergent, show that the preceding series is identical to the Fourier series of  $f(x)$ .

Show in particular that, for  $0 < x < 2\pi$

$$\log (2 \sin \tfrac{1}{2}x) = -\cos x - \frac{\cos 2x}{2} - \dots - \frac{\cos mx}{m} - \dots$$

the series of the second member being the Fourier series of its sum (use problem 17 of Chap. VI applied to the Taylor series of  $\log (1 - z)$  at the point 0).

29. Let  $F$  be a function analytic in an open subset of  $\mathbf{C}$  containing the closed strip  $0 \leq \Re z \leq 1$ . Suppose that

$$\lim_{t \rightarrow \pm \infty} e^{-2\pi|t|} F(s + it) = 0$$

*uniformly* for  $0 \leq s \leq 1$ . Consider the Fourier series of  $F$  in the form of multiples of  $2\pi x$

$$F(x) = A_0 + \sum_{m=1}^{\infty} (A_m \cos 2\pi mx + B_m \sin 2\pi mx)$$

an equality which is valid for  $0 < x < 1$ . If

$$p(s, t) = \tfrac{1}{2}(F(s + it) + F(s - it)), \quad q(s, t) = \tfrac{1}{2i}(F(s + it) - F(s - it))$$

show that, for  $m \geq 1$

$$A_m = 2 \int_0^{+\infty} e^{-2\pi mt} (q(1, t) - q(0, t)) dt$$

$$B_m = -2 \int_0^{+\infty} e^{-2\pi mt} (p(1, t) - p(0, t)) dt$$

(use the same method as in problem 10).

In particular, we have the Fourier series of  $\log \Gamma(x)$ :

$$\log \Gamma(x) = \tfrac{1}{2} \log 2\pi + \sum_{m=1}^{\infty} \left( \frac{\cos 2\pi mx}{2m} + (\gamma + \log 2m\pi) \frac{\sin 2\pi mx}{\pi m} \right)$$

(use Raabe's integral and the expression for  $\Gamma'(x)$  obtained by differentiating the Eulerian integral of the second kind).

30. Let  $f$  be a function piecewise-continuous in  $[0, 2\pi]$ , and let  $c_m$  ( $m \in \mathbf{Z}$ ) be its Fourier coefficients. Show that

$$P_N(x) = \frac{1}{2\pi} \int_0^{2\pi} \frac{\sin((N + \frac{1}{2})(x - t))}{\sin((x - t)/2)} f(t) dt.$$

(a) A necessary and sufficient condition that the Fourier series of  $f$  converge at a point  $x_0$  where  $f$  is continuous to the sum  $f(x_0)$  is that

$$\lim_{N \rightarrow \infty} \frac{1}{2\pi} \int_0^{2\pi} \frac{\sin Nu}{u} (f(x_0 + u) - f(x_0)) du = 0.$$

If this is the case, deduce that, for every piecewise-continuous function  $g$  equal to  $f$  in a neighbourhood of  $x_0$ , the Fourier series of  $g$  also converges at the point  $x_0$  to  $g(x_0)$ .

(b) Using (a), give a new proof of the fact that if  $f$  possesses a continuous derivative in a neighbourhood of  $x_0$ , the Fourier series of  $f$  converges to  $f(x_0)$  at the point  $x_0$  (integrate by parts).

(c) For every integer  $p > 1$ , denote by  $g_p(x)$  the function equal to

$$2^{-p/2} \sin \frac{2^{2p} + 1}{2} x$$

for

$$\frac{2\pi}{2^{2p} + 1} \leq x \leq \frac{2^p \pi}{2^{2p} + 1}$$

and to 0 elsewhere in  $[0, 2\pi]$ . Show that one can determine by induction a strictly increasing sequence of integers  $(p_k)$  such that the following condition is satisfied: if

$$h_n(x) = g_{p_1}(x) + g_{p_2}(x) + \cdots + g_{p_n}(x)$$

then

$$\frac{2^{2p_n+1}}{2^{2p_n+1} + 1} < \frac{2}{2^{2p_n+1}} \quad \text{and} \quad \left| \int_0^{2\pi} \frac{\sin Nu}{u} h_n(u) du \right| \leq 1$$

for  $N > 2^{2p_n+1-1}$ . Deduce that the function  $h(x) = \lim_{n \rightarrow \infty} h_n(x)$  is continuous in  $[0, 2\pi]$ , but that its Fourier series does not converge at the point  $x = 0$ .

(d) If  $f$  is continuous and *odd* its Fourier series converges at the point 0. Use this fact to define a continuous function  $f$  whose Fourier series converges at every point, although the sums  $|P_N(x)|$  are not uniformly bounded in  $[0, 2\pi]$ . (Instead of the functions  $g_p(x)$ , use the functions

$$g_p\left(x - \frac{2^{p+2}\pi}{n^2}\right)$$

to define the function in  $[0, \pi]$ , then take  $f(x) = -f(-x)$  in  $[-\pi, 0]$ .)

31. The function  $f(x) = \sin(1/x)$  is continuous in  $]0, 2\pi]$ , but not continuous at the point 0. Show that its Fourier series is defined and converges to 0 at the point 0 (use problem 14 of Chap. IV).

32. Show that the function  $f(x) = \sum_{n=1}^{\infty} \frac{\sin nx}{n^2}$  is continuous and has a Fourier series such that  $\sum_{m=-\infty}^{+\infty} |mc_m|^2$  is convergent, but is not differentiable at the point  $x = 0$  (cf. problem 28).

33. Let  $f$  be a function continuously differentiable and periodic of period  $2\pi$ . Show that if  $\int_0^{2\pi} f(t) dt = 0$

$$\int_0^{2\pi} |f(t)|^2 dt < \int_0^{2\pi} |f'(t)|^2 dt$$

unless  $f(x) = a \cos x + b \sin x$ .

34. (a) Prove the identities

$$\cos t + \cos 2t + \cdots + \cos nt = \frac{\sin \frac{nt}{2} \cos \frac{(n+1)t}{2}}{\sin \frac{t}{2}} = \frac{1}{2} \sin (n+1)t \cot \frac{t}{2} - \cos^2 \frac{(n+1)t}{2}.$$

(b) Show that in the interval  $0 \leq t \leq \pi$ , the trigonometric polynomial

$$A(n, t) = \sin t + \frac{1}{2} \sin 2t + \cdots + \frac{1}{n} \sin nt$$

has a relative maximum at each of the points  $\pi/(n+1), 3\pi/(n+1), \dots, (2q-1)\pi/(n+1)$  where  $q = [(n+1)/2]$  and a relative minimum at each of the points  $2\pi/n, 4\pi/n, \dots, 2(q-1)\pi/n$ .

(c) Using (a) show that

$$A\left(n, (2\nu+1) \frac{\pi}{n+1}\right) - A\left(n, (2\nu-1) \frac{\pi}{n+1}\right) \leq \frac{1}{2} \int_{\frac{(2\nu-1)\pi}{n+1}}^{\frac{(2\nu+1)\pi}{n+1}} \sin (n+1)t \cot \frac{t}{2} dt$$

and deduce that

$$A\left(n, (2\nu+1) \frac{\pi}{n+1}\right) < A\left(n, (2\nu-1) \frac{\pi}{n+1}\right) \quad \text{for } 1 \leq \nu \leq q.$$

(d) Deduce from (b) and (c) that

$$A\left(n, \frac{\pi}{n+1}\right) > A\left(n-1, \frac{\pi}{n}\right),$$

and show that the increasing sequence of the  $A(n, \pi/(n+1))$  has for a limit the number

$$\int_0^\pi \frac{\sin t}{t} dt = 1.8519 \dots$$

(consider  $A(n, \pi/(n+1))$  as a Riemann sum). This number is greater than the maximum of the function

$$\frac{\pi-t}{2} = \sum_{n=1}^{\infty} \frac{\sin nt}{n}$$

in the interval  $]0, 2\pi[$  (*Gibbs phenomenon*).

# Conformal mapping

## 1. Characterization of conformal mappings

(1.1) Let  $D$  be an open set in  $\mathbf{R}^2$ . Recall that a *continuously differentiable* mapping of  $D$  in  $\mathbf{R}^2$  is a mapping

$$F: (x, y) \rightarrow (P(x, y), Q(x, y))$$

such that  $P$  and  $Q$  have continuous partial derivatives. Let us consider such a mapping and let  $z_0 = (x_0, y_0)$  be a point of  $D$ . Corresponding to each path  $\gamma: t \rightarrow (u(t), v(t))$  defined in an interval  $I = [0, a]$  of  $\mathbf{R}$  and such that  $\gamma(0) = z_0$ , there is a path  $t \rightarrow F(\gamma(t)) = (P(u(t), v(t)), Q(u(t), v(t)))$  defined in  $I$  and passing through  $F(z_0)$ . Suppose  $\gamma$  differentiable at the point 0 and such that  $(u'(0), v'(0)) \neq (0, 0)$ , so that  $\gamma$  has a *tangent* at the point  $z_0$  of directional parameters  $(u'(0), v'(0))$ . The path  $F \circ \gamma$  thus has a tangent at the point  $F(z_0)$  of directional parameters

$$(1.1.1) \quad \begin{aligned} &P'_x(x_0, y_0)u'(0) + P'_y(x_0, y_0)v'(0), \\ &Q'_x(x_0, y_0)u'(0) + Q'_y(x_0, y_0)v'(0) \end{aligned}$$

provided these two numbers are not simultaneously zero. We are thus led to associate with  $F$  at the point  $z_0$  the *linear mapping*

$$J(F)(z_0): (\xi, \eta) \rightarrow J(F)(z_0) \cdot (\xi, \eta)$$

of  $\mathbf{R}^2$  into itself, where  $J(F)(z_0)$  is the matrix

$$(1.1.2) \quad \begin{pmatrix} P'_x(x_0, y_0) & P'_y(x_0, y_0) \\ Q'_x(x_0, y_0) & Q'_y(x_0, y_0) \end{pmatrix}$$

called the *Jacobian matrix* of  $F$  at the point  $z_0$ . This mapping is also called the *linear tangent mapping* to  $F$  at the point  $z_0$ . If the determinant of  $J(F)(z_0)$  is not zero, we say that the linear tangent mapping to  $F$  is *regular* at the point  $z_0$ ; then if the numbers  $u'(0), v'(0)$  are not both zero, neither are both the numbers of (1.1.1). We can thus say that the linear tangent mapping to  $F$  at the point  $z_0$  transforms the *tangent vectors* at  $z_0$  to the paths  $\gamma$  passing through  $z_0$  into *tangent vectors* at  $F(z_0)$  to the transformed paths  $F \circ \gamma$ . Thus if two paths  $\gamma_1, \gamma_2$  have the same tangent vector at  $z_0$ , the paths  $F \circ \gamma_1, F \circ \gamma_2$  both have for tangent vector at  $F(z_0)$  the image of this vector by  $J(F)(z_0)$ .

(1.2) A continuously differentiable mapping  $F$  is said to be *conformal* at a point  $z_0$  if the linear tangent mapping to  $F$  at this point is a (regular) *direct similarity*. It has been seen in Algebra that these mappings can be characterized by the property of the *preservation of the (oriented) angle of two lines*, or equivalently of preserving the orientation and of transforming two orthogonal lines into orthogonal lines. If we agree to say that the (oriented) *angle* of two paths  $\gamma_1, \gamma_2$  passing through  $z_0$  is (by definition) the (oriented) *angle of the tangents* to  $\gamma_1$  and  $\gamma_2$  at the point  $z_0$ , then the property that  $F$  is conformal at the point signifies that it *preserves the (oriented) angles of paths passing through  $z_0$*  (Fig. 61).



FIGURE 61

In particular, if two paths are *orthogonal* (in other words, if their angle is  $\pm\pi/2$ ), their images are *orthogonal*.

It has been seen in Algebra that the matrices of direct similarities are of the form

$$\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$$

Thus for  $F$  to be conformal at the point  $z_0$ , it is necessary and sufficient, by virtue of the expression (1.1.2) for the Jacobian matrix, that at the point  $z_0$

$$(1.2.1) \quad \frac{\partial P}{\partial x}(x_0, y_0) = \frac{\partial Q}{\partial y}(x_0, y_0), \quad \frac{\partial P}{\partial y}(x_0, y_0) = -\frac{\partial Q}{\partial x}(x_0, y_0)$$

and that  $\frac{\partial P}{\partial x}(x_0, y_0)$  and  $\frac{\partial P}{\partial y}(x_0, y_0)$  be not both zero.

(1.3) *In order that a continuously differentiable mapping  $F$  of an open set  $D \subset \mathbf{C}$  be conformal at every point of  $D$ , it is necessary and sufficient that  $F$  be analytic in  $D$  and that  $F'(z) \neq 0$  in  $D$ .*

The conditions (1.2.1) are just the Cauchy conditions (VII, 9.3.5) and the proposition thus follows from (VII, 9.4) and from the preceding, taking into account the expression

$$F'(z) = \frac{\partial P}{\partial x}(x, y) + i \frac{\partial Q}{\partial x}(x, y)$$

for the derivative of an analytic function.

Observe that when  $F$  is analytic and  $F'(z_0) \neq 0$ , the linear tangent mapping at the point  $z_0$  is simply expressed as the *complex homothety*

$$(1.3.1) \quad \zeta \rightarrow F'(z_0) \cdot \zeta$$

of  $\mathbf{C}$  onto itself.

(1.4) Let  $F$  be a non-constant analytic function in an open set  $D \subset \mathbf{C}$ ; it has been seen (VIII, 8.1) that the image  $F(D)$  is an *open* set in  $\mathbf{C}$ , and that if  $F$  is *injective*,  $F'(z) \neq 0$  in

D. Thus in this last case,  $F$  is a conformal mapping in  $D$ ; we then say that  $F$  is a *conformal mapping of  $D$  onto  $F(D)$* . It is clear that the inverse function  $w \rightarrow F^{-1}(w)$  is analytic in  $F(D)$  and is a conformal mapping of  $F(D)$  onto  $D$  (VIII, 8.1).

(1.5) When  $F'(z_0) = 0$  at a point  $z_0 \in D$ , and when  $F$  is not constant, let  $k$  be the smallest integer  $> 0$  such that  $F^{(k)}(z_0) \neq 0$ ; it follows immediately that if  $\alpha \in [0, 2\pi]$  is a measure of the oriented angle of two paths  $\gamma_1, \gamma_2$  passing through  $z_0$ , the oriented angle of the paths  $F \circ \gamma_1, F \circ \gamma_2$  at the point  $F(z_0)$  has a measure equal to  $k\alpha$ .

## 2. Problems of conformal mapping

(2.1) The *direct* problem of conformal mapping, formulated in a rather vague way, is the following: given a non-constant function  $F$  analytic in an open set  $D \subset \mathbf{C}$ , is  $F$  *injective* in  $D$ , and if so, can we determine the image  $F(D)$ ? We recall that the condition  $F'(z) \neq 0$  is *necessary* for  $F$  to be injective, but it is *not sufficient*, as is shown by the example  $F(z) = z^2$  in the sector  $1 \leq r \leq 2, -3\pi/4 \leq \theta \leq 3\pi/4$ . In simple cases, of which some examples are given below, we can determine explicitly the curves  $x \rightarrow F(x + iy_0)$  and  $y \rightarrow F(x_0 + iy)$ , the images of the portions of lines parallel to the axes which are contained in  $D$ , or the curves  $r \rightarrow F(re^{i\theta_0})$  and  $\theta \rightarrow F(r_0 e^{i\theta})$ , the images of the portions of half-lines of initial point 0 and of the portions of circles of centre 0 which are contained in  $D$ . In this way we verify whether or not  $F$  is injective.

(2.2) Another important case is that where  $F$  is analytic in an open set  $D_0$  containing  $D$  and its boundary  $L$ , and where this boundary is the set of points of a *loop*  $\gamma$  such that  $D$  is the set of all points  $z$  of the complement of  $L$  in  $\mathbf{C}$ , for which  $j(z; \gamma) = 1$ . Then, if  $\Gamma$  is the composed loop  $t \rightarrow F(\gamma(t))$ , a necessary and sufficient condition for  $F$  to be injective in  $D$  is that  $j(w; \Gamma) = 0$  or  $j(w; \Gamma) = 1$  for every point  $w \notin F(L)$ , and  $F(D)$  is the set of points  $w$  such that  $j(w; \Gamma) = 1$  (VIII, 6.2.2).

Finally it may happen that the boundary  $L$  of  $D$  is not connected, but that  $D$  is a union of a sequence  $(D_n)$  of open sets of which each is of the preceding type. Having “guessed” what  $F(D)$  must be, it is even sufficient to show that for every point  $w$  of this set  $j(w; \Gamma_n) = 1$  as soon as  $n$  is sufficiently large ( $\Gamma_n$  being the composed loop corresponding to  $D_n$ ). For example, for the function  $F(z) = e^z + z$ , the reasoning of (VIII, 6.3) shows that  $F$  is injective in the strip  $B: 0 < \Im z < \pi$ , and that  $F(B)$  is the half-plane  $\Im w > 0$  “cut” along the half-line  $\Re w = \pi, \Im w \leq -1$ .

(2.3) The *inverse* problem of conformal mapping consists in being given two open sets  $D, D'$  in the plane  $\mathbf{C}$ , and in trying to determine a function  $F$  analytic in  $D$  and which is a *conformal mapping of  $D$  onto  $D'$* . This is not always possible: for example, if  $D$  is simply connected, then  $D'$  must also be; however this condition is not entirely sufficient, for there is no conformal representation of the whole plane  $\mathbf{C}$  onto a *bounded* open set  $D'$ , as is shown by Liouville's theorem (VII, 8.2). A theorem first stated by Riemann, but which we shall not prove, states that the above problem always has a solution when both  $D$  and  $D'$  are simply connected and distinct from  $\mathbf{C}$ . In particular, for every simply connected open set  $D \neq \mathbf{C}$ , there exists a function  $F$  analytic in the open unit disc  $D_0: |z| < 1$ , which is a conformal mapping of  $D_0$  onto  $D$ . The “approximate” determination of  $F$  is a difficult problem of numerical calculus.

(2.4) The inverse problem of conformal mapping has important applications in all questions of Physics in which a *Newtonian* potential occurs, that is a function  $V$  defined in an open subset of  $\mathbf{C}$  and satisfying in this set the Laplace equation (VII, 9.5.1)

$$(2.4.1) \quad \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0.$$

This occurs in very different physical theories (electro-magnetic theory, gravitational fields, theory of heat, hydrodynamics); actually in these theories it is the Laplace equation in *three* variables

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

which occurs, but in numerous cases one can suppose, at least as a first approximation, that  $V$  does not depend on  $z$ .

A typical problem solved by the theory of conformal mapping is the following: consider for example an open set  $E$  in  $\mathbf{C}$  of points *exterior* to a closed set  $K$ , of boundary  $L$  (Fig. 62). Suppose a function  $F$  analytic in  $E$  is known which is a conformal mapping of  $E$  onto the *exterior*  $|z| > 1$  of the unit disc; suppose further that  $F$  can be continued to a function continuous in  $E \cup L$ ,  $F(L)$  being the circle  $|z| = 1$  (this is not always possible, for example when  $K$  is the union of the closed unit disc and a segment of a ray (Fig. 63)).



FIGURE 62

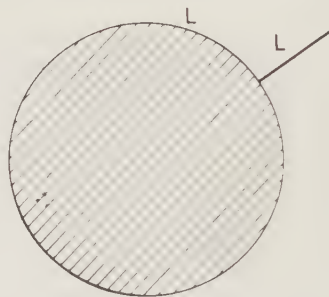


FIGURE 63

Then the function  $V(x, y) = \log |F(x + iy)|$  is a Newtonian potential in  $E$ , which can be continued by continuity into  $E \cup L$ , and is *constant* on  $L$  (which is called an *equipotential line*). The knowledge (at least approximate) of a potential having this property is important in numerous applications.

### 3. Homographic transformation

(3.1) A non-degenerate *homographic function* is a rational function of the form

$$(3.1.1) \quad F(z) = \frac{az + b}{cz + d}$$

where  $a, b, c$  and  $d$  are four complex constants such that

$$(3.1.2) \quad ad - bc \neq 0.$$

If  $c = 0$ , then necessarily  $a \neq 0$  and  $F$  is an affine linear function  $z \rightarrow az + b$  which is not constant. If  $c \neq 0$ ,  $F$  has a simple pole  $-d/c$ . Let us designate by  $D$  the plane  $\mathbf{C}$  when  $c = 0$ , the open set  $\mathbf{C} - \{-d/c\}$  in the contrary case. Then  $F$  is a *bijection* of  $D$  onto  $D' = \mathbf{C}$  when  $c = 0$ , of  $D$  onto  $D' = \mathbf{C} - \{a/c\}$  when  $c \neq 0$ . The inverse bijection is also a non-degenerate *homographic function*:

$$(3.1.3) \quad w \rightarrow \frac{-dw + b}{cw - a}.$$

(3.2) When  $c = 0$ , we can write  $F(z) = a(z + (b/a))$ ; when  $c \neq 0$ , we can write

$$(3.2.1) \quad F(z) = \frac{a}{c} + \frac{bc - ad}{c^2} \cdot \frac{1}{z + (d/c)}$$

and  $F$  can always be obtained by *composing* homographic functions of three particularly simple types:

$$(3.2.2) \quad z \rightarrow z + a$$

$$(3.2.3) \quad z \rightarrow kz \quad (k \neq 0)$$

$$(3.2.4) \quad z \rightarrow \frac{1}{z}$$

These three conformal mappings have a simple geometric interpretation: the transformation (3.2.2) is a *translation* of vector  $a$ ; the transformation (3.2.3) is a *direct similarity of ratio*  $\rho = |k|$  and of angle  $\alpha = \arg k$ ; finally (3.2.4) is composed of the symmetry  $z \rightarrow \bar{z}$  with respect to the real axis and the inversion  $z \rightarrow 1/\bar{z}$  of pole 0 and of power 1 (Fig. 64).

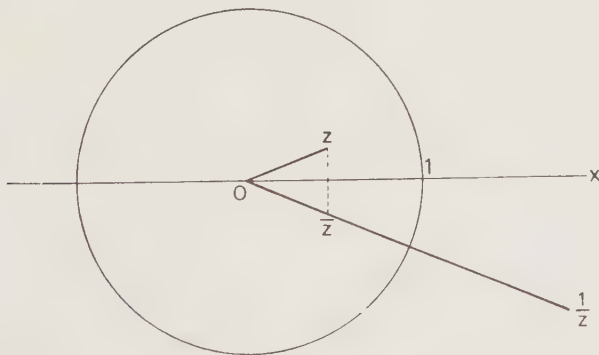


FIGURE 64

From this decomposition of the homographic function it follows immediately that the image of a *circle*  $\Gamma$  under  $F$  is a *circle* if  $\Gamma$  does not contain the point  $-d/c$ , and in the contrary case the image of  $\Gamma - \{-d/c\}$  is a *line*.

It is sufficient to verify this for (3.2.4);  $\Gamma$  is the set of points such that  $z = z_0(1 + \rho e^{i\theta})$  with  $0 \leq \theta \leq 2\pi$  and it follows immediately that if  $\rho \neq 1$

$$\frac{1}{1 + \rho e^{i\theta}} = \frac{1}{1 - \rho^2} - \frac{\rho}{1 - \rho^2} \frac{\rho + e^{i\theta}}{1 + \rho e^{i\theta}}.$$

Since

$$|\rho + e^{i\theta}| = |\rho + e^{-i\theta}| = |e^{-i\theta}(1 + \rho e^{i\theta})| = |1 + \rho e^{i\theta}|,$$

$$\left| \frac{1}{z_0(1 + \rho e^{i\theta})} - \frac{1}{z_0(1 - \rho^2)} \right|$$

is constant. If on the contrary  $\rho = 1$

$$\frac{1}{1 + e^{i\theta}} = \frac{1}{2} - i \tan \frac{\theta}{2}.$$

It is at once deduced from the preceding that if  $D$  is an *open disc* or an *open half-plane* not containing the point  $-d/c$ ,  $F(D)$  is an open disc or an open half-plane. If on the contrary  $-d/c \in D$ ,  $F(D - \{-d/c\})$  is the exterior of a disc. If  $D$  is the exterior of a disc, and if  $-d/c \notin D$ ,  $F(D)$  is an open disc or an open half-plane. If on the contrary  $-d/c \in D$ ,  $F(D - \{-d/c\})$  is of the form  $D' - \{a/c\}$ , where  $D'$  is the exterior of a disc.

In particular:

(3.3) For every complex number  $a$  such that  $\mathcal{I}a > 0$ , the homographic function

$$(3.3.1) \quad z \rightarrow \frac{z - a}{z - \bar{a}}$$

is a *conformal mapping* of the open half-plane  $\mathcal{I}z > 0$  onto the open disc  $|z| < 1$ , mapping the point  $a$  into the point 0.

(3.4) For every complex number  $a$  such that  $|a| < 1$  and every real number  $\alpha$ , the homographic function

$$(3.4.1) \quad z \rightarrow e^{i\alpha} \frac{z - a}{\bar{a}z - 1}$$

is a *conformal mapping* of the disc  $|z| < 1$  onto itself, mapping the point  $a$  into the point 0. It can be shown that there are no other conformal mappings of the unit disc onto itself (problem 8).

#### 4. Examples of conformal mappings

(4.1) Let  $\alpha$  be a real number  $\geq \frac{1}{2}$ , and let  $D$  be the *open angular sector* formed by the points  $z = re^{i\theta}$  such that

$$(4.1.1) \quad r > 0, \quad 0 < \theta < \frac{\pi}{\alpha}.$$

Then the image of an open half-line  $r \rightarrow re^{i\theta}$  contained in  $D$  ( $r$  varying in  $]0, +\infty[$ ,  $\theta$

fixed) under the mapping  $z \rightarrow e^{i\pi\alpha}(-z)^\alpha$  analytic in  $D$  (VIII, 9.6) is the half-line  $r \rightarrow r^\alpha e^{i\alpha\theta}$ . It follows at once that the function  $z \rightarrow e^{i\pi\alpha}(-z)^\alpha$  is a *conformal mapping of the open angular sector  $D$  onto the open half-plane  $\mathcal{I}z > 0$* . By composition with (3.3.1), an explicit solution for  $D$  of Riemann's problem (2.3) is obtained, i.e. a conformal mapping of  $D$  onto the unit disc. For example for  $\alpha = 2$ ,  $z \rightarrow (z^2 - i)/(z^2 + i)$  is a conformal representation of the "right angle" onto the unit disc. For  $\alpha = 1$ ,

$$z \rightarrow \frac{-1 + (-z)^{1/2}}{(-z)^{1/2} + 1}$$

is a conformal mapping of the plane cut along the positive real axis onto the unit disc. The inverse conformal mapping

$$(4.1.2) \quad w \rightarrow -c \left( \frac{w+1}{w-1} \right)^2$$

maps the unit disc onto the plane cut along  $\mathbf{R}_+$ , for every real number  $c > 0$ .

(4.2) Consider the function

$$(4.2.1) \quad F(z) = \frac{1}{2} \left( z + \frac{1}{z} \right)$$

which is meromorphic in  $\mathbf{C}$ , with a simple pole at  $z = 0$ . Since  $F(1/z) = F(z)$ , the image under  $F$  of the exterior of the unit disc is identical to the image of the (open) unit disc less the point 0 ("punctured disc"). If  $z = re^{i\theta}$  ( $r > 0$ ),  $F(z) = u + iv$ , it follows that

$$(4.2.2) \quad u = \frac{1}{2} \left( r + \frac{1}{r} \right) \cos \theta, \quad v = \frac{1}{2} \left( r - \frac{1}{r} \right) \sin \theta.$$

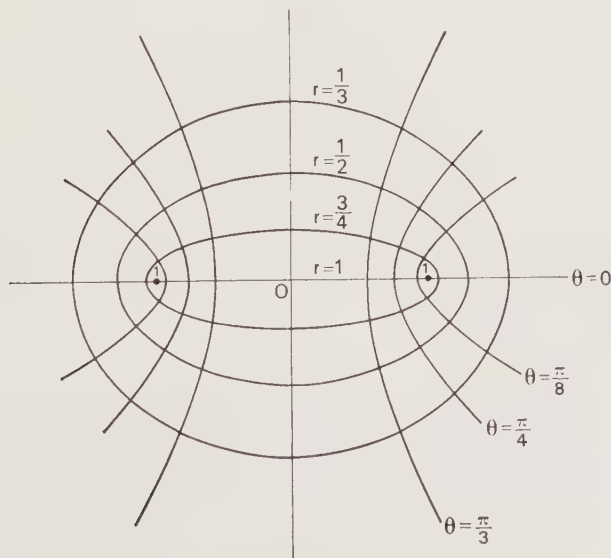


FIGURE 65

Thus the image under  $F$  of the circle  $|z| = r$  is the ellipse of foci  $\pm 1$  and semi-axes  $\frac{1}{2}(r + (1/r))$ ,  $\frac{1}{2}|r - (1/r)|$  for  $r \neq 1$ . For  $r = 1$ , this image degenerates into the segment of endpoints  $\pm 1$ . The image under  $F$  of a half-line  $r \rightarrow re^{i\theta}$  ( $r > 0$ ) is (for  $\theta$  distinct from  $0, \pm\pi/2$  and  $\pi$ ) a branch of a hyperbola of foci  $\pm 1$  (Fig. 65). It is concluded from these remarks that  $F$  is a conformal mapping of the exterior of the unit disc  $|z| > 1$  onto the plane "cut" along the segment of endpoints  $\pm 1$ .

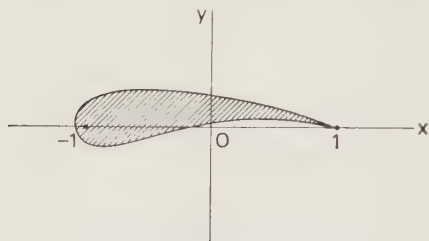


FIGURE 66

$|\Im z| < \pi$ , a half-line  $r \rightarrow re^{i\theta}$  being mapped into the line  $r \rightarrow \log r + i\theta$  parallel to the real axis, and a circle  $\theta \rightarrow re^{i\theta}$  less the point  $-r$  into the open segment  $\theta \rightarrow \log r + i\theta$  ( $-\pi < \theta < \pi$ ) parallel to the imaginary axis. The function  $\log z$  is thus a conformal mapping of an open angular sector  $\theta_1 < \theta < \theta_2$  onto the strip  $\theta_1 < \Im z < \theta_2$ ; the exponential function effects the inverse conformal mapping.

(4.4) The function  $z \rightarrow \sin z = (1/2i)(e^{iz} - e^{-iz})$  can be considered as the composition of the four functions

$$z \rightarrow iz$$

$$z \rightarrow e^z$$

$$z \rightarrow -iz$$

$$z \rightarrow \frac{1}{2} \left( z + \frac{1}{z} \right)$$

which have already been studied. It is easily deduced for example that

$$z \rightarrow \sin z$$

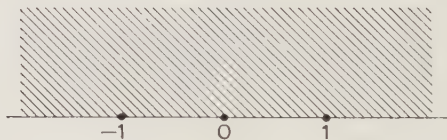
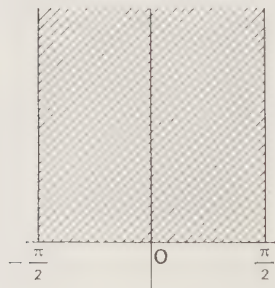


FIGURE 67

is a conformal mapping of the half-strip

$$-\frac{\pi}{2} < \Re z < \frac{\pi}{2}, \quad \Im z > 0$$

onto the half-plane  $\Im z > 0$  (Fig. 67).

## 5. The Schwarz-Christoffel transformation

(5.1) Consider a finite increasing sequence of real numbers

$$a_1 < a_2 < \dots < a_n \quad (n \geq 3)$$

and a sequence  $(\mu_h)_{1 \leq h \leq n}$  of  $n$  real numbers satisfying the conditions

$$(5.1.1) \quad 0 < \mu_h < 1 \quad \text{for all } h, \quad \sum_{h=1}^n \mu_h = 2.$$

Let  $D_0$  be the plane cut along the half-lines

$$\Re z = a_h, \quad \Im z \leq 0 \quad \text{for } 1 \leq h \leq n$$

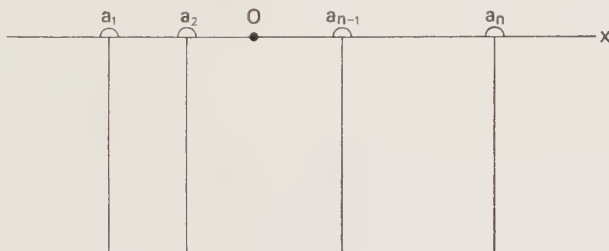


FIGURE 68

(Fig. 68). Changing the notation of (VIII, 9.6), designate by  $(z - a_h)^{\mu_h}$  the function analytic in  $D_0$  and equal to

$$|x - a_h|^{\mu_h} e^{i\mu_h\pi}$$

for  $x$  real and  $x < a_1$ . Because of (5.1.1), the function

$$(5.1.2) \quad f(z) = (z - a_1)^{\mu_1} (z - a_2)^{\mu_2} \dots (z - a_n)^{\mu_n}$$

is thus *real* and  $> 0$  for  $z$  real and  $x < a_1$ . When  $x$  increases from  $-\infty$  to  $+\infty$ , the argument of  $f(x)$  increases by  $-\mu_n\pi$  as  $x$  passes through the point  $a_n$ , so  $f(z)$  is still *real* and  $> 0$  for  $z$  real and  $x > a_n$ .

(5.2) By a translation it may always be supposed that  $a_1 > 0$ . Consider then the function

$$(5.2.1) \quad F(z) = \int_0^z \frac{du}{f(u)} = \int_0^z \frac{du}{(u - a_1)^{\mu_1} (u - a_2)^{\mu_2} \dots (u - a_n)^{\mu_n}}$$

which is evidently analytic in the open half-plane  $D: \Im z > 0$ , and in fact also in  $D_0$ . To see that it is sufficient to show that  $D_0$  is simply connected; now, we have a homotopy  $(t, s) \rightarrow \varphi(\gamma(t), s)$  of every loop  $\gamma$  in  $D_0$  into a loop in  $D$  by taking  $\varphi(x + iy, s) = x + iy + is$  for  $y \geq 0$  and  $\varphi(x + iy, s) = x + i(1 - s)y + is$  for

$y \leq 0$  ( $0 \leq s \leq 1$ ). This being so, under the hypotheses of (5.2.1), the function  $F$  is a conformal mapping of  $D$  onto a connected set  $P$  whose boundary  $L$  is a polygon of  $n$  sides making angles  $(1 - \mu_h)\pi$  ( $1 \leq h \leq n$ ) ("Schwarz-Christoffel transformation").

Observe that the improper integrals

$$\int_{a_{h-1}}^{a_h} \frac{dx}{|f(x)|} \quad (2 \leq h \leq n), \quad \int_{a_n}^{+\infty} \frac{dx}{|f(x)|} \quad \text{and} \quad \int_{-\infty}^{a_1} \frac{dx}{|f(x)|}$$

are all convergent by virtue of (5.1.1). Apply the method of (2.2) to the loops  $\Gamma_v$  whose images are formed of a semicircle of centre 0 and of large radius  $R$ , of semicircles of centres  $a_h$ , and of segments of the real axis (Fig. 69). Cauchy's theorem shows that the

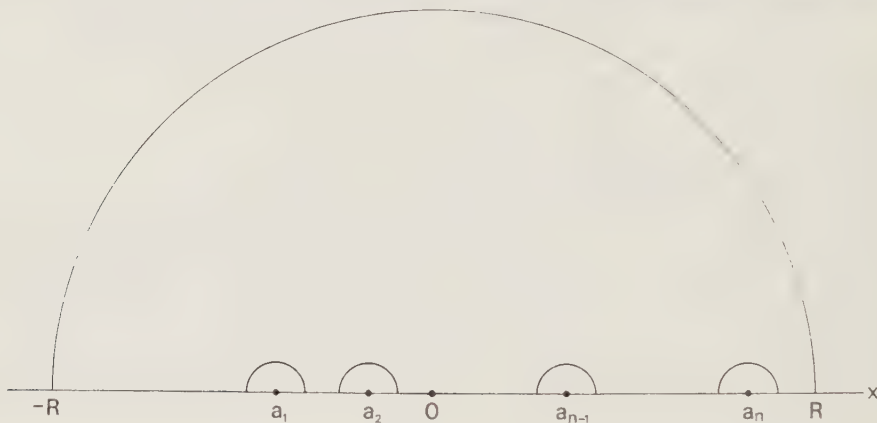


FIGURE 69

integral along this closed path is zero. It follows immediately that the loops  $F \circ \Gamma_v$  tend in the limit to the loop  $x \rightarrow F(x)$  for  $x \in \mathbf{R}$  (continuing  $F$  by continuity to the points  $a_h$  and  $\pm\infty$ ). Because of the change of argument of  $f(x)$  as real  $x$  passes through one of the  $a_h$ , it is at once seen that the image  $L$  of the preceding closed path is a juxtaposition of the segments of initial points  $c_h = F(a_h)$  and terminal points  $c_{h+1} = F(a_{h+1})$  for  $1 \leq h \leq n-1$  and of the segment (contained in  $\mathbf{R}$ ) of initial point  $c_n$  and terminal point  $c_1$ . Also,  $(c_{h+1} - c_h)/(c_h - c_{h-1})$  has argument  $\mu_h\pi$ . Application of (2.2) is made by studying the sense of the variation of the real and imaginary parts of  $F(x)$ , and it is easily seen that every line parallel to the imaginary axis meets  $L$  at two points at most, except in the case where one of the preceding segments is parallel to this axis; (VII, 6.6.6) can therefore be applied.

Since the half-plane is invariant under every homographic transformation

$$z \rightarrow \frac{az + b}{cz + d}$$

where  $a, b, c$  and  $d$  are real and  $ad - bc = 1$ ,  $F(D)$  does not change when  $F$  is replaced by

$$\int_0^z \frac{du}{(cu + d)^2 f\left(\frac{au + b}{cu + d}\right)}.$$

In particular, taking the homographic transformation  $z \rightarrow a_n - (1/z)$ , instead of  $F$  we can consider (up to similarity) the conformal representation

$$(5.2.2) \quad z \rightarrow \int_0^z \frac{du}{(u - b_1)^{\mu_1} \dots (u - b_{n-1})^{\mu_{n-1}}}$$

where the  $b_h$  are any real numbers and the  $\mu_h$  satisfy the first of the conditions (5.2.1)

and the condition  $\sum_{h=1}^{n-1} \mu_h < 2$ .

The determination of the  $b_h$  in (5.2.2) when the vertices of the polygon  $L$  are given is a difficult problem of numerical analysis for  $n \geq 5$ . For  $n = 3$ , we can reduce in (5.2.2) to the case where  $b_1 = 0$ ,  $b_2 = 1$ , by translation and homothety. If we put  $\mu_1 = 1 - \alpha$ ,  $\mu_2 = 1 - \beta$ ,  $P$  is then a triangle whose angles are  $\alpha\pi, \beta\pi, \gamma\pi$  ( $\gamma = 1 - \alpha - \beta$ ) and the side opposite the angle  $\gamma\pi$  has length

$$c = \int_0^1 x^{\alpha-1} (1-x)^{\beta-1} dx = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} = \frac{1}{\pi} \sin \gamma\pi \Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma).$$

(5.3) One can also study the function  $F$  where no condition is imposed on the real numbers  $\mu_h$ ; but  $F$  is then no longer necessarily injective in  $D$ , although the boundary of  $F(D)$  is still a union of segments (problem 23).

## 6. The symmetry principle

(6.1) Given a line  $L$  in  $\mathbf{C}$ , the image of  $\mathbf{R}$  under an affine linear function  $\gamma: t \rightarrow at + b$  ( $a$  and  $b$  complex,  $a \neq 0$ ), an *open segment* on  $L$  is defined as the image under  $\gamma$  of a non-empty, bounded open interval of  $\mathbf{R}$ ; the endpoints are by definition the images under  $\gamma$  of the endpoints of this interval. Similarly an *open half-line* on  $L$  is defined as the image under  $\gamma$  of an open interval unbounded on the right or on the left.

(6.2) Let  $P$  be an open half-plane in  $\mathbf{C}$ ,  $L$  the boundary line of  $P$ ,  $D$  an open set contained in  $P$ ; suppose that the boundary of  $D$  contains a set  $S$  which is an open segment or an open half-line on  $L$ . Let  $f$  be a complex function defined and continuous in  $D \cup S$ , analytic in  $D$ , and such that  $f(S)$  is contained in a line  $L'$ . Let  $\sigma, \sigma'$  be the symmetries with respect to  $L$  and  $L'$ . Then there is a function  $g$  analytic in the open union  $U$  of  $D$ , of  $S$  and of  $\sigma(D)$ , coinciding with  $f$  in  $D \cup S$  and such that

$$(6.2.1) \quad g(\sigma(z)) = \sigma'(g(z))$$

for every  $z \in D$ .

By affine linear mappings on  $z$  and on  $f(z)$ , it can always be supposed that  $L$  and  $L'$  are both the real axis,  $\sigma$  and  $\sigma'$  then being the symmetry  $z \rightarrow \bar{z}$  (Fig. 70). Let  $x_0 \in S$ ; the hypothesis on  $f(S)$  signifies that  $f$  takes real values in  $S$ . Consider then, in  $\sigma(D) \cup S$ , the function  $f_1: z \rightarrow \overline{f(\bar{z})}$ ; it is continuous in  $\sigma(D) \cup S$ , analytic in  $\sigma(D)$  and coincides with  $f$  in  $S$ ; there is therefore a function  $g$  continuous in the open set  $U = D \cup S \cup \sigma(D)$ , coinciding with  $f$  in  $D$ , with  $f_1$  in  $\sigma(D)$ , so analytic in  $D$  and in  $\sigma(D)$ . It follows from (VIII, 9.8.1) that  $g$  is analytic in  $U$ .

*Remark (6.3)* It may happen that the boundary of  $D$  contains several segments  $S_j$  ( $1 \leq j \leq k$ ) having the property stated, each of the  $f(S_j)$  being contained in a line

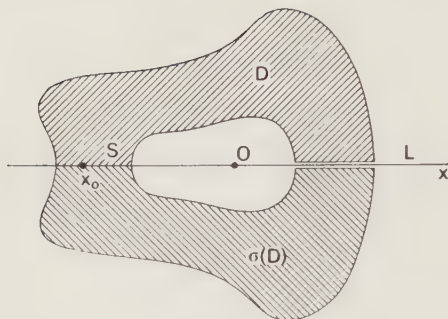


FIGURE 70

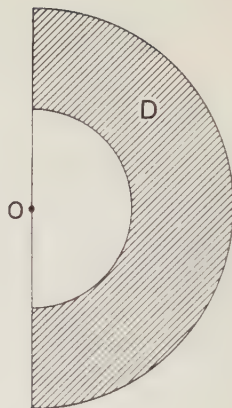


FIGURE 71

$L'_j$ . Take care not to think that  $f$  can be continued into the open union of  $D$ , of  $\sigma(D)$  and of all the segments  $S_j$ . As a counter-example, let  $D$  be the “semi-annulus”  $1 < |z| < 2$ ,  $\Re z > 0$  (Fig. 71), and  $f(z) = z^\lambda$  with  $\lambda$  not an integer (VIII, 9.6). However we can obtain such a continuation when all the lines  $L'_j$  are the same.

## 7. Conformal mapping and elliptic functions

(7.1) We propose to find a conformal mapping  $w \rightarrow z = F(w)$  of the *open half-plane*  $D: \Im w > 0$  onto a *rectangle*. The problem is solved by the Schwarz-Christoffel transformation (5.2.1) with  $n = 4$ ,  $\mu_h = \frac{1}{2}$  for every  $h$ ; in other words we can take

$$(7.1.1) \quad z = F(w) = \int_0^w \frac{du}{\sqrt{(1-u^2)(1-k^2u^2)}} \quad \text{with } 0 < k < 1$$

the determination of the integrand being taken equal to 1 for  $u = 0$ . The vertices of the rectangle  $R_0 = F(D)$  are the points  $-K$ ,  $K$ ,  $K + iK'$ ,  $-K + iK'$ , where  $K$  and  $K'$  are positive real numbers given by

$$(7.1.2) \quad K = \int_0^1 \frac{du}{\sqrt{(1-u^2)(1-k^2u^2)}}, \quad K' = \int_1^{1/k} \frac{du}{\sqrt{(1-u^2)(1-k^2u^2)}},$$

(Fig. 72). Denote by

$$(7.1.3) \quad z \rightarrow \operatorname{sn} z$$

the inverse function of  $F$ , analytic in  $R_0$ , and which is therefore a conformal mapping of  $R_0$  onto  $D$ . Note that for  $w = it$  purely imaginary

$$F(it) = i \int_0^t \frac{dt}{\sqrt{(1+t^2)(1+k^2t^2)}}$$

so that  $F(it)$  is purely imaginary and the symmetry principle can be applied to the imaginary half-axis  $\Re w = 0$ ,  $\Im w > 0$  (6.2), which shows that in  $R_0$

$$(7.1.4) \quad \operatorname{sn}(-\bar{z}) = -\overline{\operatorname{sn} z}.$$

It will be seen below that the function  $z \rightarrow \operatorname{sn} z$  can be continued to a meromorphic function in  $\mathbf{C}$  having the property of double periodicity

$$(7.1.5) \quad \operatorname{sn}(z + 4K) = \operatorname{sn} z, \quad \operatorname{sn}(z + 2iK') = \operatorname{sn} z.$$

(7.2) Let  $\sigma_1: z \rightarrow \bar{z}$ ,  $\sigma_2: z \rightarrow 2K - \bar{z}$  be the symmetries with respect to the real axis  $L_1: \Im z = 0$  and the line  $L_2: \Re z = K$  parallel to the imaginary axis. Note that  $\sigma_1\sigma_2 = \sigma_2\sigma_1$  is the symmetry  $z \rightarrow 2K - z$  with respect to the point  $K$ . The symmetry principle can be applied to the function  $F$  for the segment  $] -1, 1[$  and the segment  $]1, 1/k[$  of the real axis (6.2). In this way is obtained on the one hand a continuation  $F_1$  of  $F$  to  $D_1$ , the plane cut along the two half-lines  $\Im w = 0, \Re w \geq 1$  and  $\Im w = 0, \Re w \leq -1$ , such that  $F_1$  is still injective in  $D_1$ ;  $F_1(D_1)$  is the rectangle  $R_1: -K < \Re z < K, -K' < \Im z < K'$ . On the other hand there is a continuation  $F_2$  of  $F$  to  $D_2$ , the plane cut along the two half-lines  $\Im w = 0, \Re w \leq 1$  and  $\Im w = 0, \Re w \geq 1/k$ , such that  $F_2$  is again injective in  $D_2$ ;  $F_2(D_2)$  is the rectangle  $R_2: -K < \Re z < 3K, 0 < \Im z < K'$ . We can thus continue  $\operatorname{sn} z$  to a function analytic in  $R_1 \cup R_2$ , and for  $z \in R_0$

$$(7.2.1) \quad \operatorname{sn} \bar{z} = \overline{\operatorname{sn} z}, \quad \operatorname{sn}(2K - \bar{z}) = \overline{\operatorname{sn} z}.$$

Finally the symmetry principle is applied twice more to the function  $\operatorname{sn} z$  in  $R_1 \cup R_2$  and to the line segments  $\Re z = K, 0 \leq \Im z \leq K'$ , and  $\Im z = 0, -K \leq \Re z \leq K$ . If  $R$  is the rectangle defined by  $-K < \Re z < 3K, -K' < \Im z < K'$ , two continuations of  $\operatorname{sn} z$  are obtained, one to the complement in  $R$  of the segment  $\Im z = 0, K \leq \Re z < 3K$ , the other to the complement in  $R$  of the segment  $\Re z = K, -K' < \Im z \leq 0$ . Moreover it follows from (7.2.1) that these two continuations coincide in the rectangle  $\sigma_1\sigma_2(R_0)$  symmetric of  $R_0$  with respect to the point  $K$ , and an analytic function  $\operatorname{sn} z$  in  $R - \{K\}$  has been defined satisfying the relations (7.2.1) and their consequence

$$(7.2.2) \quad \operatorname{sn}(2K - z) = \operatorname{sn} z.$$

Furthermore the definition of  $F(w)$  shows that for  $\varepsilon > 0$  arbitrarily small, there exists  $r > 0$  such that for  $w \in D$  and  $|w| > r$ , we have  $|F(w) - iK'| \leq \varepsilon$ . Therefore for the points  $z \in R_0$  satisfying  $|z - K| < \varepsilon$ , we certainly have  $|\operatorname{sn} z| \leq r$ . Successive continuations of  $\operatorname{sn} z$  then show that for  $|z - K| \leq \varepsilon$  and  $z \neq K$ , we also have  $|\operatorname{sn} z| \leq r$ . We conclude that the function  $\operatorname{sn} z$  can be continued by continuity to the point  $z = K$  and is then analytic in  $R$  (VIII, 3.3). It clearly satisfies (7.2.2), (7.2.1) as well as the consequence of (7.1.4) and (7.2.1)

$$(7.2.3) \quad \operatorname{sn}(z + 2K) = \operatorname{sn}(-z) = -\operatorname{sn} z \quad \text{for } z \in R_1.$$

(7.3) Now let  $\sigma_3: z \rightarrow 2iK' - \bar{z}$  be the symmetry with respect to the line  $L_3: \Im w = K'$ . The symmetry principle can be applied to  $F$  for the open half-lines  $\Im w = 0, \Re w > 1/k$ , or  $\Im w = 0, \Re w < -1/k$ . Since the images of these half-lines are both on  $L_3$ , it is seen that a continuation  $F_3$  of  $F$  is obtained to  $D_3$ , the plane cut along the segment  $\Im w = 0, -1/k \leq \Re w \leq 1/k$ , and  $F_3$  is again injective in  $D_3$  and such that  $F_3(D_3)$  is the complement of the point  $iK'$  in the rectangle  $R_3: -K < \Re z < K, 0 < \Im z < 2K'$ . Thus one can again continue  $\operatorname{sn} z$  to  $R_3 - \{iK'\}$ , and, for  $z \in R_0$

$$(7.3.1) \quad \operatorname{sn}(2iK' - \bar{z}) = \overline{\operatorname{sn} z}.$$

We have thus defined  $\operatorname{sn} z$  in  $(R_1 \cup R_3) - \{iK'\}$ , and see moreover that for  $z \in \sigma_1(R_0)$ , as a consequence of (7.2.1) and (7.3.1)

$$(7.3.2) \quad \operatorname{sn}(z + 2iK') = \operatorname{sn} z.$$

(7.4) The final stage in the continuation of  $\operatorname{sn} z$  consists in defining it as a *meromorphic function* in  $\mathbf{C}$  by the formula

$$(7.4.1) \quad \operatorname{sn}(z + 2mK + 2inK') = (-1)^m \operatorname{sn} z$$

for every  $z \in R_1$  and every pair of integers  $(m, n)$ . Every point of  $\mathbf{C}$  which is not on one of the lines  $\Re z = (2m + 1)K$  or  $\Im z = (2n + 1)K'$  can be written *uniquely* in the form  $z + 2mK + 2inK'$  with  $z \in R_1$ , so the formula (7.4.1) defines  $\operatorname{sn} z$  except on these lines. But since there is a continuation of  $\operatorname{sn} z$  to  $R$  satisfying (7.2.3), so analytic for  $\Re z = K$ ,  $-K' < \Im z < K'$ , it is seen that  $\operatorname{sn} z$  can be continued by continuity to all of the lines  $\Re z = (2m + 1)K$  except perhaps at the points of intersection of these lines with the lines  $\Im z = (2n + 1)K'$ . Similarly, since there is an analytic continuation of  $\operatorname{sn} z$  to  $(R_1 \cup R_3) - \{iK'\}$  satisfying (7.3.2) in  $\sigma_1(R_0)$ ,  $\operatorname{sn} z$  can be continued by continuity to all of the lines  $\Im z = (2n + 1)K'$  except perhaps at the points of intersection of these lines with the lines  $\Re z = (2m + 1)K$  and at the points  $2mK + (2n + 1)iK'$ . The same reasoning as in (7.2) then shows that  $\operatorname{sn} z$  is bounded in the neighbourhood of the points  $(2m + 1)K + i(2n + 1)K'$ , and is therefore analytic in the *complement of the set of points*  $2mK + i(2n + 1)K'$ .

It remains to show that these last points are *simple poles* of  $\operatorname{sn} z$ ; because of (7.4.1) it is sufficient to consider the point  $iK'$ . Using (VIII, 3.3) we must prove that as  $z$  tends to  $iK'$ , the product  $(z - iK')\operatorname{sn} z$  remains *bounded* in absolute value. Now this product is equal to

$$(7.4.2) \quad -w \int_w^\infty \frac{du}{\sqrt{(1-u^2)(1-k^2u^2)}}$$

the integral being taken along the half-line  $u = tw$  where  $t \geq 1$ , and it must be shown that the expression (7.4.2) remains bounded as  $|w|$  tends to  $+\infty$ . But this integral is equal to

$$\int_1^{+\infty} \frac{dt}{\sqrt{(1-t^2w^2)(1-k^2t^2w^2)}}$$

and as soon as  $|w| > 2/k$

$$|\sqrt{(1-t^2w^2)(1-k^2t^2w^2)}| \geq \sqrt{(t^2-1)(4t^2-1)}$$

hence the conclusion, the integral

$$\int_1^{+\infty} \frac{dt}{\sqrt{(t^2-1)(4t^2-1)}}$$

being convergent.

(7.5) To summarize the results obtained, a *meromorphic function*  $\operatorname{sn} z$  in  $\mathbf{C}$  has been defined satisfying (7.4.1) for every  $z$  distinct from the poles, and hence also the relation of *double periodicity*

$$(7.5.1) \quad \operatorname{sn}(z + 4mK + 2inK') = \operatorname{sn} z.$$

By analytic continuation, from (7.1.4) and (7.2.1)

$$(7.5.2) \quad \operatorname{sn}(-z) = -\operatorname{sn} z$$

$$(7.5.3) \quad \operatorname{sn}(\bar{z}) = \overline{\operatorname{sn} z}.$$

A rectangle such as  $R$ , whose sides are  $4K$  and  $2K'$ , is called the *fundamental rectangle of periods* of  $\operatorname{sn} z$ . The study made in (7.2) shows that we have a *bijection onto the whole plane*  $\mathbf{C}$  by restricting  $\operatorname{sn} z$  to the set  $R'_1$ , the union of  $R_1$ , of the segment  $\Re z = K$ ,  $0 \leq \Im z \leq K'$ , of the segment  $\Im z = K'$ ,  $-K < \Re z < 0$ , of the segment  $\Im z = K'$ ,  $0 < \Re z \leq K$  and of the segment  $\Re z = -K$ ,  $0 \leq \Im z \leq K'$  (Fig. 72). The equation

$$(7.5.4) \quad \operatorname{sn} z = w_0$$

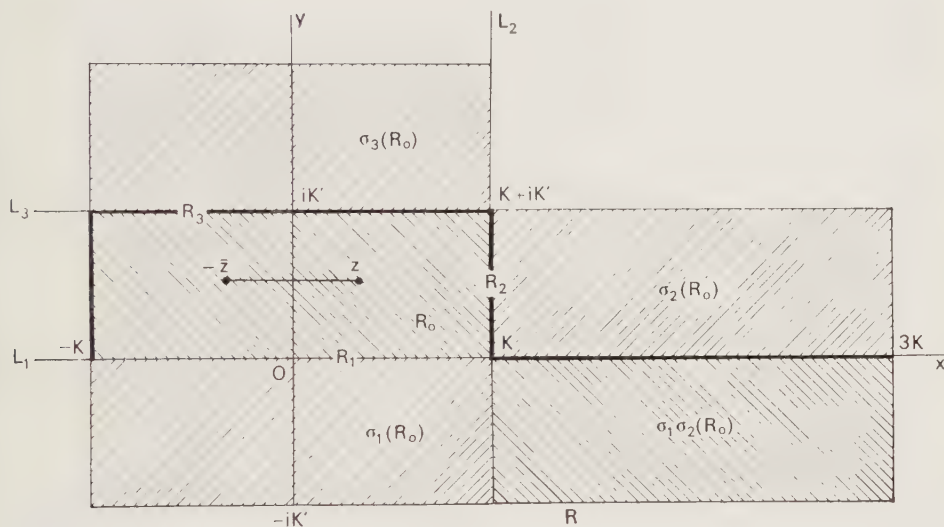


FIGURE 72

has exactly *two simple roots* in  $R' = R'_1 \cup (R'_1 + 2K)$  *except* for the four values

$$w_0 = \pm 1, \quad w_0 = \pm \frac{1}{k}.$$

The equation  $\operatorname{sn} z = 1$  has infinitely many *double roots* at the points

$$K + 4mK + 2inK'$$

and the equation  $\operatorname{sn} z = 1/k$  has infinitely many *double roots* at the points

$$K + iK' + 4mK + 2inK'.$$

The fact that the roots are double roots is immediately seen by observing that the angles of the paths passing through these points are *doubled* by the transformation  $z \rightarrow \operatorname{sn} z$  (1.5). One can obtain the roots of  $\operatorname{sn} z = -1$  and of  $\operatorname{sn} z = -1/k$  with the help

of the preceding and (7.5.2). The zeros of the derivative  $(\operatorname{sn} z)'$  are thus the points  $(2m + 1)K + inK'$ .

(7.6) From the above, the meromorphic functions

$$1 - \operatorname{sn}^2 z, \quad 1 - k^2 \operatorname{sn}^2 z$$

have all their poles *double* and all their zeros *double*. Hence (VIII, 9.8), one can define two *meromorphic* functions in  $\mathbf{C}$  by the conditions

$$(7.6.1) \quad \operatorname{cn}^2 z = 1 - \operatorname{sn}^2 z, \quad \operatorname{cn} 0 = 1$$

$$(7.6.2) \quad \operatorname{dn}^2 z = 1 - k^2 \operatorname{sn}^2 z, \quad \operatorname{dn} 0 = 1.$$

The three functions  $\operatorname{sn}$ ,  $\operatorname{cn}$ ,  $\operatorname{dn}$ , are called the *Jacobian elliptic functions*; they are all three doubly periodic of principal periods  $4K$  and  $2iK'$  and have the same (simple) poles. Moreover, it is clear that in  $R_0$  the function  $\operatorname{sn} z$  is a solution of the differential equation

$$(7.6.3) \quad w'^2 = (1 - w^2)(1 - k^2 w^2)$$

hence by analytic continuation, for  $z$  distinct from the poles of  $\operatorname{sn} z$

$$(7.6.4) \quad \left( \frac{d}{dz} \operatorname{sn} z \right)^2 = \operatorname{cn}^2 z \cdot \operatorname{dn}^2 z.$$

But by virtue of the definition of  $\operatorname{sn} z$ , its derivative at the origin is equal to 1 (7.1), hence from (7.6.4)

$$(7.6.5) \quad \frac{d}{dz} \operatorname{sn} z = \operatorname{cn} z \operatorname{dn} z.$$

## PROBLEMS

1. Let  $f(z) = (az + b)/(cz + d)$  be a homographic function not identically equal to  $z$ ; the solutions of the equation  $f(z) = z$  are called the *fixed points* of the conformal mapping  $w = f(z)$ . If there are no fixed points, the transformation  $f$  is called *parabolic*, and the relations  $w = f(z)$  is equivalent to

$$\frac{1}{w - \alpha} = \frac{1}{z - \alpha} + h$$

or to

$$w = z + h.$$

If there are one or two fixed points, the relation  $w = f(z)$  is equivalent to

$$w - \alpha = k(z - \alpha)$$

or to

$$\frac{w - \alpha}{w - \beta} = k \frac{z - \alpha}{z - \beta} \quad (\alpha \neq \beta)$$

with  $k \neq 1$ .

We say that  $f$  is *hyperbolic* if  $k$  is real and  $> 0$ , *elliptic* if  $k = e^{i\theta}$ ,  $\theta$  real, *loxodromic* for the other values of  $k$ . Write the conditions on  $a, b, c, d$ , which give any one of these four cases.

## 2. The transformations

$$w = e^{t\alpha} \frac{z - a}{\bar{a}z - 1} \quad (\alpha \text{ real})$$

mapping the disc  $|z| < 1$  onto itself cannot be other than elliptic, hyperbolic or the identity; for which values of  $a$  do we have each of these cases?

3. (a) Let  $f$  be a function meromorphic in an open set containing the disc  $|z| \leq 1$ ; suppose that  $|f(z)|$  is constant on the circle  $|z| = 1$ . Show that  $f$  has the form

$$f(z) = c \prod_{j=1}^m \frac{z - a_j}{\bar{a}_j z - 1} \cdot \prod_{k=1}^n \frac{\bar{b}_k z - 1}{z - b_k} \quad (c \text{ constant}).$$

(Let  $a_1, \dots, a_m$  be the zeros,  $b_1, \dots, b_n$  the poles of  $f$  in the disc  $|z| < 1$  counted with their orders of multiplicity; show that the quotient of  $f(z)$  and the product of the second member is necessarily constant, by using problem 11(b) of Chap. VI.)

(b) If further  $f$  has no poles in the disc  $|z| < 1$ , show that for each  $w$  satisfying  $|w| < 1$ , the equation  $f(z) = w$  has exactly  $m$  roots (counted with their order of multiplicity) in the disc  $|z| < 1$  (consider the function  $f(z) - w$  and apply Rouché's theorem).

(c) Deduce from (b) that the only conformal mappings of the unit disc onto itself which can be continued to a function analytic in a disc  $|z| < R$  with  $R > 1$ , are the homographic functions (3.4.1) (cf. problem 8).

4. (a) Show that the function

$$f(z) = \frac{4a}{(z + 1)^2}$$

is a conformal mapping of the unit disc  $|z| < 1$  onto the exterior of a parabola.

(b) Show that the function

$$f(z) = \frac{z}{(1 - z)^2}$$

is a conformal mapping of the disc  $|z| < 1$  onto the plane cut along the half-line  $\mathcal{J}w = 0$ ,  $\mathcal{R}w \leq -\frac{1}{4}$ .

5. Show that the function

$$f(z) = \frac{z}{\sqrt{z^2 + 1}}$$

is a conformal mapping of the half-plane  $\mathcal{J}z > 0$  cut along the half-line  $\mathcal{R}z = 0$ ,  $\mathcal{J}z \geq 1$ , onto the half-plane  $\mathcal{J}w > 0$ .

6. Let  $f(z) = a_0 + a_1 z + \dots + a_n z^n + \dots$  be a function analytic in the disc  $|z| < R$ . Let  $D(f)$  be the least upper bound of the numbers

$$|f(z_1) - f(z_2)| \quad \text{for} \quad |z_1| < R, \quad |z_2| < R.$$

Show that

$$|a_1 R| \leq \frac{1}{2} D(f).$$

(Observe that for  $0 < r < R$  we have

$$4\pi r a_1 = \int_0^{2\pi} (f(re^{i\theta}) - f(-re^{i\theta})) e^{-i\theta} d\theta.)$$

7. Let  $S$  be an annulus  $r \leq |z| \leq R$ ,  $f$  a function analytic in an open set  $D$  containing  $S$  and injective in  $S$ ; show that if  $f(z) = \sum_{n=-\infty}^{+\infty} a_n z^n$  is the Laurent development of  $f$ , the area of  $f(S)$  is given by

$$\pi \sum_{n=-\infty}^{+\infty} n |a_n|^2 (R^{2n} - r^{2n})$$

(Use the formula for change of variables in the double integrals (K-R, p. 151).)

8. Let  $D$  be an open disc  $|z| < R$ ,  $f$  an injective analytic function in  $D$ ; suppose that the area  $A(f(D))$  is finite. For each  $r$  satisfying  $0 < r < R$  let  $D_r$  be the disc  $|z| < r$ ,  $A(f(D_r))$  the area of  $f(D_r)$ . Show that

$$\frac{A(f(D))}{A(f(D_r))} \geq \frac{R^2}{r^2}$$

and that equality can occur only if  $f$  is a polynomial of degree 1. In particular,  $A(f(D)) \geq \pi R^2 |f'(0)|^2$ , equality only occurring if  $f$  is a polynomial of the first degree.

Deduce that if  $f$  is a conformal mapping of the unit disc  $|z| < 1$  onto itself such that  $f(0) = 0$ , we have  $f(z) = z e^{i\alpha}$  (apply the preceding to  $f$  and to its inverse function). Every conformal mapping of the unit disc onto itself is thus of the form (3.4.1).

9. Let  $f$  be a function analytic in the unit disc  $D: |z| < 1$ , such that  $|f(z)| < M$  in  $D$ . Let  $a_1, \dots, a_n$  be zeros of  $f$  in  $D$ , counted with their order of multiplicity. Show that

$$(1) \quad |f(z)| \leq M \left| \prod_{k=1}^n \frac{z - a_k}{\bar{a}_k z - 1} \right|.$$

(If  $h$  is the quotient of  $f$  with the product of the second member, observe that for each  $\varepsilon > 0$ , there exists  $r$  satisfying  $1 - \varepsilon < r < 1$  such that  $|h(z)| \leq M(1 + \varepsilon)$  for  $|z| = r$ , then use the maximum principle.)

In particular

$$(2) \quad |f(0)| \leq \prod_{k=1}^n |a_k|.$$

More particularly, if  $|f(z)| \leq M$  in  $D$  and if  $f(0) = 0$

$$(3) \quad |f(z)| \leq M|z|$$

in  $D$  (Schwarz lemma). Hence, we also have in this case

$$(4) \quad |f'(0)| \leq M.$$

10. Let  $g_1, g_2$  be two conformal mappings of the unit disc  $D: |z| < 1$  onto the open sets  $U_1, U_2$  of  $\mathbf{C}$ , and let  $g_1(0) = c_1, g_2(0) = c_2$ . For each  $r$  satisfying  $0 < r < 1$  let  $U_1(r)$  and  $U_2(r)$  be the images of the disc  $D_r: |z| < r$  under  $g_1$  and  $g_2$  respectively. Show that if  $f$  is a function analytic in  $U_1$  and such that  $f(U_1) \subset U_2$  and  $f(c_1) = c_2$ , then, for each  $r$  satisfying  $0 < r < 1$

$$f(U_1(r)) \subset U_2(r)$$

(apply problem 9 to the function  $g_2^{-1} \circ f \circ g_1$ ).

11. Let  $f$  be a function analytic in the unit disc  $D: |z| < 1$  and such that  $|f(z)| < 1$  in  $D$ . Show that for every point  $a \in D$ ,

$$(1) \quad \left| \frac{f(z) - f(a)}{1 - \overline{f(z)}f(a)} \right| \leq \left| \frac{z - a}{1 - \bar{a}z} \right|$$

(cf. problem 10). In particular, for every  $z \in D$

$$(2) \quad |f'(z)| \leq \frac{1 - |f(z)|^2}{1 - |z|^2}.$$

More particularly

$$(3) \quad \frac{|f(0)| - |z|}{1 - |f(0)| \cdot |z|} \leq |f(z)| \leq \frac{|f(0)| + |z|}{1 + |f(0)| \cdot |z|}.$$

12. (a) With the same hypotheses as in problem 11, show that for  $|z_1| \leq r$  and  $|z_2| \leq r$  with  $0 < r < 1$

$$\left| \frac{f(z_1) - f(z_2)}{z_1 - z_2} \right| \leq \frac{1}{1 - r^2}$$

(use inequality (2) of problem 11).

(b) If  $f(z_1) = f(z_2) = \beta$  and  $|z_1| = |z_2| = \rho < 1$ , and further if  $f(0) = 0$ , show that

$$\left| \frac{f(z) - \beta}{1 - \bar{\beta}f(z)} \right| \leq \left| \frac{z - z_1}{1 - \bar{z}_1 z} \right| \cdot \left| \frac{z - z_2}{1 - \bar{z}_2 z} \right|$$

and deduce that  $|\beta| \leq \rho^2$ .

(c) Under the same hypotheses as in (b), show that if  $|f'(0)| = \alpha \neq 0$

$$\rho(\alpha - \rho) \leq (1 - \alpha\rho)|f(z)|$$

for every  $z$  such that  $|z| = \rho$  (apply relation (3) of problem 11 to  $f(z)/z$ ).

(d) Deduce from (c) that  $f$  is injective in the disc  $|z| < \rho_0$ , where  $\rho_0 = \alpha/(1 + \sqrt{1 - \alpha^2})$ . (Observe, with the help of Rouché's theorem, that if  $r$  is the smallest number  $> 0$  such that there exist  $z_1$  and  $z_2$  distinct in  $D$  satisfying  $|z_1| = r$ ,  $|z_2| \geq r$  and  $f(z_1) = f(z_2)$ , then necessarily  $|z_2| = r$ , by *reductio ad absurdum*.)

13. Let  $f$  be a function analytic for  $|z| < R$ , and let  $A(R) = \sup_{|z| \leq R} \Re f(z)$ . Show that, if for  $r < R$  we put  $A(r) = \sup_{|z| \leq r} \Re f(z)$

$$A(r) \leq \frac{R - r}{R + r} A(0) + \frac{2r}{R + r} A(R)$$

(apply problem 10).

14. Let  $f$  be a function analytic and bounded for  $|z| < R$ , and let

$$M(R) = \sup_{|z| < R} |f(z)|.$$

Show that if  $f(z) \neq 0$  for  $|z| < R$  and if  $M(r) = \sup_{|z| < r} |f(z)|$

$$M(r) \leq M(0)^{(R-r)/(R+r)} M(R)^{2r/(R+r)}$$

for  $0 < r < R$  (apply problem 13).

15. Let  $(f_n)$  be a sequence of functions analytic in  $|z| < 1$  and satisfying

$$|f_n(z)| \leq 1 \quad \text{and} \quad f_n(z) \neq 0$$

for every  $n$  and for  $|z| < 1$ . Show that if  $\lim_{n \rightarrow \infty} f_n(0) = 0$ , then  $\lim_{n \rightarrow \infty} f_n(z) = 0$  for all  $z$  such that  $|z| < 1$ , the convergence being uniform in every disc  $|z| \leq r < 1$  (use problem 14).

16. Let  $f$  be a function analytic in the unit disc  $D: |z| < 1$ , such that  $f(0)$  is real and  $\Re f(z) \geq 0$  in  $D$ . Show that

$$f(0) \frac{1 - |z|}{1 + |z|} \leq \Re f(z) \leq f(0) \frac{1 + |z|}{1 - |z|}, \quad |\Im f(z)| \leq f(0) \frac{2|z|}{1 - |z|^2}$$

$$f(0) \frac{1 - |z|}{1 + |z|} \leq |f(z)| \leq f(0) \frac{1 + |z|}{1 - |z|}$$

(apply problem 10 taking  $U_1 = D$  and for  $U_2$  the half-plane  $\Re z > 0$ ).

17. Let  $f$  be a function analytic in  $D: |z| < 1$  and such that  $f(0) = 0$  and  $|\Re f(z)| < 1$  in  $D$ . Show that for  $z \in D$

$$|\Re f(z)| \leq \frac{4}{\pi} \arctan |z|, \quad |\Im f(z)| \leq \frac{2}{\pi} \log \frac{1 + |z|}{1 - |z|}$$

(apply problem 10 taking  $U_1 = D$  and for  $U_2$  the strip  $|\Re z| < 1$ ).

18. Let  $f$  be a function analytic and bounded in  $D: |z| < 1$ , and not identically zero. Show that if there exist infinitely many zeros  $a_n$  ( $n \geq 1$ ) of  $f$  in  $D$ , the series

$$\sum_{n=1}^{\infty} \log |a_n|$$

is convergent, and hence also the series  $\sum_{n=1}^{\infty} (1 - |a_n|)$  (cf. problem 9).

19. Let  $f$  be a function analytic in the exterior  $E: |z| > 1$  of the unit disc, and suppose that its Laurent series is of the form

$$f(z) = z + b_0 + \frac{b_1}{z} + \cdots + \frac{b_n}{z^n} + \cdots$$

Show that if  $f$  is *injective* in  $E$ , then necessarily

$$(1) \quad |b_1|^2 + 2|b_2|^2 + \cdots + n|b_n|^2 + \cdots \leq 1$$

(use problem 7). In particular

$$(2) \quad |b_1| \leq 1$$

and equality can only occur if  $f$  is of the form

$$f(z) = z + b_0 + \frac{e^{i\alpha}}{z}.$$

Deduce from (1) that in  $E$

$$|f'(z)| \leq \frac{|z|^2}{|z|^2 - 1}$$

(use the Cauchy-Schwarz inequality for series).

20. In this problem we shall admit that if  $h$  is an analytic function in an open simply connected set  $D$  such that  $h(z) \neq 0$  in  $D$ , there exists a function  $g$  analytic in  $D$  such that  $(g(z))^2 = h(z)$  in  $D$ .† In particular, suppose that  $f$  satisfies the hypotheses of problem 19 and that  $f(z) \neq 0$  in  $E$ . Then there exists a function  $g$  analytic in  $E$ , such that  $(g(z))^2 = f(z^2)$  and such that the Laurent development of  $g$  is of the form  $z + b'_0 + \cdots$  (applying the preceding to the function  $zf(1/z)$  for  $|z| < 1$ ). Deduce with the help of the inequality (2) of problem 19 applied to  $g$ , that then  $|b'_0| \leq 2$ .

† See [FA], X, section 2, problem 6.

21. Let  $f$  be a function analytic in the unit disc  $D: |z| < 1$  and such that  $f(0) = 0, f'(0) = 1$ , so that

$$f(z) = z + a_2 z^2 + \cdots + a_n z^n + \cdots$$

Suppose that  $f$  is *injective* in  $D$ .

(a) Show that then  $|a_2| \leq 2$ , equality being possible only if  $f$  is of the form

$$(*) \quad f(z) = \frac{z}{(1 + e^{i\alpha} z)^2} \quad (\alpha \text{ real}).$$

(Apply problem 20 to the function  $(f(z^{-1}))^{-1}$ .)

(b) The open set  $f(D)$  cannot be the whole plane, by virtue of Liouville's theorem, therefore its boundary  $F$  is not empty. Show that  $d(0, F) \geq \frac{1}{4}$ , equality being attained only by functions of the form  $(*)$ , in which case  $f(D)$  is the plane cut along the half-line  $t \rightarrow e^{i\alpha} t$  ( $t \geq \frac{1}{4}$ ). (If  $c \in F$ , consider the function

$$\frac{cf(z)}{c - f(z)}.)$$

22. Let  $g$  be an analytic function injective in an open set  $D$ , and let  $h = g^{-1}$  be the inverse mapping of  $g(D)$  onto  $D$ . For every path  $\gamma$  in  $D$  and every function  $F$  continuous in  $g(D)$

$$\int_{g \circ \gamma} F(w) h'(w) dw = \int_{\gamma} F(g(z)) dz.$$

Apply this to the case where  $D$  contains the closed unit disc  $|z| \leq 1$  and where  $\gamma$  is the closed path  $t \rightarrow e^{it}$  ( $0 \leq t \leq 2\pi$ ). Deduce from Cauchy's formula

$$2\pi i f(0) = \int_{\gamma} \frac{f(z)}{z} dz$$

by application of the conformal transformation

$$g(z) = \frac{R(z - \zeta)}{R^2 - \bar{\zeta}z},$$

the *Poisson formula* for every function  $f$  analytic in an open set containing  $|z| \leq R$

$$f(re^{i\varphi}) = \frac{1}{2\pi} \int_0^{2\pi} f(Re^{i\theta}) \frac{R^2 - r^2}{R^2 - 2Rr \cos(\theta - \varphi) + r^2} d\theta \quad (0 < r < R).$$

23. Suppose in the formula (5.2.1) that  $-1 < \mu_h < 1$  for every  $h$ , and  $\sum_h \mu_h \geq 2$ . Show that when the  $\mu_h$  and all the  $a_h$ , except one,  $a_j$ , remain fixed,  $a_j$  varying between  $a_{j-1}$  and  $a_{j+1}$ , the lengths  $|c_{h+1} - c_h|$  are continuous functions of  $a_j$ . As  $a_j$  tends to  $a_{j-1}$ , these lengths tend to limits  $\neq 0$ , except for  $|c_j - c_{j-1}|$ , which tends to 0. Deduce from this example where the mapping  $F$  is not injective (take  $n = 10$  and all of the  $\mu_h$  equal to  $\frac{1}{2}$ , except  $\mu_3 = \mu_4 = \mu_5 = -\frac{1}{2}$ ).

24. Show that the function

$$\int_{z_0}^z \frac{\sqrt{1-u^4}}{u^2} du \quad (z_0 \neq 0)$$

is a conformal mapping of the unit disc  $|z| < 1$  onto the exterior of a square.

25. What are the images of the unit disc under the functions

$$\int_0^z \frac{du}{(1-u^n)^{2/n}}, \quad \int_0^z \frac{(1-u^n)^\lambda du}{(1+u^n)^{\lambda+(2/n)}} \quad \left(-1 < \lambda < 1 - \frac{2}{n}\right)?$$

Calculate the radius of the smallest disc of centre 0 containing the image.

26. What is the image of the disc  $|z| < 1$  under a function of the form

$$f(z) = \frac{1}{z} \prod_{k=1}^n (z - a_k)^{\lambda_k}$$

where  $|a_k| = 1$ ,  $0 < \lambda_k < 1$  and  $\sum_{k=1}^n \lambda_k = 2$ ?

27. With the notations of no. 7 let

$$q = \exp(-\pi K'/K) < 1, \quad \zeta = \exp(\pi iz/2K).$$

The infinite product  $\prod_{m=0}^{\infty} (1 - q^m \zeta)$  is then strictly convergent for every  $\zeta$  distinct from the  $q^{-m}$ . Show that the function meromorphic in  $\mathbf{C}$

$$f(z) = \zeta \frac{\prod_{m=0}^{\infty} (1 - q^{2m} \zeta^{-2})(1 - q^{2m} \zeta^2)}{\prod_{m=0}^{\infty} (1 - q^{2m+1} \zeta^{-2})(1 - q^{2m+1} \zeta^2)}$$

has the periods  $4K$  and  $2iK'$ , and has the same zeros and poles as  $\operatorname{sn} z$ , with the same multiplicities. With the help of Liouville's theorem, deduce that  $\operatorname{sn} z = Cf(z)$ , where  $C$  is a constant. Determine  $C$  with the help of the relations  $\operatorname{sn} K = 1$ ,  $\operatorname{sn}(K + iK') = k^{-1}$ , and show that

$$C = -iq^{1/4}k^{-1/2}.$$

28. Put for  $|q| < 1$  and any  $\zeta \in \mathbf{C}$

$$\begin{aligned} \varphi_n(\zeta) &= (1 + q\zeta)(1 + q\zeta^{-1})(1 + q^2\zeta)(1 + q^2\zeta^{-1}) \dots (1 + q^{2n-1}\zeta)(1 + q^{2n-1}\zeta^{-1}) \\ &= c_0 + c_1(\zeta + \zeta^{-1}) + c_2(\zeta^2 + \zeta^{-2}) + \dots + c_n(\zeta^n + \zeta^{-n}). \end{aligned}$$

(a) Determine the coefficients  $c_n$  using the relation between  $\varphi_n(q^2\zeta)$  and  $\varphi_n(\zeta)$ .

(b) Deduce from (a) by passage to the limit, the relation

$$\prod_{n=1}^{\infty} (1 + q^{2n-1}\zeta)(1 + q^{2n-1}\zeta^{-1})(1 - q^{2n}) = \sum_{n=-\infty}^{+\infty} q^{n^2} z^n.$$

Deduce from this the identities

$$\prod_{n=1}^{\infty} (1 - q^n) = \sum_{n=-\infty}^{+\infty} (-1)^n q^{(3n^2+n)/2}$$

$$\frac{\prod_{n=1}^{\infty} (1 - q^{2n})}{\prod_{n=1}^{\infty} (1 - q^{2n-1})} = \sum_{n=0}^{\infty} q^{n(n+1)/2}$$

$$\frac{\prod_{n=1}^{\infty} (1 - q^n)}{\prod_{n=1}^{\infty} (1 + q^n)} = 1 - 2q + 2q^4 + \dots + (-1)^n 2q^{n^2} + \dots$$

(substitute suitable values for  $q$  and  $z$ ).

(c) Put

$$\theta(z) = \sum_{n=-\infty}^{+\infty} (-1)^n q^{n^2} \zeta^{2n} = \sum_{n=-\infty}^{+\infty} (-1)^n \exp\left(\frac{n\pi iz - n^2\pi K'}{K}\right)$$

Show that

$$\theta(z + 2K) = \theta(z), \quad \theta(z + 2iK') = -\frac{1}{q} e^{-\pi iz/K} \theta(z).$$

Express  $\operatorname{sn} z$  with the aid of the function  $\theta(z)$ .

# Differential equations

## 1. Solutions and approximate solutions

(1.1) Given a real function  $(t, x) \rightarrow f(t, x)$  defined and *continuous* in an open set  $D \subset \mathbf{R}^2$ , a *solution* of the differential equation of the first order

$$(1.1.1) \quad x' = f(t, x)$$

in an open interval  $I \subset \mathbf{R}$  is a real function  $t \rightarrow u(t)$  defined, continuous and differentiable in  $I$ , such that  $(t, u(t)) \in D$  for every  $t \in I$  and such that  $u'(t) = f(t, u(t))$  for every  $t \in I$ . Geometrically, we can associate with every point  $(t, x) \in D$  the vector  $v(t, x)$  of components  $(1, f(t, x))$ . To say  $u$  is a solution of (1.1.1) signifies that the graph of  $u$  is contained in  $D$  and has at each point  $(t, x)$  a tangent for which  $v(t, x)$  is a direction vector (Fig. 73). Simple examples show that one must expect infinitely many solutions to exist depending on a parameter and that supplementary conditions can therefore be

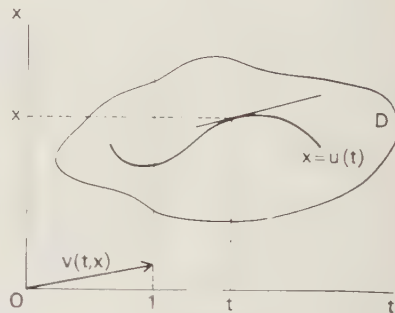


FIGURE 73

imposed on the solutions. The most frequent of these conditions consists in choosing a point  $(t_0, x_0) \in D$  and seeking the solutions whose graph *contains this point*; this is the same as imposing on the desired solution the condition

$$(1.1.2) \quad u(t_0) = x_0.$$

The problem of the existence and uniqueness of a solution of (1.1.1) satisfying (1.1.2) is called the *Cauchy problem* for (1.1.1), and (1.1.2) is an *initial condition* for this equation. This word is an indication of the fact that in applications the variable  $t$  often represents time.

(1.2) To say that a solution  $u$  of (1.1.1) satisfies the Cauchy condition (1.1.2) also signifies that  $u$  satisfies in  $I$  the *integral equation*

$$(1.2.1) \quad u(t) = x_0 + \int_{t_0}^t f(s, u(s)) \, ds.$$

The Cauchy problem is more accessible in this form, by virtue of the fact that integration is easier to handle than differentiation (I, 3.7).

(1.3) It would be extremely naive to believe, merely on the example of a few particularly well-known simple cases (such as the linear equation of the first order), that one can generally obtain solution of (1.1.1) by “quadratures”, that is applying a certain number of times the operation consisting of the calculation of a primitive of a known function. Here as everywhere else, the aim of Analysis is the formation of algorithms for the APPROXIMATION of the solutions sought, in every sense which can be given to this word.

(1.4) We are thus led to introduce a new notion besides that of a “solution” of a differential equation. Given an open interval  $I \subset \mathbf{R}$  and a number  $\varepsilon > 0$ , we shall say that a real function  $v$ , continuous in  $I$  and which is the primitive in  $I$  of a *piecewise-continuous function*  $v'$ , is an *approximate solution within  $\varepsilon$*  of (1.1.1) if, for every  $t \in I$ , we have  $(t, v(t)) \in D$  and if, *except at the points of discontinuity of  $v'$*

$$(1.4.1) \quad |v'(t) - f(t, v(t))| \leq \varepsilon.$$

It will be seen below that there are several methods for the formation of approximate solutions for given equations (1.1.1) which are quite general. Another case to which this notion applies is that where one can find solutions of an equation  $x' = g(t, x)$  “close” to the equation (1.1.1); to be precise, if  $f$  and  $g$  are continuous in  $D$  and satisfy

$$|f(t, x) - g(t, x)| \leq \varepsilon$$

it is clear that for every solution  $v$  of  $x' = g(t, x)$ , we have the relation (1.4.1).

## 2. Comparison of approximate solutions

(2.1) The notion of approximate solution is evidently of interest only if, as its name suggests, it effectively “approximates” a solution in the interval in which it is defined. For the sake of precision the arbitrary parameter which figures in the solutions of (1.1.1) must be eliminated, which is to say that the Cauchy problem must be considered. Moreover it must be insisted that the approximate solution  $v$  also satisfies the condition (1.1.2), or more generally that  $|v(t_0) - x_0|$  be smaller than a given number. The problem then is to *majorize*, for  $t \in I$ , the absolute value  $|u(t) - v(t)|$  of the difference between the solution  $u$  (assumed to exist) of the Cauchy problem and the “approximate solution within  $\varepsilon$ ”  $v$ , as a function of what is given, i.e. of  $f$ ,  $\varepsilon$  and  $|v(t_0) - x_0|$ . This problem admits no solution if the only hypothesis on  $f$  is that of continuity (3.8.2). The notion of a *Lipschitz* function must be introduced.

(2.2) A function  $f$  defined and continuous in  $D$  and satisfying the properties of (1.1) is said to be *Lipschitz (in  $x$ )* for the constant  $k > 0$ , if for any points  $(t, x_1)$  and  $(t, x_2)$  of  $D$  having the same abscissa

$$(2.2.1) \quad |f(t, x_1) - f(t, x_2)| \leq k|x_1 - x_2|.$$

An important case where this property occurs is that where, for every  $t \in \text{pr}_1(D)$ , the

function  $x \rightarrow f(t, x)$  is the primitive of a piecewise-continuous function  $\partial f / \partial x$  and where there exists a number  $k$  independent of  $t$  such that

$$(2.2.2) \quad \left| \frac{\partial f}{\partial x}(t, x) \right| \leq k$$

for  $(t, x) \in D$ ; the relation (2.2.1) is then a consequence of the theorem of the mean.

The function  $f(t, x) = \sqrt{|x|}$  defined in  $\mathbf{R}^2$  is an example of a function which is *not Lipschitz*, the ratio  $f(t, x)/x$  tending to  $+\infty$  as  $x$  tends to 0 through values  $> 0$  (cf. (3.8.2)).

The solution of the problem formulated in (2.1) is then given by the following fundamental proposition, which more generally compares two approximate solutions (without ever prejudging the *existence* of a “true” solution):

(2.3) *Let  $f$  be a real function defined and continuous in an open set  $D \subset \mathbf{R}^2$  and satisfying the inequality (2.2.1). Let  $u_1, u_2$  be two real functions continuous in an interval  $I \subset \mathbf{R}$ , which are primitives in this interval of piecewise-continuous functions and satisfy in  $I$  the conditions  $(t, u_1(t)) \in D$  and  $(t, u_2(t)) \in D$ , as well as the inequalities*

$$(2.3.1) \quad |u'_1(t) - f(t, u_1(t))| \leq \varepsilon_1, \quad |u'_2(t) - f(t, u_2(t))| \leq \varepsilon_2$$

*except at the points of discontinuity of the derivatives. Let  $t_0 \in I$  and suppose that*

$$(2.3.2) \quad |u_1(t_0) - u_2(t_0)| \leq \delta.$$

*Then, for every  $t \in I$*

$$(2.3.3) \quad |u_1(t) - u_2(t)| \leq \delta e^{k|t-t_0|} + \frac{\varepsilon}{k} (e^{k|t-t_0|} - 1),$$

*with*

$$\varepsilon = \varepsilon_1 + \varepsilon_2.$$

To prove this inequality, first establish a very useful lemma in this kind of problem, which may be described as a “solution of a linear integral inequality”:

(2.3.4) (Gronwall’s lemma) *In a closed interval  $[0, c]$  of  $\mathbf{R}$ , let  $\varphi, \psi, w$  be three functions  $\geq 0$ , piecewise-continuous and satisfying the inequality*

$$(2.3.5) \quad w(t) \leq \varphi(t) + \int_0^t \psi(s)w(s) \, ds$$

*except at the points of discontinuity. Then, except at the points of discontinuity*

$$(2.3.6) \quad w(t) \leq \varphi(t) + \int_0^t \varphi(s)\psi(s) \exp \left( \int_s^t \psi(\xi) \, d\xi \right) ds.$$

Put

$$(2.3.7) \quad y(t) = \int_0^t \psi(s)w(s) \, ds,$$

a continuous function, which is the primitive of the piecewise-continuous function  $\psi(t)w(t)$ . Multiplying the two members of (2.3.5) by  $\psi(t)$

$$(2.3.8) \quad y'(t) - \psi(t)y(t) \leq \varphi(t)\psi(t)$$

except at the points of discontinuity.

The method of integrating a linear equation of the first order introduces the function

$$(2.3.9) \quad z(t) = y(t) \exp \left( - \int_0^t \psi(s) ds \right)$$

and by multiplying the two members of (2.3.8) by  $\exp \left( - \int_0^t \psi(s) ds \right)$ ,

$$(2.3.10) \quad z'(t) \leq \varphi(t)\psi(t) \exp \left( - \int_0^t \psi(s) ds \right)$$

except at the points of discontinuity.

On the other hand  $z(0) = y(0) = 0$ , and therefore

$$z(t) \leq \int_0^t \varphi(s)\psi(s) \exp \left( - \int_0^s \psi(\xi) d\xi \right) ds.$$

Hence, substituting into (2.3.9)

$$y(t) \leq \int_0^t \varphi(s)\psi(s) \exp \left( \int_s^t \psi(\xi) d\xi \right) ds$$

and finally, since (2.3.5) implies  $w(t) \leq \varphi(t) + y(t)$ , we have (2.3.6).

The lemma being established, note that from the inequalities

$$|u'_i(t) - f(t, u_i(t))| \leq \varepsilon_i \quad (i = 1, 2)$$

valid except at the points of discontinuity, we deduce, by the theorem of the mean, for  $t \geq t_0$ ,

$$\left| u_i(t) - u_i(t_0) - \int_{t_0}^t f(s, u_i(s)) ds \right| \leq \varepsilon_i(t - t_0) \quad (i = 1, 2)$$

and hence

$$|u_1(t) - u_2(t) - (u_1(t_0) - u_2(t_0)) - \int_{t_0}^t (f(s, u_1(s)) - f(s, u_2(s))) ds| \leq \varepsilon(t - t_0).$$

But since  $f$  is *Lipschitz*, it is deduced (2.2.1) that

$$|f(s, u_1(s)) - f(s, u_2(s))| \leq k|u_1(s) - u_2(s)|$$

and by putting  $w(t) = |u_1(t) - u_2(t)|$ , the inequality

$$(2.3.11) \quad w(t) \leq w(t_0) + \varepsilon(t - t_0) + k \int_{t_0}^t w(s) ds$$

is therefore obtained. It is sufficient to apply Gronwall's lemma to the latter to obtain (2.3.3) for  $t \geq t_0$ ; the case  $t \leq t_0$  is deduced from this by the change of variable  $t' = -t$ .

### 3. Cauchy-Lipschitz method

(3.1) Suppose that the function  $f$  is *bounded* in  $D$ , let  $|f(t, x)| \leq M$ , and *uniformly continuous*, i.e. (0, 5.6) such that for each  $\varepsilon > 0$  there exists a number  $\delta > 0$  (depending only on  $\varepsilon$ ) such that the relations

$$(t_1, x_1) \in D, \quad (t_2, x_2) \in D, \quad |t_1 - t_2| \leq \delta, \quad |x_1 - x_2| \leq \delta$$

imply

$$|f(t_1, x_1) - f(t_2, x_2)| \leq \varepsilon.$$

Note that this condition is satisfied if  $D$  is convex and if  $f$  has *bounded and continuous partial derivatives*  $\partial f / \partial t$  and  $\partial f / \partial x$ , by virtue of the theorem of the mean (I, 3.6).

For each  $\varepsilon > 0$ , we shall form solutions of (1.1.1) *approximate within  $\varepsilon$*  and satisfying the Cauchy condition (1.1.2). To do this, consider any point  $(t_1, x_1) \in D$ , and consider, in the interval  $[t_1, t_1 + h]$ , the *affine linear function*

$$u: t \rightarrow x_1 + f(t_1, x_1)(t - t_1)$$

in other words, the affine linear function whose graph has for its slope precisely the *value of  $f$  at the point  $(t_1, x_1)$* . To find out whether this function is an approximate solution within  $\varepsilon$  in the interval  $[t_1, t_1 + h]$ , we must form the difference

$$f(t_1, x_1) - f(t, x_1 + f(t_1, x_1)(t - t_1))$$

which is majorized in absolute value by  $\varepsilon$ , provided that

$$h \leq \delta \quad \text{and} \quad Mh \leq \delta$$

and further provided that *the point  $(t, u(t))$  is in  $D$  for  $t_1 \leq t \leq t_1 + h$* .

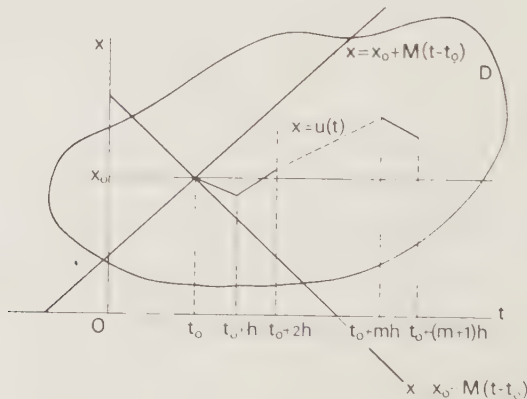


FIGURE 74

(3.2) This leads to the formation of the desired approximate solution by the following procedure, called the *Cauchy-Lipschitz method*; choose a number  $h$  satisfying  $0 \leq h \leq \inf(\delta, \delta/M)$  and consider the points  $t_0 + mh$  ( $m \in \mathbf{Z}$ ) in  $\mathbf{R}$ . Then apply, for  $t \geq t_0$ , the procedure described above, to form successively *affine linear* functions having the following properties (Fig. 74):

In  $[t_0, t_0 + h]$  the function  $u_1$  is such that

$$u_1(t_0) = x_0, \quad u'_1(t) = f(t_0, x_0).$$

In  $[t_0 + h, t_0 + 2h]$  the function  $u_2$  is such that

$$u_2(t_0 + h) = u_1(t_0 + h), \quad u'_2(t) = f(t_0 + h, u_1(t_0 + h));$$

. . . . .

In  $[t_0 + mh, t_0 + (m+1)h]$  the function  $u_{m+1}$  is such that

$$u_{m+1}(t_0 + mh) = u_m(t_0 + mh), \quad u'_{m+1}(t) = f(t_0 + mh, u_m(t_0 + mh))$$

. . . . .

The procedure can be continued *as long as the points*  $(t, u_m(t))$  *belong to*  $D$ . Proceed in the same manner for  $t \leq t_0$  (reducing to the preceding case by the change of variable  $t' = -t$ ). In this way a “piecewise-linear function” has been obtained which is an approximate solution within  $\varepsilon$  to (1.1.1) in the precise sense of (1.4).

(3.3) The fact that we must verify at each step of the above construction that we have not left the open set  $D$ , poses the problem of knowing what can be said about the *maximum* interval  $[t_0 - ph, t_0 + qh]$  in which the above construction is possible. It is easy to define explicitly an interval *contained* in this one in which one can be sure that the construction does not stop. Indeed, the piecewise-linear function  $u$  constructed in (3.2) is such that  $|u(t) - x_0| \leq M|t - t_0|$ , as is immediately seen by applying the theorem of the mean. If the rectangle

$$|t - t_0| \leq a, \quad |x - x_0| \leq b$$

is contained in  $D$ , one can then be certain that the approximate solution  $u$  will be defined in the interval  $|t - t_0| \leq c - h$ , where  $c = \inf(a, b/M)$  (Fig. 74).

If nothing else is known about the geometry of  $D$ , it is not possible to improve this bound; but in many cases  $u$  can be defined in a much larger interval.

(3.4) Having in this way obtained an approximate solution within  $\varepsilon$  for every  $\varepsilon > 0$ , it is natural to ask whether, when  $f$  is a Lipschitz function, one can deduce the existence of a “true” solution in the interval where the approximate solution is defined, or at least in a smaller interval, and how much this solution differs from the approximate solution obtained. The natural idea is to give to  $\varepsilon$  a sequence of values tending to 0, and to “pass to the limit” with the help of (2.3). However one must be sure that the approximate solutions thus constructed are all defined in a fixed interval. This will be the case if this interval is taken to be contained in the interval  $|t - t_0| \leq c$  considered in (3.3). However *if in fact an approximate solution to  $v$  is already known*, defined in some open interval  $I$  containing  $t_0$ , one can deduce, from the knowledge of the approximate solution (when

$f$  is a Lipschitz function), an interval where every approximate solution within  $\varepsilon$  is certainly defined (the interval being in general larger than that defined in (3.3)):

(3.5) Suppose that  $f$  satisfies the conditions of (3.1) and is Lipschitz for the constant  $k$ ; let  $v$  be an approximate solution of (1.1.1) within  $\alpha$  ( $\alpha \geq 0$ ), defined in a bounded open interval  $I \subset \text{pr}_1(D)$ . For  $\varepsilon > 0$ , put

$$(3.5.1) \quad \varphi(t) = \frac{\alpha + \varepsilon}{k} (e^{k|t-t_0|} - 1)$$

and let  $J_\varepsilon \subset I$  be the largest interval in which the point  $(t, v(t) + \theta\varphi(t))$  belongs to  $D$  for  $|\theta| \leq 1$ . Then there exists an approximate solution  $u$  within  $\varepsilon$  of (1.1.1), defined in  $J_\varepsilon$ , such that  $(t, u(t)) \in D$  for  $t \in J_\varepsilon$  and  $u(t_0) = v(t_0)$ . Moreover, for every approximate solution  $u$  having these properties

$$(3.5.2) \quad |u(t) - v(t)| \leq \varphi(t)$$

in  $J_\varepsilon$ .

If  $u$  is constructed by the Cauchy-Lipschitz method, it is sufficient to show by induction on  $m$  that if  $u$  is defined for  $t_0 \leq t \leq t_0 + mh$  and satisfies there  $(t, u(t)) \in D$ , and if the interval  $[t_0 + mh, t_1]$ , with  $t_1 \leq t_0 + (m+1)h$ , is still contained in  $J_\varepsilon$ , then the point  $(t, u(t))$  is again in  $D$  for  $t_0 + mh \leq t \leq t_1$ . In the contrary case, there is a smallest number  $t_2$  such that  $t_0 + mh < t_2 < t_1$  and such that  $(t_2, u(t_2)) \notin D$ . (By virtue of the continuity of  $\xi \rightarrow u(\xi)$  in the open interval  $]t_0 + mh, t_1[$  and of the fact that  $D$  is open in  $\mathbf{R}^2$ , the set of points  $\xi$  of this interval such that  $(\xi, u(\xi)) \notin D$  is closed and non-empty in  $\mathbf{R}$ , and it is sufficient to take for  $t_2$  its greatest lower bound.) But by virtue of (2.3), for  $t_0 \leq t < t_2$ ,  $|u(t) - v(t)| \leq \varphi(t)$ , hence by continuity  $|u(t_2) - v(t_2)| \leq \varphi(t_2)$ , which contradicts the hypothesis made on  $J_\varepsilon$ . The validity of (3.5.2) in  $J_\varepsilon$  then follows from (2.3).

The existence and uniqueness of solutions of (1.1.1) under the conditions of (3.5) can now be proved:

(3.6) The hypotheses and notations being those of (3.5), for each  $\varepsilon > 0$ , there exists one and only one solution  $u$  of (1.1.1) defined in  $J_\varepsilon$  satisfying  $u(t_0) = v(t_0)$ , and this solution satisfies (3.5.2) in  $J_\varepsilon$ .

By definition,  $J_{\varepsilon_2} \subset J_{\varepsilon_1}$  for  $\varepsilon_1 < \varepsilon_2$ . Replace  $\varepsilon$  by  $\varepsilon/2^n$  and let  $u_n$  be an approximate solution within  $\varepsilon/2^n$  of (1.1.1), defined in  $J_\varepsilon$  and satisfying  $u_n(t_0) = v(t_0)$ . It follows from (2.3) that

$$|u_{n+1}(t) - u_n(t)| \leq \frac{\varepsilon}{2^{n-1}k} (e^{k|t-t_0|} - 1)$$

for every  $t \in J_\varepsilon$ . The series of general term  $u_{n+1}(t) - u_n(t)$  is therefore normally convergent in  $J_\varepsilon$ , which implies that the sequence  $(u_n)$  converges uniformly in  $J_\varepsilon$  to a limit  $u$  continuous in  $J_\varepsilon$  and such that  $u(t_0) = v(t_0)$ . Moreover, by virtue of (2.2.1)

$$|f(t, u(t)) - f(t, u_n(t))| \leq k|u(t) - u_n(t)|$$

for every  $t \in J_\varepsilon$ . Hence the sequence of functions  $t \rightarrow f(t, u_n(t))$  also converges uniformly in  $J_\varepsilon$  to  $f(t, u(t))$ . Applying (V, 3.4), for every  $t \in J_\varepsilon$ ,  $u(t) = u(t_0) + \int_{t_0}^t f(s, u(s)) ds$ ,

hence the existence of the solution of (1.1.1) satisfying the Cauchy condition  $u(t_0) = v(t_0)$ . The uniqueness follows from (2.3) applied with  $\delta = \varepsilon_1 = \varepsilon_2 = 0$ .

(3.7) In practice, one often has to consider functions  $f$  which are only supposed *continuous and locally Lipschitz* in  $D$ ; this means that for every point  $(t_1, x_1) \in D$ , there is a closed square

$$C_{(t_1, x_1)}: |t - t_1| \leq a(t_1, x_1), \quad |x - x_1| \leq a(t_1, x_1)$$

contained in  $D$  and in which  $f$  is *Lipschitz* (for a constant  $k$  depending on  $(t_1, x_1)$ ) (an example is given by  $f(t, x) = x^2$  in  $\mathbf{R}^2$ ). Since it is then further known that  $f$  is *bounded and uniformly continuous* in  $C_{(t_1, x_1)}(0, 5.6)$ , there exists an open interval  $]t_1 - \alpha, t_1 + \alpha[$  (with  $\alpha < a$ ) and a solution  $u$  of (1.1.1) defined in this interval such that  $u(t_1) = x_1$ . Moreover, if a second solution  $v$  of (1.1.1) is defined in an interval  $]t_1 - \beta, t_1 + \beta[$  and such that  $v(t_1) = x_1$ , it *coincides with  $u$*  in the smaller of the two intervals ((3.2), (3.3) and (3.6)).

This being so, we assume *only* that  $f$  is *continuous and locally Lipschitz* in  $D$ , and start from any point  $(t_0, x_0) \in D$ . If  $J_1, J_2$  are two open intervals containing  $t_0$  and contained in  $\text{pr}_1(D)$ ,  $u_i$  a solution of (1.1.1) defined in  $J_i$  and satisfying  $u_i(t_0) = x_0$  ( $i = 1, 2$ ), it follows again from (2.3) that  $u_1$  and  $u_2$  *coincide* in  $J_1 \cap J_2$ . Otherwise, the set of  $t \in J_1 \cap J_2$  such that  $u_1(s) = u_2(s)$  for  $s \leq t$  has for example a *least upper bound*  $\beta$  belonging to  $J_1 \cap J_2$ , and by continuity then  $u_1(\beta) = u_2(\beta)$ ; but because of the foregoing, there is an interval  $J' \subset J_1 \cap J_2$  containing  $\beta$  and in which  $u_1(t) = u_2(t)$ , which contradicts the definition of the least upper bound. We conclude from this that the *union*  $J_0$  of all open intervals  $J \subset \mathbf{R}$  containing  $t_0$  and in each of which there exists a solution of (1.1.1) satisfying the Cauchy condition (1.1.2) is the *largest* of these intervals and that the solution of (1.1.1) satisfying (1.1.2) in  $J_0$  is *unique*.

It may happen that the initial point  $b$  of  $J_0$  is  $-\infty$  or that its terminal point  $c$  is  $+\infty$ . If for example  $c$  is *finite*, two cases can occur:

(I) The improper integral  $\int_{t_0}^c f(s, u(s)) ds$  is not convergent, and hence, by virtue of (1.2.1), the function  $u(t)$  *does not tend to a finite limit as  $t$  tends to  $c$* .

(II) The improper integral  $\int_{t_0}^c f(s, u(s)) ds$  is *convergent* and the left limit  $u(c-)$  therefore exists and is finite. It can then be stated in addition that the point  $(c, u(c-))$  is a *boundary point* of  $D$ . In the contrary case, there would exist a solution  $w$  of (1.1.1) defined in an interval  $]c - h, c + h[$  and such that  $w(c) = u(c-)$ ; it coincides with  $u$  in the interval  $]c - h, c[$  by virtue of (2.3), and hence the function equal to  $u$  in  $]b, c[$ , to  $w$  in  $]c - h, c + h[$ , is a solution of (1.1.1) in  $]b, c + h[$  satisfying the Cauchy condition (1.1.2), contradicting the definition of  $c$ .

In particular, when  $f$  is *bounded* in  $D$ , the case (I) cannot occur and we therefore have that either  $c = +\infty$  or  $c$  is finite and  $(c, u(c-))$  equal to a boundary point of  $D$ .

*Remarks* (3.8.1) In general it is not possible to determine the largest interval  $J_0$  defined in (3.7) simply by inspection of the equation (1.1.1) and the initial condition (1.1.2). For example, for the equation  $x' = x^2$ , with  $D = \mathbf{R}^2$ , the solution taking the value  $x_0 > 0$  at the point  $t_0 = 0$  is  $x_0/(1 - tx_0)$ , and  $J$  is the interval  $] -\infty, 1/x_0[$ .

(3.8.2) When the function  $f$  is not assumed to be Lipschitz in  $x$ , it may happen that

there are *several* solutions of (1.1.1) in an interval of  $\mathbf{R}$ , all satisfying the Cauchy condition (1.1.2). For example, take  $f(t, x) = 2|x|^{1/2}$ , and for every  $c \geq 0$ , denote by  $u_c$  the continuously differentiable function in  $\mathbf{R}$ , equal to 0 for  $t \leq c$ , to  $(t - c)^2$  for  $t \geq c$ . It is clear that all the functions  $u_c$  satisfy the differential equation  $x' = 2|x|^{1/2}$  and the Cauchy condition  $u_c(0) = 0$ .

#### 4. Extension to differential systems and to differential equations of higher order

(4.1) Let  $n$  be any integer  $> 0$ ,  $D$  an open set in  $\mathbf{R}^{n+1}$ ,  $f_j$  ( $1 \leq j \leq n$ )  $n$  continuous real functions in  $D$ . A solution of the *system of  $n$  differential equations* of the first order

$$(4.1.1) \quad \begin{cases} x'_1 = f_1(t, x_1, \dots, x_n) \\ x'_2 = f_2(t, x_1, \dots, x_n) \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ x'_n = f_n(t, x_1, \dots, x_n) \end{cases}$$

in an open interval  $I \subset \mathbf{R}$  is a system of  $n$  real functions

$$t \rightarrow u_j(t) \quad (1 \leq j \leq n)$$

defined, continuous and differentiable in  $I$ , such that

$$(t, u_1(t), \dots, u_n(t)) \in D$$

for every  $t \in I$  and

$$u'_j(t) = f_j(t, u_1(t), \dots, u_n(t))$$

for every  $t \in I$  and for  $1 \leq j \leq n$ .

It is very convenient to use geometrical language, denoting by  $\mathbf{x}$  the vector  $(x_1, \dots, x_n) \in \mathbf{R}^n$ , and by  $(t, \mathbf{x}) \rightarrow \mathbf{f}(t, \mathbf{x})$  the mapping

$$(t, x_1, \dots, x_n) \rightarrow (f_1(t, x_1, \dots, x_n), \dots, f_n(t, x_1, \dots, x_n))$$

of the open set  $D \subset \mathbf{R} \times \mathbf{R}^n$  into  $\mathbf{R}^n$ . The system (4.1.1) can then be written in the form of a single *vector differential equation*

$$(4.1.2) \quad \mathbf{x}' = \mathbf{f}(t, \mathbf{x})$$

and a *solution* of this equation is a vector mapping

$$t \rightarrow \mathbf{u}(t) = (u_1(t), \dots, u_n(t))$$

of  $I \subset \mathbf{R}$  into  $\mathbf{R}^n$ , continuous and differentiable in  $I$  and such that  $\mathbf{u}'(t) = \mathbf{f}(t, \mathbf{u}(t))$  for every  $t \in I$ . The *Cauchy problem* is here posed in the following way: given a point  $(t_0, \mathbf{x}_0) \in D$  (with  $\mathbf{x}_0 = (x_{01}, \dots, x_{0n}) \in \mathbf{R}^n$ ), solutions are required satisfying in addition the *initial (vector) condition*

$$(4.1.3) \quad \mathbf{u}(t_0) = \mathbf{x}_0$$

in an open interval containing  $t_0$ , which is of course equivalent to the  $n$  “initial (scalar) conditions”  $u_j(t_0) = x_{0j}$  for  $1 \leq j \leq n$ .

The definition of the integral of a vector function (0, 4.4) then enables us to formulate the search for a solution of (4.1.2) satisfying the Cauchy condition (4.1.3) in the following equivalent form:  $\mathbf{u}$  must satisfy in  $I$  the (vector) *integral equation*

$$(4.1.4) \quad \mathbf{u}(t) = \mathbf{x}_0 + \int_{t_0}^t \mathbf{f}(s, \mathbf{u}(s)) ds.$$

The case where  $\mathbf{f}$  is defined in an open set  $D$  of  $\mathbf{R} \times \mathbf{C}^n$  is a particular case of the preceding, since  $\mathbf{C}^n = \mathbf{R}^{2n}$ .

(4.2) The analogy in the notations suggests that to pursue the theory of systems of differential equations we *transcribe* step by step the theory of one scalar equation. We therefore first define the notion of an *approximate solution within  $\varepsilon$*  of (4.1.2) in  $I \subset \mathbf{R}$  as a vector function  $\mathbf{v}: I \rightarrow \mathbf{R}^n$ , continuous in  $I$ , which is the *primitive* in  $I$  of a *piecewise-continuous* vector function  $\mathbf{v}'$  such that  $(t, \mathbf{v}(t)) \in D$  for every  $t \in I$  and

$$(4.2.1) \quad \|\mathbf{v}'(t) - \mathbf{f}(t, \mathbf{v}(t))\| \leq \varepsilon$$

except at the points of discontinuity;  $\|\mathbf{x}\|$  is the *norm* introduced in (I, 1.6.1). We then introduce as in (2.2) the notion of a function  $\mathbf{f}$  (defined and continuous in  $D$ ) *Lipschitz* (in  $\mathbf{x}$ ) for the constant  $k$ , by the condition

$$(4.2.2) \quad \|\mathbf{f}(t, \mathbf{x}_1) - \mathbf{f}(t, \mathbf{x}_2)\| \leq k \|\mathbf{x}_1 - \mathbf{x}_2\|$$

for any points  $(t, \mathbf{x}_1), (t, \mathbf{x}_2)$  of  $D$  having the same first coordinate. By virtue of (I, 3.6.1) this condition is satisfied when  $D$  is convex and the partial derivatives  $\partial f_i / \partial x_j$  ( $1 \leq i \leq n, 1 \leq j \leq n$ ) are *bounded* in  $D$ . This gives the proposition corresponding to (2.3):

(4.3) *Let  $\mathbf{f}$  be a function with values in  $\mathbf{R}^n$ , defined and continuous in an open set  $D \subset \mathbf{R} \times \mathbf{R}^n$  and satisfying (4.2.2). Let  $\mathbf{u}_1, \mathbf{u}_2$  be two functions continuous in an open interval  $I \subset \mathbf{R}$ , which are *primitives* in this interval of *piecewise-continuous* functions, and satisfy in  $I$  the conditions  $(t, \mathbf{u}_1(t)) \in D$  and  $(t, \mathbf{u}_2(t)) \in D$ , as well as the inequalities*

$$(4.3.1) \quad \|\mathbf{u}'_1(t) - \mathbf{f}(t, \mathbf{u}_1(t))\| \leq \varepsilon_1, \quad \|\mathbf{u}'_2(t) - \mathbf{f}(t, \mathbf{u}_2(t))\| \leq \varepsilon_2$$

*except at the points of discontinuity. Let  $t_0 \in I$  and suppose that*

$$(4.3.2) \quad \|\mathbf{u}_1(t_0) - \mathbf{u}_2(t_0)\| \leq \delta.$$

*Then for every  $t \in I$*

$$(4.3.3) \quad \|\mathbf{u}_1(t) - \mathbf{u}_2(t)\| \leq \delta e^{k|t-t_0|} + \frac{\varepsilon}{k} (e^{k|t-t_0|} - 1),$$

*with*

$$\varepsilon = \varepsilon_1 + \varepsilon_2.$$

The proof is *identical* to that of (2.3), replacing everywhere absolute values by norms and using the “vector” version of the theorem of the mean (I, 3.5.4).

(4.4) Assuming  $\mathbf{f}$  bounded and uniformly continuous in  $D$ , a “piecewise-linear”

approximate solution of (4.1.2) can now be constructed by the Cauchy-Lipschitz method developed in (3.1): here  $\mathbf{u}_{m+1}$  will be in  $[t_0 + mh, t_0 + (m+1)h]$  the vector function

$$t \rightarrow \mathbf{u}_m(t_0 + mh) + (t - t_0 - mh)\mathbf{f}(t_0 + mh, \mathbf{u}_m(t_0 + mh)).$$

It is proved as in (3.3) that the construction makes sense at least in an interval  $|t - t_0| \leq c - h$ , when it is supposed that  $D$  contains the "parallelootope"  $|t - t_0| \leq a$ ,  $\|\mathbf{x} - \mathbf{x}_0\| \leq b$ , and when  $c = \inf(a, b/M)$ .

The following result corresponds to (3.5):

(4.5) *Suppose that  $\mathbf{f}$  is bounded, uniformly continuous in  $D$  and Lipschitz for the constant  $k$ ; let  $\mathbf{v}$  be an approximate solution within  $\alpha$  of (4.1.2), defined in a bounded open interval  $I$ . Denote by  $\varphi(t)$  the function (3.5.1), and let  $J_\varepsilon \subset I$  be the largest open interval in which the point  $(t, \mathbf{v}(t) + c\varphi(t))$  belongs to  $D$  for every  $\mathbf{c} \in \mathbf{R}^n$  satisfying  $\|\mathbf{c}\| \leq 1$ . Then there exists an approximate solution  $\mathbf{u}$  within  $\varepsilon$  of (4.1.2), defined in  $J_\varepsilon$  and such that  $\mathbf{u}(t_0) = \mathbf{v}(t_0)$ ; moreover, for every approximate solution  $\mathbf{u}$  having these properties*

$$(4.5.1) \quad \|\mathbf{u}(t) - \mathbf{v}(t)\| \leq \varphi(t)$$

in  $J_\varepsilon$ .

The proof is the same as that of (3.5) replacing absolute values by norms.

The existence and uniqueness of a solution of the Cauchy problem for (4.1.2) is now deduced:

(4.6) *The hypotheses and notations being those of (4.5), for each  $\varepsilon > 0$ , there exists one and only one solution  $\mathbf{u}$  of (4.1.2) defined in  $J_\varepsilon$  and such that  $\mathbf{u}(t_0) = \mathbf{v}(t_0)$ , and this solution satisfies (4.5.1) in  $J_\varepsilon$ .*

Again it is sufficient to argue as in (3.6).

(4.7) Lastly, the existence of a *largest* open interval  $J_0$  in which the solution of the Cauchy problem for (4.1.2) is defined, when  $\mathbf{f}$  is only assumed continuous and locally Lipschitz in  $\mathbf{x}$ , is proved as in (3.7), and the examination of what happens at the end-points of  $J_0$  is carried out in the same way.

(4.8) Given a real function  $f$  continuous in an open set  $D \subset \mathbf{R}^{n+1}$ , a solution of the differential equation of order  $n$

$$(4.8.1) \quad x^{(n)} = f(t, x, x', x'', \dots, x^{(n-1)})$$

in an open interval  $I \subset \mathbf{R}$ , is a real function  $u$  defined, continuous and  $n$  times differentiable in  $I$ , such that

$$(t, u(t), u'(t), \dots, u^{(n-1)}(t)) \in D$$

for every  $t \in I$  and

$$u^{(n)}(t) = f(t, u(t), \dots, u^{(n-1)}(t))$$

for every  $t \in I$ .

The search for the solutions of such an equation can be immediately reduced to that for a *system of  $n$  differential equations of the first order*. If  $u$  is a solution of (4.8.1) in  $I$ , the system of  $n$  functions

$$v_1 = u, \quad v_2 = u', \quad \dots, \quad v_n = u^{(n-1)}$$

is formed of functions continuous and differentiable in  $I$  and is a solution of the system

$$(4.8.2) \quad \begin{cases} y'_1 = y_2 \\ y'_2 = y_3 \\ \dots \\ y'_{n-1} = y_n \\ y'_n = f(t, y_1, y_2, \dots, y_n) \end{cases}$$

Conversely, if  $(v_1, v_2, \dots, v_n)$  is a solution of this system, the function  $u = v_1$  is continuous and  $n$  times differentiable in  $I$  and is a solution of (4.8.1).

The *Cauchy problem* for (4.8.1) is thus by definition the Cauchy problem for the system (4.8.2), i.e. we seek a solution  $u$  of (4.8.1) in an open interval  $I$  containing  $t_0$ , satisfying in addition the *initial conditions*

$$(4.8.3) \quad u(t_0) = x_0, \quad u'(t_0) = x'_0, \quad \dots, \quad u^{(n-1)}(t_0) = x_0^{(n-1)}$$

the point  $(t_0, x_0, x'_0, \dots, x_0^{(n-1)})$  belonging to  $D$ .

Similarly one can reduce to a system of the type (4.1.1) a system of differential equations of order  $\geq 1$  of the form

$$(4.8.4) \quad \begin{aligned} x_1^{(n_1)} &= f_1(t, x_1, \dots, x_1^{(n_1-1)}, \dots, x_r, \dots, x_r^{(n_r-1)}) \\ \dots \\ x_r^{(n_r)} &= f_r(t, x_1, \dots, x_1^{(n_1-1)}, \dots, x_r, \dots, x_r^{(n_r-1)}) \end{aligned}$$

One takes as unknown functions all the functions and all the derivatives figuring in the second members, and obtains a system of

$$n_1 + n_2 + \dots + n_r$$

differential equations of the first order, whose solutions correspond in a one-to-one way with those of (4.8.4).

(4.9) In a vector differential equation (4.1.2) one often has to effect a *change of variable* or a *change of unknown function*. We confine ourselves to the case where  $D = I \times H$ , where  $I$  is an open interval of  $\mathbf{R}$ ,  $H$  an open subset of  $\mathbf{R}^n$ . Let  $I_1$  be an open interval of  $\mathbf{R}$  and  $\varphi: s \rightarrow \varphi(s)$  a continuously differentiable mapping of  $I_1$  into  $I$ ; the transformed equation of (4.1.2) by the change of variable  $t = \varphi(s)$  is the differential equation

$$(4.9.1) \quad \mathbf{x}' = \mathbf{f}(\varphi(s), \mathbf{x})\varphi'(s)$$

where the second member is defined in  $I_1 \times H$ . For every solution  $\mathbf{u}$  of (4.1.2) defined in  $J \subset I$ , the function  $s \rightarrow \mathbf{u}(\varphi(s))$  is a solution of (4.9.1) defined in  $\varphi^{-1}(J) \subset I_1$ ; if the function  $\varphi$  is *bijective* and if its inverse function is continuously differentiable in  $I$ , we deduce conversely a solution of (4.1.2) from every solution of (4.9.1).

For the change of unknown function, we confine ourselves here to the case where  $H = \mathbf{R}^n$  and to *linear* changes. Let  $A(t)$  be a square *invertible* matrix of order  $n$  defined in  $I$ , with real elements, continuously differentiable in  $I$ . The transformed equation of (4.1.2) by the change of unknown function  $y = A(t) \cdot x$  is the differential equation

$$(4.9.2) \quad y' = A(t) \cdot f(t, A^{-1}(t) \cdot y) + A'(t) A^{-1}(t) \cdot y$$

where the second member is defined and continuous in  $\mathbf{R}^n$ . For every solution  $u$  of (4.1.2) defined in  $J \subset I$ , the function  $v: t \rightarrow A(t) \cdot u(t)$  is a solution of (4.9.2) defined in  $J$ , and conversely, if  $v$  is such a solution,  $u: t \rightarrow A^{-1}(t) \cdot v(t)$  is a solution of (4.1.2) defined in  $J$ .

## 5. Differential equations in the complex domain

(5.1) Let  $D$  be an open set in the space  $\mathbf{C}^n$ ; a *complex* function  $f$  defined and continuous in  $D$  is said to be *analytic* (or *holomorphic*) in  $D$  if for each point  $z_0 = (z_{10}, z_{20}, \dots, z_{n0}) \in D$ . For each of the indices  $j$  ( $1 \leq j \leq n$ ), the function

$$z_j \rightarrow f(z_{10}, \dots, z_{j-1,0}, z_j, z_{j+1,0}, \dots, z_{n0})$$

is analytic in a neighbourhood of the point  $z_{j0}$  and if the partial derivatives  $\frac{\partial f}{\partial z_j}(z_1, \dots, z_n)$  (which exist in  $D$  by hypothesis) are continuous in  $D$  (it can be proved that this last condition is a consequence of the others). The usual proof of the theorem of differentiation of composed functions shows that when  $w_1, \dots, w_n$  are analytic functions of a complex variable  $z$  in an open subset  $U$  of  $\mathbf{C}$ , such that

$$(w_1(z), \dots, w_n(z)) \in D$$

for every  $z \in U$ , then the composed function

$$z \rightarrow f(w_1(z), \dots, w_n(z)) = g(z)$$

is analytic in  $U$  and has a derivative given by

$$(5.1.1) \quad g'(z) = \sum_{j=1}^n w'_j(z) \frac{\partial f}{\partial z_j}(w_1(z), \dots, w_n(z)).$$

An analytic function in  $D$  with values in  $\mathbf{C}^r$  is similarly defined by the condition that its  $r$  components are analytic in  $D$ ; the partial derivatives of such a function  $f = (f_k)_{1 \leq k \leq r}$  are the analytic vector functions

$$\frac{\partial f}{\partial x_j} = \left( \frac{\partial f_k}{\partial x_j} \right)_{1 \leq k \leq r}.$$

Suppose that  $D$  contains the *product*  $\Delta$  of  $n$  closed discs

$$|z_j - z_{j0}| \leq r_j \quad (1 \leq j \leq n)$$

and let  $M$  be the least upper bound of the  $n$  continuous functions  $|\partial f / \partial z_j|$  in  $\Delta$ . Then for any complex vectors  $z'$  and  $z''$  in  $\Delta$

$$(5.1.2) \quad |f(z') - f(z'')| \leq nM \|z' - z''\|.$$

Indeed, the point  $\mathbf{z}' + t(\mathbf{z}' - \mathbf{z}'')$  belongs to  $\Delta$  when  $t$  varies in a sufficiently small neighbourhood  $V$  of the segment of endpoints 0 and 1 in  $\mathbf{C}$ . The function

$$t \rightarrow g(t) = f(\mathbf{z}' + t(\mathbf{z}'' - \mathbf{z}'))$$

is thus analytic in  $V$  and in particular  $g$  is continuously differentiable for  $0 \leq t \leq 1$  and

$$g'(t) = \sum_{j=1}^n (z_j'' - z_j') \frac{\partial f}{\partial z_j} (\mathbf{z}' + t(\mathbf{z}'' - \mathbf{z}')).$$

The inequality (5.1.2) thus follows from the formula of the mean.

(5.2) Let  $D$  be an open set in  $\mathbf{C}^{n+1}$ , and consider  $n$  complex functions  $f_j$  ( $1 \leq j \leq n$ ) analytic in  $D$ ; a solution of the *system of  $n$  differential equations of the first order*

$$(5.2.1) \quad \begin{cases} w_1' = f_1(z, w_1, \dots, w_n) \\ w_2' = f_2(z, w_1, \dots, w_n) \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ w_n' = f_n(z, w_1, \dots, w_n) \end{cases}$$

in an open set  $H \subset \mathbf{C}$  is a system of  $n$  complex functions  $z \rightarrow u_j(z)$  ( $1 \leq j \leq n$ ) *analytic* in  $H$ , such that

$$(z, u_1(z), \dots, u_n(z)) \in D$$

for every  $z \in H$  and

$$u_j'(z) = f_j(z, u_1(z), \dots, u_n(z))$$

for every  $z \in H$  and for  $1 \leq j \leq n$ .

Denoting by  $\mathbf{w}$  the complex vector  $(w_1, \dots, w_n) \in \mathbf{C}^n$ ,  $\mathbf{f}(z, \mathbf{w})$  the vector  $(f_1(z, w_1, \dots, w_n), \dots, f_n(z, w_1, \dots, w_n))$ , the system (5.2.1) may also be written

$$(5.2.2) \quad \mathbf{w}' = \mathbf{f}(z, \mathbf{w}),$$

and the Cauchy problem for this vector equation is formulated as in (4.1).

Let  $H$  be an open *simply connected* set in  $\mathbf{C}$ ,  $\mathbf{u}$  a function analytic in  $H$  with values in  $\mathbf{C}^n$ ; if  $\mathbf{u}$  is a solution of (5.2.2) such that  $\mathbf{u}(z_0) = \mathbf{w}_0$  for a point  $z_0 \in H$ , we also have

$$(5.2.3) \quad \mathbf{u}(z) = \mathbf{w}_0 + \int_{\alpha(z)} \mathbf{f}(s, \mathbf{u}(s)) \, ds$$

for every path  $\alpha(z)$  joining  $z_0$  to  $z$  in  $H$  (VII, 5.2). Conversely if a vector function  $\mathbf{u}$  *analytic* in  $H$  is such that  $(z, \mathbf{u}(z)) \in D$  for every  $z \in H$  and satisfies (5.2.3), it is a solution of (5.2.2) satisfying the initial condition  $\mathbf{u}(z_0) = \mathbf{w}_0$ , for the function  $\mathbf{f}(z, \mathbf{u}(z))$  is then analytic in  $H$  (5.1) and  $\mathbf{u}(z)$  is the primitive of this function taking the value  $\mathbf{w}_0$  at the point  $z_0$ .

(5.3) Unexpected difficulties arise when one studies the global behaviour of the solutions of a differential equation (5.2.2) in the complex domain. For example, take  $D = \mathbf{C}$  for the equation  $w' = -\frac{1}{2}w^3$ ; however there exists no analytic solution in  $\mathbf{C}$ , nor even a

meromorphic solution in  $\mathbf{C}$  (all the solutions have "branch points"). Whereas we were able to obtain some information about the behaviour of a solution of a differential equation in the real domain in the whole interval where it was defined (3.7), we shall confine ourselves here to a *local* existence and uniqueness theorem, without attempting to examine the problem of the analytic continuation of such a solution.

(5.4) Let  $D$  be the product of  $n + 1$  open discs

$$|z - z_0| < a, \quad |w_j - w_{j0}| < b \quad (1 \leq j \leq n)$$

in  $\mathbf{C}^{n+1}$ . Let  $(z, \mathbf{w}) \rightarrow \mathbf{f}(z, \mathbf{w})$  be a vector function analytic in  $D$  with values in  $\mathbf{C}^n$ , and suppose that  $\|\mathbf{f}(z, \mathbf{w})\| \leq M$  and  $\|\partial \mathbf{f} / \partial z_j\| \leq A$  ( $1 \leq j \leq n$ ) in  $D$ . Then, in the disc  $\Delta: |z - z_0| < c = \inf(a, b/M)$ , there exists one and only one analytic solution  $\mathbf{u}$  of (5.2.2) such that  $\mathbf{u}(z_0) = \mathbf{w}_0 = (w_{j0})$ .

The general idea of iteration (or successive approximations) (II, 3.1) is utilized here: we shall prove that one can define by induction a sequence of vector functions  $(\mathbf{u}_m)$  analytic in  $\Delta$ , by the conditions

$$(5.4.1) \quad \mathbf{u}_0(z) = \mathbf{w}_0$$

$$(5.4.2) \quad \mathbf{u}_{m+1}(z) = \mathbf{w}_0 + \int_{\alpha(z)} \mathbf{f}(s, \mathbf{u}_m(s)) \, ds \quad \text{for } m \geq 0,$$

$\alpha(z)$  being a path in  $\Delta$  of initial point  $z_0$  and terminal point  $z$  (for example the segment joining these two points); we shall then prove that the sequence  $(\mathbf{u}_m)$  converges *uniformly* in  $\Delta$  to the required solution  $\mathbf{u}$ , and that this solution is unique.

Suppose  $\mathbf{u}_m$  defined and satisfying

$$\|\mathbf{u}_m(z) - \mathbf{w}_0\| < b$$

for every  $z \in \Delta$ ; then the second member of (5.4.2) is defined for every  $z \in \Delta$ , and to continue the induction process it is necessary to prove also that  $\|\mathbf{u}_{m+1}(z) - \mathbf{w}_0\| \leq b$  for every  $z \in \Delta$ . Now, since  $\|\mathbf{f}(z, \mathbf{u}_m(z))\| \leq M$  for every  $z \in \Delta$ , it follows from (VII, 2.1.3) that

$$\left\| \int_{\alpha(z)} \mathbf{f}(s, \mathbf{u}_m(s)) \, ds \right\| \leq M|z - z_0| < b,$$

hence our assertion. On the other hand, by (5.1.2)

$$(5.4.3) \quad \|\mathbf{f}(z, \mathbf{w}') - \mathbf{f}(z, \mathbf{w}'')\| \leq k\|\mathbf{w}' - \mathbf{w}''\|$$

with  $k = nA$ , for any points  $(z, \mathbf{w}')$  and  $(z, \mathbf{w}'')$  in  $D$  having the same first projection. Thus from (5.4.2), for every  $m \geq 1$ ,

$$\begin{aligned} \|\mathbf{u}_{m+1}(z) - \mathbf{u}_m(z)\| &= \left\| \int_{\alpha(z)} (\mathbf{f}(s, \mathbf{u}_m(s)) - \mathbf{f}(s, \mathbf{u}_{m-1}(s))) \, ds \right\| \\ &\leq k \left| \int_{\alpha(z)} \|\mathbf{u}_m(s) - \mathbf{u}_{m-1}(s)\| \, ds \right| \end{aligned}$$

Since

$$\|\mathbf{u}_1(z) - \mathbf{u}_0(z)\| = \|\mathbf{u}_1(z) - \mathbf{w}_0\| = \left\| \int_{\alpha(z)} \mathbf{f}(s, \mathbf{w}_0) \, ds \right\| \leq M|z - z_0|,$$

it is deduced, by induction on  $m$  (taking a segment for  $\alpha(z)$ ), that

$$(5.4.4) \quad \|\mathbf{u}_{m+1}(z) - \mathbf{u}_m(z)\| \leq \frac{Mk^m}{(m+1)!} |z - z_0|^{m+1} \leq \frac{M}{k} \frac{(kc)^{m+1}}{(m+1)!}$$

which proves that the series of general term  $\mathbf{u}_{m+1}(z) - \mathbf{u}_m(z)$  is *normally convergent* in  $\Delta$ . We conclude, because of (5.4.3), that the sequence of analytic vector functions  $z \rightarrow \mathbf{f}(z, \mathbf{u}_m(z))$  is also uniformly convergent in  $\Delta$  to  $z \rightarrow \mathbf{f}(z, \mathbf{u}(z))$ . From (VII, 10.1) the limit  $\mathbf{u}$  of the sequence  $(\mathbf{u}_m)$  is analytic in  $\Delta$ , and from (V, 3.4) for every  $z \in \Delta$ ,

$$\mathbf{u}(z) = \mathbf{w}_0 + \int_{\alpha(z)} \mathbf{f}(s, \mathbf{u}(s)) ds$$

which, by virtue of (5.2), completes the proof of the existence of the solution  $\mathbf{u}$ . The uniqueness can be deduced directly from (4.6) by observing that for each  $\theta$  satisfying  $0 \leq \theta \leq 2\pi$ , the vector function of the *real* variable  $t$

$$\mathbf{v}: t \rightarrow \mathbf{u}(z_0 + te^{i\theta})$$

is a solution of the vector differential equation

$$\mathbf{v}'(t) = e^{i\theta} \mathbf{f}(z_0 + te^{i\theta}, \mathbf{v}(t))$$

in the interval  $] -c, c[$  and satisfies the Cauchy condition  $\mathbf{v}(0) = \mathbf{w}_0$ . One can also argue directly by observing that if  $\mathbf{g}$  is a solution of (5.2.3) in  $\Delta$ , for every integer  $m \geq 0$

$$\begin{aligned} \|\mathbf{u}_{m+1}(z) - \mathbf{g}(z)\| &= \left\| \int_{\alpha(z)} (\mathbf{f}(s, \mathbf{u}_m(s)) - \mathbf{f}(s, \mathbf{g}(s))) ds \right\| \\ &\leq k \left| \int_{\alpha(z)} \|\mathbf{u}_m(s) - \mathbf{g}(s)\| ds \right| \end{aligned}$$

and on the other hand, for  $m = 0$ ,

$$\|\mathbf{u}_0(z) - \mathbf{g}(z)\| = \|\mathbf{w}_0 - \mathbf{g}(z)\| = \left\| \int_{\alpha(z)} \mathbf{f}(s, \mathbf{g}(s)) ds \right\| \leq M|z - z_0|.$$

Hence, as above, by induction on  $m$

$$\|\mathbf{u}_{m+1}(z) - \mathbf{g}(z)\| \leq \frac{Mk^m}{(m+1)!} |z - z_0|^{m+1}$$

and passing to the limit we indeed obtain  $\mathbf{g} = \mathbf{u}$ .

Q.E.D.

(5.5) The problem of *change of variables* or *change of unknown function* in the complex domain is treated as in the real domain (4.9), simply replacing  $\mathbf{R}$  by  $\mathbf{C}$  and  $\mathbf{I}$  by an open subset of  $\mathbf{C}$  and considering only *analytic* functions. One should observe that after a change of variable  $z = \varphi(w)$  a solution of the initial equation can be deduced from a solution of the transformed equation only if  $\varphi$  is *bijective*. In the contrary case, it is convenient to decompose the domains of definition of the variables  $w$  and  $z$  into several others where the condition of bijectivity is fulfilled, and to study in detail the manner of passing from one of these partial domains to another.

## 6. Dependence of the solution relative to the initial conditions and parameters

(6.1) As we have said (5.3), we shall not treat the difficult problem of the analytic continuation of the solutions of a differential equation (5.2.2) in the complex domain. However we shall see that when a solution is supposed *already* known in an open subset of  $\mathbf{C}$ , we can assert the existence in the *same* open set of a solution of a “neighbouring” differential equation satisfying initial conditions “near” those satisfied by the given solution.

(6.2) In the space  $\mathbf{C}^{n+k+1}$ , envisaged as the product space  $\mathbf{C} \times \mathbf{C}^n \times \mathbf{C}^k$ , consider an open set  $D$  and an analytic vector function

$$(z, \mathbf{w}, \mathbf{t}) \rightarrow \mathbf{f}(z, \mathbf{w}, \mathbf{t})$$

defined in  $D$  and with values in  $\mathbf{C}^n$ . We propose to study the solutions of the differential equation

$$(6.2.1) \quad \mathbf{w}' = \mathbf{f}(z, \mathbf{w}, \mathbf{t})$$

depending on the complex vector “parameter”  $\mathbf{t}$ . Suppose  $D$  to be of the form  $V \times \Delta$ , where  $V$  is an open set in  $\mathbf{C}^{n+1}$  and  $\Delta$  is a product of discs  $\|\mathbf{t} - \mathbf{t}_0\| < R$ . Suppose also that there exists a solution  $\mathbf{g}$  of the differential equation

$$(6.2.2) \quad \mathbf{w}' = \mathbf{f}(z, \mathbf{w}, \mathbf{t}_0)$$

defined and analytic in an open set  $H_0 \subset \text{pr}_1(V)$  (with, of course,  $\mathbf{g}(H_0) \subset V$ ). Then let  $F$  be a bounded closed set contained in  $H_0$  and let  $H$  be its interior. Assume  $H$  to be *simply connected*.

This being so, since  $\mathbf{g}(F)$  is a bounded closed subset of  $V$ , there exists a number  $\delta_0 > 0$  such that, for every  $\delta$  satisfying  $0 \leq \delta \leq \delta_0$ , the open set  $U_\delta$  of points  $(\zeta, \mathbf{v}, \mathbf{t}) \in \mathbf{C}^{n+k+1}$  satisfying the conditions

$$(6.2.3) \quad \zeta \in H, \quad \|\mathbf{v} - \mathbf{g}(\zeta)\| + \|\mathbf{t} - \mathbf{t}_0\| < \delta$$

is contained in  $D$ , as also is its boundary; we then prove the following result:

(6.3) *Under the hypotheses of (6.2), there exists a number  $\delta$  such that  $0 < \delta < \delta_0$ , and such that for every point  $(\zeta, \mathbf{v}, \mathbf{t}) \in U_\delta$ , there exists one and only one solution  $z \mapsto \mathbf{u}(z, \zeta, \mathbf{v}, \mathbf{t})$  of (6.2.1), defined in  $H$  and satisfying the initial condition*

$$(6.3.1) \quad \mathbf{u}(\zeta, \zeta, \mathbf{v}, \mathbf{t}) = \mathbf{v}.$$

Moreover, the function  $(z, \zeta, \mathbf{v}, \mathbf{t}) \mapsto \mathbf{u}(z, \zeta, \mathbf{v}, \mathbf{t})$  is analytic (with respect to the  $n+k+2$  complex variables on which it depends) in  $H \times U_\delta$ .

This proposition will be proved by a modification of the process of iteration used in (5.4). It will be seen that by taking  $\delta$  sufficiently small, one can define in  $H \times U_\delta$  a sequence of analytic vector functions  $(z, \zeta, \mathbf{v}, \mathbf{t}) \mapsto \mathbf{u}_m(z, \zeta, \mathbf{v}, \mathbf{t})$ , so that  $(z, \mathbf{u}_m(z, \zeta, \mathbf{v}, \mathbf{t})) \in V$  for every  $m$ , proceeding by induction in the following manner:

$$(6.3.2) \quad \mathbf{u}_0(z, \zeta, \mathbf{v}, \mathbf{t}) = \mathbf{g}(z) + \mathbf{v} - \mathbf{g}(\zeta)$$

and for  $m \geq 0$

$$(6.3.3) \quad \mathbf{u}_{m+1}(z, \zeta, \mathbf{v}, \mathbf{t}) = \mathbf{v} + \int_{\alpha_z} \mathbf{f}(s, \mathbf{u}_m(s, \zeta, \mathbf{v}, \mathbf{t}), \mathbf{t}) \, ds$$

where  $\alpha_z: [0, \lambda] \rightarrow H$  is a polygonal line of initial point  $\zeta$  and terminal point  $z$ . It is known that the integral does not depend on the path joining  $\zeta$  to  $z$ , since  $H$  is simply connected. To prove the existence of the  $\mathbf{u}_m$  we may confine ourselves to allowing  $z$  to vary in a *closed disc*

$H_1 \subset H$  of centre  $z_1$ . Then take for  $\alpha_z$  the juxtaposition of a path *independent of*  $z$ , of initial point  $\zeta$  and terminal point  $z_1$ , and of the segment of initial point  $z_1$  and terminal point  $z$ . Suppose  $\alpha_z$  to be parametrized as a function of  $\xi$ , the latter varying in an interval  $[0, a_z]$  of length bounded by a *fixed* number  $c$  for  $z \in H_1$ , and  $|\alpha'_z(\xi)| = 1$  for  $0 \leq \xi \leq a_z$  except at a finite number of points. In the first place

$$(6.3.4) \quad \|u_0(z, \zeta, \nu, t) - g(z)\| = \|\nu - g(\zeta)\|;$$

therefore, for every  $\delta \leq \delta_0$ , the relation  $(\zeta, \nu, t) \in U_\delta$  implies  $(z, u_0(z, \zeta, \nu, t), t) \in U_\delta$  for all  $z \in H$ . Secondly we have, by definition of  $g$ ,

$$\|u_1(z, \zeta, \nu, t) - u_0(z, \zeta, \nu, t)\| = \left\| \int_{\alpha_z} (f(s, u_0(s, \zeta, \nu, t), t) - f(s, g(s), t_0)) ds \right\|.$$

Since the partial derivatives are *bounded* in the bounded closed union of  $U_\delta$  and its boundary (0, 5.6), there exists a constant  $k > 0$  such that

$$(6.3.5) \quad \|f(z, w', t') - f(z, w'', t'')\| \leq k(\|w' - w''\| + \|t' - t''\|)$$

for two points  $(z, w', t')$ ,  $(z, w'', t'')$  of  $U_{\delta_0}$  (5.1.2). Thus, by (6.3.4), (6.3.5) and the theorem of the mean, for  $0 \leq \xi \leq a_z$  and for  $(\zeta, \nu, t) \in U_\delta$

$$(6.3.6) \quad \|u_1(\alpha_z(\xi), \zeta, \nu, t) - u_0(\alpha_z(\xi), \zeta, \nu, t)\| \leq k \delta \xi.$$

Take  $\delta$  such that

$$(6.3.7) \quad \delta e^{kc} < \delta_0.$$

Then in particular  $\delta(1 + k\xi) < \delta_0$  for  $0 \leq \xi \leq a_z$  and from (6.3.4) and (6.3.6)

$$(\alpha_z(\xi), u_1(\alpha_z(\xi), \zeta, \nu, t), t) \in U_{\delta_0}$$

for  $0 \leq \xi \leq a_z$  and  $(\zeta, \nu, t) \in U_\delta$ . Now prove by induction on  $m$  that

$$(6.3.8) \quad \|u_m(\alpha_z(\xi), \zeta, \nu, t) - u_{m-1}(\alpha_z(\xi), \zeta, \nu, t)\| \leq \delta \frac{k^m \xi^m}{m!}$$

for  $0 \leq \xi \leq a_z$  and  $(\zeta, \nu, t) \in U_\delta$ . By virtue of (6.3.7) and of the inequality

$$1 + \frac{k\xi}{1!} + \frac{k^2\xi^2}{2!} + \dots + \frac{k^m\xi^m}{m!} \leq e^{k\xi} \leq e^{ka_z}$$

this implies that

$$(\alpha_z(\xi), u_m(\alpha_z(\xi), \zeta, \nu, t), t) \in U_{\delta_0}$$

for  $0 \leq \xi \leq a_z$  and for  $(\zeta, \nu, t) \in U_\delta$ . Hence  $u_{m+1}$  can be defined by the formula (6.3.3). The inequality (6.3.8) having been proved for  $m = 1$ , suppose it true for an  $m > 1$  and note that

$$\begin{aligned} & \|u_{m+1}(\alpha_z(\xi), \zeta, \nu, t) - u_m(\alpha_z(\xi), \zeta, \nu, t)\| \\ &= \left\| \int_{0,1}^{\xi} (f(\alpha_z(\eta), u_m(\alpha_z(\eta), \zeta, \nu, t), t) - f(\alpha_z(\eta), u_{m-1}(\alpha_z(\eta), \zeta, \nu, t), t)) \alpha'_z(\eta) d\eta \right\| \end{aligned}$$

Hence, by virtue of (6.3.5) and (6.3.8)

$$\|u_{m+1}(\alpha_z(\xi), \zeta, \nu, t) - u_m(\alpha_z(\xi), \zeta, \nu, t)\| \leq \delta \frac{k^{m+1} \xi^{m+1}}{(m+1)!}$$

which justifies the induction and defines completely the  $\mathbf{u}_m$ . It has been shown moreover that for every  $z \in H_1$  and for  $(\zeta, \mathbf{v}, \mathbf{t}) \in U_\delta$

$$(6.3.9) \quad \|\mathbf{u}_m(z, \zeta, \mathbf{v}, \mathbf{t}) - \mathbf{u}_{m-1}(z, \zeta, \mathbf{v}, \mathbf{t})\| \leq \delta \frac{k^m c^m}{m!}$$

which establishes the *uniform convergence* of the sequence  $(\mathbf{u}_m)$  in  $H_1 \times U_\delta$ . The limit  $\mathbf{u}$  of this sequence is thus analytic in  $H_1 \times U_\delta$  (VII, 10.1): complete the proof as in (5.4).

(6.4) The same proof can be applied in the real domain, taking for  $D$  an open subset of  $\mathbf{R}^{n+1} \times \mathbf{R}^k$  of the form  $V \times \Delta$ , where  $V$  is open in  $\mathbf{R}^{n+1}$  and  $\Delta$  is a product of intervals, replacing the analytic functions by the continuously differentiable functions, and  $H_0$  and  $H$  by open intervals of  $\mathbf{R}$ . Complete the reasoning by invoking (V, 3.4) to pass to the limit in the two members of (6.3.3).

## PROBLEMS

1. Consider the scalar differential equation

$$x' = |x|^{-3/4}x + t \sin\left(\frac{\pi}{t}\right) = f(t, x)$$

where the second member is defined in  $\mathbf{R}^2$ ; take for initial condition  $t_0 = 0$ ,  $x_0 = 0$ , and consider the approximate solutions formed by the Cauchy-Lipschitz procedure for the values  $h = (n + \frac{1}{2})^{-1}$  ( $n$  tending to  $+\infty$ ). Designate by  $u_n(t)$  the approximate solution corresponding to  $n$ .

(a) Show that if  $n$  is *even*

$$u_n(h) = 0, \quad u_n(2h) = h^2 \quad \text{and} \quad u_n(3h) \geq \frac{1}{2}h^{3/2} > (3h)^{3/2}/16.$$

There exists moreover a constant  $c > 0$  such that if for an integer  $m$  satisfying  $mh < c$ , we have  $u_n(mh) > \frac{1}{16}(mh)^{3/2}$ , we also have  $u_n(ph) > \frac{1}{16}(ph)^{3/2}$  for  $mh < ph < c$ . (Reason by induction showing that

$$f(ph, u_n(ph)) > (u_n(ph))^{1/4} - ph > \frac{1}{2}(ph)^{3/8} - ph > \frac{1}{10}(ph)^{3/8}$$

and note that for  $t < c$  we have  $\frac{1}{10}t^{3/8} > \frac{d}{dt}(\frac{1}{16}t^{3/2})$ ).

(b) Reason similarly for  $n$  *odd*, showing that then

$$u_n(mh) < -\frac{1}{16}(mh)^{3/2} \quad \text{for } 3h \leq mh < c.$$

Conclude that the approximate solutions  $u_n(t)$  do not tend to any limit as  $n$  tends to  $+\infty$ .

2. Let  $f(t, x)$  be a continuous function with real values defined in the set  $|t| \leq a$ ,  $|x| \leq b$  of  $\mathbf{R}^2$ , such that  $f(t, x) < 0$  for  $tx > 0$  and  $f(t, x) > 0$  for  $tx < 0$ . Show that the differential equation  $x' = f(t, x)$  satisfying the initial condition  $x(0) = 0$  has for a unique solution the function which is identically zero. (Reason by *reductio ad absurdum*, considering in an interval  $[0, c]$  where a solution  $u$  is defined, the points where  $u$  attains its maximum and its minimum.)

3. Let  $f(t, x)$  be a continuous function with real values, defined in  $\mathbf{R}^2$  by the following conditions:  $f(t, x) = -2t$  for  $x \geq t^2$ ,  $f(t, x) = -2(x/t)$  for  $|x| < t^2$ ,  $f(t, x) = 2t$  for  $x \leq -t^2$ . Let  $(u_n)$  be the sequence of functions defined by  $u_0(t) = t^2$ ,  $u_n(t) = \int_0^t f(s, u_{n-1}(s)) ds$  for  $n \geq 1$ . Show that the sequence  $(u_n(t))$  does not converge for any value of  $t \neq 0$ , although the

differential equation  $x' = f(t, x)$  has a unique solution  $u = 0$  satisfying the initial condition  $u(0) = 0$  (problem 2).

4. Let  $D \subset \mathbf{R} \times \mathbf{R}^n$  be an open set containing a "parallelopete"

$$P: |t - t_0| < a, \quad \|x - x_0\| < b$$

and let  $f$  be a function continuous and locally Lipschitz in  $x$  in  $D$ .

(a) Let  $h(s, z)$  be a function  $\geq 0$  of the real variables  $s, z$ , defined and continuous for  $0 \leq s \leq a$  and  $0 \leq z \leq b$ , such that for every  $s \in [0, a]$ , the mapping  $z \rightarrow h(s, z)$  is increasing. Suppose that in the parallelopete  $P$  we have

$$\|f(t, x)\| \leq h(|t - t_0|, \|x - x_0\|).$$

Let  $\varphi$  be a primitive of a piecewise-continuous function in an interval  $0 \leq t < c$  (with  $c < a$ ), with real values in  $[0, b[$  and such that  $\varphi(0) = 0$  and in  $[0, c[$

$$\varphi'(s) > h(s, \varphi(s))$$

except at the points of discontinuity of  $\varphi'$ . Show that the largest interval  $J_0$ , in which the solution  $u$  of the equation  $x' = f(t, x)$  satisfying the Cauchy condition  $u(t_0) = x_0$  is defined, contains the interval  $]t_0 - c, t_0 + c[$ , and that in this interval

$$\|u(t) - x_0\| \leq \varphi(|t - t_0|).$$

(b) Suppose that  $D$  contains the open set  $I \times \mathbf{R}^n$ , where  $I$  is an open interval in  $\mathbf{R}$ , and that in  $D$ ,  $\|f(t, x)\| \leq h(\|x\|)$ , where  $h(z)$  is a function of the real variable  $z$ , defined, continuous, increasing and  $> 0$  for every  $z \geq 0$  and such that

$$\int_0^{+\infty} \frac{dz}{h(z)} = +\infty.$$

Show that every solution of the equation  $x' = f(t, x)$  is defined in the whole of  $I$ .

5. (a) In the rectangle  $P: |t - t_0| < a, |x - x_0| < b$ , let  $g$  and  $h$  be two real functions continuous and locally Lipschitz in  $x$  satisfying

$$g(t, x) < h(t, x)$$

in  $P$ . Let  $u$  (resp.  $v$ ) be the solution of  $x' = g(t, x)$  (resp.  $x' = h(t, x)$ ) such that  $u(t_0) = x_0$  (resp.  $v(t_0) = x_0$ ), defined for  $t_0 \leq t < t_0 + c$ . Show that  $u(t) < v(t)$  for  $t_0 < t < t_0 + c$  (consider the least upper bound of the set of all  $t$  for which this inequality holds).

(b) With the same notations, suppose only that  $g(t, x) \leq h(t, x)$  in  $P$ . Show then that  $u(t) \leq v(t)$  for  $t_0 \leq t < t_0 + c$  (consider the solution  $v_n$  of  $x' = h(t, x) + (1/n)$  such that  $v_n(t_0) = x_0$ , then let  $n$  tend to  $+\infty$ , using (3.6)).

6. Let  $J$  be the largest interval of initial point 0 in  $\mathbf{R}$  in which there is defined the solution  $u$  of the equation

$$x' = \lambda + \frac{x^2}{1 + t^2} \quad (\lambda \text{ real constant}).$$

(a) Show that if  $\lambda \leq \frac{1}{4}$ , we have  $J = [0, +\infty[$  (use problem 4(a), noting that there then exist real numbers  $c$  such that  $c \leq \lambda + c^2$ ).

(b) Show that if  $\lambda > \frac{1}{4}$ , we have  $J = [0, a[$ , with

$$\sinh \frac{\pi}{2\sqrt{\lambda}} < a < \sinh \frac{\pi}{\sqrt{4\lambda - 1}}$$

(put  $x = y(1 + t^2)^{1/2}$ , and use problem 5).

# Linear differential equations

## 1. Domain of existence of a solution of a linear differential equation

(1.1) A system of  $n$  differential equations of the first order is said to be *linear* if it has the form

$$(1.1.1) \quad \begin{cases} x'_1 = a_{11}(t)x_1 + \cdots + a_{1n}(t)x_n + b_1(t) \\ x'_2 = a_{21}(t)x_1 + \cdots + a_{2n}(t)x_n + b_2(t) \\ \vdots \\ x'_n = a_{n1}(t)x_1 + \cdots + a_{nn}(t)x_n + b_n(t) \end{cases}$$

It is supposed either that the variable  $t$  lies in an open interval  $I \subset \mathbf{R}$  and that the functions  $a_{jk}$  and  $b_k$  are *continuous* in  $I$ , the functions  $x_k$  being real, or that the variable  $t$  lies in an open set  $D \subset \mathbf{C}$  and that the functions  $a_{jk}$  and  $b_k$  are *analytic* in  $D$ , the functions  $x_k$  being complex. In the first case we seek continuously differentiable solutions, in the second case, analytic solutions.

Denote by  $\mathbf{x}$  the vector  $(x_1, \dots, x_n)$  of  $\mathbf{R}^n$  (resp.  $\mathbf{C}^n$ ), by  $\mathbf{b}(t)$  the vector  $(b_1(t), \dots, b_n(t))$  of  $\mathbf{R}^n$  (resp.  $\mathbf{C}^n$ ), by  $A(t)$  the square matrix  $(a_{jk}(t))_{1 \leq j \leq n, 1 \leq k \leq n}$ , a linear transformation of  $\mathbf{R}^n$  into  $\mathbf{R}^n$  (resp. of  $\mathbf{C}^n$  into  $\mathbf{C}^n$ ). The system can thus be written (taking the vectors for matrices with one column and  $n$  rows)

$$(1.1.2) \quad \mathbf{x}' = A(t) \cdot \mathbf{x} + \mathbf{b}(t).$$

Note that the method of (XI, 4.8) shows that a *linear differential equation of order  $n$*

$$(1.1.3) \quad x^{(n)} = a_1(t)x^{(n-1)} + a_2(t)x^{(n-2)} + \cdots + a_n(t)x + b(t)$$

is equivalent to a linear system (1.1.1).

It is often convenient to consider, for an equation (1.1.2), where the vectors  $\mathbf{b}(t)$  and  $\mathbf{x}$  are in  $\mathbf{R}^n$  and  $A(t)$  is a matrix with *real* elements, the equation  $\mathbf{y}' = A(t) \cdot \mathbf{y} + \mathbf{b}(t)$ , where  $\mathbf{y}$  is a *vector of  $\mathbf{C}^n$* ,  $A(t)$  is considered as a matrix of a linear transformation of  $\mathbf{C}^n$  into  $\mathbf{C}^n$ , and  $\mathbf{b}(t)$  as a vector of  $\mathbf{C}^n$ . If  $\mathbf{v}(t)$  is a solution of this “continued” equation, it is clear that if  $\mathbf{u}(t)$  is the vector of  $\mathbf{R}^n$  whose components are the *real parts* of those of the vector  $\mathbf{v}(t)$ , then  $\mathbf{u}$  is a solution of (1.1.2).

(1.2) In the real domain, the linear equations (1.1.2) have the remarkable property of possessing solutions in the *whole interval*  $I$  in which  $A$  and  $\mathbf{b}$  are continuous. Indeed, I

is the union of a sequence of closed intervals  $[a_m, b_m]$ , in each of which the continuous functions  $A$  and  $b$  are *bounded*. Considering  $A$  as a vector function with values in  $\mathbf{R}^{n^2}$ , evidently

$$(1.2.1) \quad \|A(t) \cdot x_1 - A(t) \cdot x_2\| \leq n \|A(t)\| \cdot \|x_1 - x_2\|$$

for any  $x_1, x_2$  in  $\mathbf{R}^n$ . We thus have the conditions for the application of (XI, 4.7). To prove, for example, that the terminal point  $c$  of the largest interval  $J_0 \subset I$  where a solution  $u$  of (1.1.2) is defined is necessarily the terminal point of  $I$ , assume on the contrary that  $c < b_m$  for some index  $m$ . Since  $A$  and  $b$  are bounded in  $[a_m, b_m]$ , we deduce from the relation

$$u(t) = u(t_0) + \int_{t_0}^t (A(s) \cdot u(s) + b(s)) ds$$

and from (1.2.1), that there exists a constant  $k > 0$  such that, for every  $t$  satisfying  $t_0 \leq t < c$

$$(1.2.2) \quad \|u(t)\| \leq k \left( 1 + \int_{t_0}^t \|u(s)\| ds \right).$$

From this inequality and from Gronwall's lemma (XI, 2.3.4)

$$\|u(t)\| \leq k(1 + e^{k(t-t_0)})$$

and hence  $\|u(t)\|$  and  $\|A(t) \cdot u(t) + b(t)\|$  remain *bounded* in the interval  $[t_0, c[$ , which is impossible by virtue of (XI, 3.7) since here  $D = I \times \mathbf{R}^n$ .

(1.3) Similarly, in the complex domain, when  $D$  is open and *simply connected*,  $A$  and  $b$  being analytic in  $D$ , then *every* solution  $u$  of (1.1.2) is defined in the *whole* of  $D$ . With the notations of (XI, 6.3), one can form for every point  $\zeta \in D$ , the sequence of functions analytic in  $D$

$$u_0(z) = w_0$$

$$(1.3.1) \quad u_{m+1}(z) = w_0 + \int_{\alpha_z} (A(s) \cdot u_m(s) + b(s)) ds \quad \text{for } m \geq 0.$$

Here the question of the *existence* of these functions does not arise, the second member of (1.1.2) being defined in the whole of  $D \times \mathbf{C}^n$ . Also with the same notations, for a suitable constant  $k$

$$\|A(\alpha_z(\xi)) \cdot (w' - w'')\| \leq k \|w' - w''\|$$

for *any*  $z$  in a closed disc  $H_1$  with *arbitrary* centre  $z_1 \in D$ , contained in  $D$ , for any  $\xi$  satisfying  $0 \leq \xi \leq a_z$ , and for *any*  $w', w''$  in  $\mathbf{C}^n$ . The reasoning of (XI, 6.3) thus gives, for every  $z \in H_1$  and  $m \geq 1$

$$(1.3.2) \quad \|u_m(z) - u_{m-1}(z)\| \leq \delta \frac{k^m \xi^m}{m!}$$

where  $\delta$  is the (finite) least upper bound of

$$\left\| \int_{\alpha_z} (A(s) \cdot w_0 + b(s)) ds \right\|$$

for  $z \in H_1$ . One concludes as in (XI, 6.3).

(1.4) Let us return to the question of changing the unknown function (XI, 4.9) only to make clear the fact that in a linear equation (1.1.2), if we make the change  $\mathbf{y} = P \cdot \mathbf{x}$ , where  $P$  is an invertible *constant* matrix, the transformed equation can be written

$$(1.4.1) \quad \mathbf{y}' = PA(t)P^{-1} \cdot \mathbf{y} + P \cdot \mathbf{b}(t).$$

By means of such a change of unknown function, one can often reduce to the case where  $A(t)$  has the form

$$\begin{pmatrix} A_1(t) & 0 \\ 0 & A_2(t) \end{pmatrix}$$

$A_1(t)$  and  $A_2(t)$  being square matrices of order  $p$  and  $n - p$  respectively. It is clear that  $\mathbf{x}$  (resp.  $\mathbf{b}(t)$ ) can be decomposed into the form  $\mathbf{x}_1 + \mathbf{x}_2$  (resp.  $\mathbf{b}_1(t) + \mathbf{b}_2(t)$ ) where the first (resp. the second) vector has zero components of index  $> p$  (resp.  $\leq p$ ), and that every solution of (1.1.2) is then of the form  $\mathbf{u}_1(t) + \mathbf{u}_2(t)$ , where  $\mathbf{u}_j$  is a solution of

$$(1.4.2) \quad \mathbf{x}'_j = A_j(t) \cdot \mathbf{x}_j + \mathbf{b}_j(t) \quad (j = 1, 2).$$

## 2. Resolvent matrix of a system of linear differential equations in the real domain

(2.1) We first place ourselves in the real domain and consider for a linear equation (1.1.2), the corresponding *homogeneous* linear equation

$$(2.1.1) \quad \mathbf{x}' = A(t) \cdot \mathbf{x}$$

the matrix  $A(t)$  being defined and continuous in an open interval  $I$ . It is then known (1.2) that for every  $\mathbf{y} \in \mathbf{R}^n$  and every  $s \in I$ , there exists one and only one solution  $t \rightarrow \mathbf{u}(t, s, \mathbf{y})$  of (2.1.1) in  $I$  such that

$$(2.1.2) \quad \mathbf{u}(s, s, \mathbf{y}) = \mathbf{y}.$$

We shall deduce from this that, for  $t$  and  $s$  fixed, the mapping

$$\mathbf{y} \rightarrow \mathbf{u}(t, s, \mathbf{y})$$

of  $\mathbf{R}^n$  into  $\mathbf{R}^n$  is *linear*. It is clear that the vector function  $t \rightarrow \alpha \mathbf{u}(t, s, \mathbf{y}_1) + \beta \mathbf{u}(t, s, \mathbf{y}_2)$  is a solution of (2.1.1) which takes the value  $\alpha \mathbf{y}_1 + \beta \mathbf{y}_2$  for  $t = s$ . The *uniqueness* of the solution of (2.1.1) satisfying this initial condition thus shows that

$$\mathbf{u}(t, s, \alpha \mathbf{y}_1 + \beta \mathbf{y}_2) = \alpha \mathbf{u}(t, s, \mathbf{y}_1) + \beta \mathbf{u}(t, s, \mathbf{y}_2)$$

which proves the assertion. Hence

$$(2.1.3) \quad \mathbf{u}(t, s, \mathbf{y}) = R(t, s) \cdot \mathbf{y}$$

where  $R$  is a matrix depending only on  $t$  and  $s$ . If  $(\mathbf{e}_j)_{1 \leq j \leq n}$  designates the canonical base of  $\mathbf{R}^n$ , the  $j^{\text{th}}$  column of  $R(t, s)$  is by definition  $R(t, s) \cdot \mathbf{e}_j$ , i.e. the solution  $\mathbf{u}_j$  of (2.1.1) such that  $\mathbf{u}_j(s) = \mathbf{e}_j$ . We say that  $R(t, s)$  is the *resolvent matrix* of the equation (2.1.1) (or (1.1.2)).

(2.2) It follows from (2.1.2) that

$$(2.2.1) \quad R(s, s) = I \quad (\text{unit matrix}).$$

Furthermore, for any  $r, s, t$  in  $I$

$$(2.2.2) \quad R(t, s)R(s, r) = R(t, r)$$

since, for every  $\mathbf{y} \in \mathbf{R}^n$ , the function  $t \rightarrow R(t, s) \cdot (R(s, r) \cdot \mathbf{y})$  is the solution of (2.1.1) which takes the value  $R(s, r) \cdot \mathbf{y}$  at the point  $s$ . But  $t \rightarrow R(t, r) \cdot \mathbf{y}$  takes precisely this value at the point  $s$ , so by the uniqueness property  $R(t, s) \cdot (R(s, r) \cdot \mathbf{y}) = R(t, r) \cdot \mathbf{y}$ . Since this is true for every  $\mathbf{y} \in \mathbf{R}^n$ , we obtain (2.2.2). In particular, by virtue of (2.2.1)

$$(2.2.3) \quad R(t, s)R(s, t) = I$$

which proves that the resolvent matrix  $R(s, t)$  is *invertible* and that

$$(2.2.4) \quad R(s, t) = (R(t, s))^{-1}.$$

It is already known that  $t \rightarrow R(t, s)$  is continuously differentiable. The relation (2.2.4) shows that the same is true of  $t \rightarrow R(s, t)$ , and hence

$$(s, t) \rightarrow R(t, s) = R(t, s_0)R(s_0, s)$$

is continuous with respect to the two variables  $s, t$  and has continuous partial derivatives in  $I \times I$ .

A *fundamental system* of solutions of (2.1.1) is by definition a system of  $n$  solutions  $\mathbf{v}_1, \dots, \mathbf{v}_n$  such that, for every  $t \in I$ , the  $n$  vectors  $\mathbf{v}_1(t), \dots, \mathbf{v}_n(t)$  are *linearly independent* in  $\mathbf{R}^n$  (and hence form a *basis* of  $\mathbf{R}^n$ ). Since  $\mathbf{v}_j(t) = R(t, s) \cdot \mathbf{v}_j(s)$  and  $R(s, t)$  is invertible, it is in fact sufficient for  $\mathbf{v}_1(s), \dots, \mathbf{v}_n(s)$  to be linearly independent at *one* point  $s \in I$  in order that the  $n$  solutions  $\mathbf{v}_j$  form a fundamental system. Moreover, if  $V(t)$  is the matrix whose columns are  $\mathbf{v}_1(t), \dots, \mathbf{v}_n(t)$ , the preceding relations show that

$$(2.2.5) \quad R(t, s) = V(t)V(s)^{-1}$$

and knowledge of the fundamental system of solutions is thus equivalent to knowing the resolvent matrix.

It is also said that  $V(t)$  is a *fundamental matrix* of (2.1.1) (or (1.1.2)); all the fundamental matrices are therefore of the form  $V(t)P$ , where  $V(t)$  is one fundamental matrix and  $P$  is a constant invertible matrix.

(2.3) If the function  $t \rightarrow R(t, s) \cdot \mathbf{y}$  is a solution of (2.1.1), writing  $R'(t, s)$  for the derivative of  $t \rightarrow R(t, s)$ , by (0, 4.1.1)

$$R'(t, s) \cdot \mathbf{y} = A(t) \cdot (R(t, s) \cdot \mathbf{y}).$$

Since this holds for every  $\mathbf{y} \in \mathbf{R}^n$

$$(2.3.1) \quad R'(t, s) = A(t)R(t, s).$$

In other words,  $t \rightarrow R(t, s)$  is the unique solution of the linear differential equation

$$(2.3.2) \quad U'(t) = A(t)U(t)$$

for the vector function  $U(t)$  with values in  $\mathbf{R}^{n^2}$ , satisfying the initial condition

$$(2.3.3) \quad U(s) = I.$$

Note that if  $\Delta(t, s) = \det(R(t, s))$

$$(2.3.4) \quad \Delta(t, s) = \exp \left( \int_s^t \text{Tr}(A(\xi)) d\xi \right).$$

Indeed, writing for simplicity  $U(t)$  instead of  $R(t, s)$  and  $\Delta(t)$  instead of  $\Delta(t, s)$ , we have for every  $t \in I$  and every sufficiently small  $h$ ,

$$\Delta(t + h) = \Delta(t) \det(U(t + h)U(t)^{-1})$$

and

$$U(t + h) = U(t) + h \cdot U'(t) + o(h).$$

Hence by virtue of (2.3.2) and the development of a determinant

$$\det(U(t + h)U(t)^{-1}) = 1 + h \cdot \text{Tr}(A(t)) + o(h).$$

Letting  $h$  tend to 0

$$(2.3.5) \quad \Delta'(t) = \Delta(t) \text{Tr}(A(t))$$

which with the relation  $\Delta(s) = 1$  implies (2.3.4).

It follows immediately from the definition of the product of matrices that one obtains *all* the solutions of (2.3.2) by taking the square matrices  $U(t)$  of which each column is a solution of (2.1.1) (these columns do not necessarily form a fundamental system). The determinant  $\det(U(t))$  of such a solution always satisfies (2.3.5), but is *identically zero* when  $U(t)$  is not a fundamental matrix.

(2.4) Knowledge of the resolvent matrix of (2.1.1) (or, what amounts to the same thing, knowing a fundamental system of  $n$  solutions of this equation) enables one to obtain all the solutions of the corresponding *non-homogeneous* equation (1.1.2) by *Lagrange's method*. For such a solution  $\mathbf{v}$  let

$$\mathbf{w}(t) = R(t, s) \cdot \mathbf{v}(t)$$

which is equivalent (2.2.4) to  $\mathbf{v}(t) = R(t, s) \cdot \mathbf{w}(t)$ ; since  $\mathbf{w}$  is continuously differentiable, by (0, 4.1.1)

$$R'(t, s) \cdot \mathbf{w}(t) + R(t, s) \cdot \mathbf{w}'(t) = A(t) \cdot (R(t, s) \cdot \mathbf{w}(t)) + \mathbf{b}(t)$$

which gives, taking into account (2.3.1),

$$\mathbf{w}'(t) = R(t, s) \cdot \mathbf{b}(t)$$

and hence

$$\mathbf{w}(t) = \mathbf{w}(s) + \int_s^t R(s, \xi) \cdot \mathbf{b}(\xi) d\xi.$$

Finally, taking into account (2.2.3) and putting  $\mathbf{w}(s) = \mathbf{x}_0$

$$(2.4.1) \quad \mathbf{v}(t) = R(t, s) \cdot \mathbf{x}_0 + \int_s^t R(t, \xi) \cdot \mathbf{b}(\xi) d\xi$$

which supposing the resolvent matrix known is an explicit expression for the unique solution of (1.1.2) taking the value  $\mathbf{x}_0$  at the point  $s$ .

Denoting by  $V(t)$  any fundamental matrix, this formula can also be written

$$(2.4.2) \quad \mathbf{v}(t) = V(t)V(s)^{-1} \cdot \mathbf{x}_0 + V(t) \cdot \int_s^t V(\xi)^{-1} \cdot \mathbf{b}(\xi) d\xi.$$

When  $I = [t_0, +\infty[$  and it is known that the improper integral

$$\int_s^{+\infty} V(\xi)^{-1} \cdot \mathbf{b}(\xi) d\xi$$

converges, one can also write (2.4.2) in the form

$$(2.4.3) \quad \mathbf{v}(t) = V(t) \cdot \mathbf{c} - V(t) \cdot \int_t^{+\infty} V(\xi)^{-1} \cdot \mathbf{b}(\xi) d\xi$$

with

$$\mathbf{c} = \int_s^{+\infty} V(\xi)^{-1} \cdot \mathbf{b}(\xi) d\xi + \mathbf{x}_0.$$

(2.5) All the preceding work applies in particular to a *linear differential equation of order  $n$*

$$(2.5.1) \quad x^{(n)} + a_1(t)x^{(n-1)} + \dots + a_n(t)x = b(t)$$

where the  $a_j$  and  $b$  are continuous in  $I$ . The method of (XI, 4.8) associates with this equation the equivalent *linear system*

$$(2.5.2) \quad \begin{cases} y'_1 = y_2 \\ y'_2 = y_3 \\ \cdot \quad \cdot \quad \cdot \\ y'_{n-1} = y_n \\ y'_n = -a_n(t)y_1 - \dots - a_1(t)y_n + b(t). \end{cases}$$

In the vector form

$$(2.5.3) \quad \mathbf{y}' = A(t) \cdot \mathbf{y} + \mathbf{b}(t)$$

with

$$(2.5.4) \quad A(t) = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \dots & 0 & 1 \\ -a_n(t) & -a_{n-1}(t) & -a_{n-2}(t) & \dots & -a_1(t) & 0 \end{pmatrix}$$

$\mathbf{b}(t)$  being the vector  $(0, 0, \dots, 0, b(t))$ . The one-to-one correspondence between solutions of (2.5.1) and (2.5.3) consists in associating with every solution  $u$  of (2.5.1) the

(vector) solution  $\mathbf{v} = (u, u', \dots, u^{(n-1)})$  of (2.5.3). A system of  $n$  solutions  $u_1, \dots, u_n$  of the *homogeneous* linear equation

$$(2.5.5) \quad x^{(n)} + a_1(t)x^{(n-1)} + \dots + a_n(t)x = 0$$

is said to be *fundamental* if the corresponding  $n$  solutions

$$\mathbf{v}_j = (u_j, u'_j, \dots, u_j^{(n-1)}) \quad (1 \leq j \leq n)$$

form a fundamental system of solutions of the homogeneous equation  $\mathbf{y}' = A(t) \cdot \mathbf{y}$ . It has been seen that a necessary and sufficient condition for this is that the determinant (called the *Wronskian* of  $u_1, \dots, u_n$ )

$$(2.5.6) \quad W(t) = \begin{vmatrix} u_1(t) & u_2(t) & \dots & u_n(t) \\ u'_1(t) & u'_2(t) & \dots & u'_n(t) \\ \dots & \dots & \dots & \dots \\ u_1^{(n-1)}(t) & \dots & \dots & u_n^{(n-1)}(t) \end{vmatrix}$$

be non-zero at *one* point  $s \in I$ , in which case it is not zero at any point of  $I$  and is given (2.3.4) by the formula

$$(2.5.7) \quad W(t) = W(s) \exp \left( - \int_s^t a_1(\xi) d\xi \right)$$

since  $\text{Tr}(A(t)) = -a_1(t)$ .

(2.6) An equivalent condition for  $u_1, \dots, u_n$  to constitute a fundamental system of solutions of (2.5.5) is that these functions be *linearly independent*, i.e. that there exist no system of  $n$  (real) *constants*  $\lambda_1, \dots, \lambda_n$  *not all zero* for which

$$(2.6.1) \quad \lambda_1 u_1(t) + \dots + \lambda_n u_n(t) = 0$$

for *every*  $t \in I$ . Indeed, if there were such constants  $\lambda_j$  not all zero, then by differentiation the identities

$$\lambda_1 u_1^{(k)}(t) + \dots + \lambda_n u_n^{(k)}(t) = 0$$

would apply for  $k \leq n-1$ , and hence  $\lambda_1 \mathbf{v}_1(t) + \lambda_2 \mathbf{v}_2(t) + \dots + \lambda_n \mathbf{v}_n(t) = 0$  for the  $n$  solutions of  $\mathbf{y}' = A(t) \cdot \mathbf{y}$  corresponding to the  $u_j$ ; the converse is evident.

It can also be said that the solutions of (2.5.5) form a *vector space of dimension  $n$*  over  $\mathbf{R}$ .

It follows then from the general theory that if  $u_1, \dots, u_n$  is a fundamental system of solutions of (2.5.5), every solution  $u$  of this equation can be written uniquely in the form

$$(2.6.2) \quad u(t) = \lambda_1 u_1(t) + \dots + \lambda_n u_n(t)$$

where the  $\lambda_j$  are constants. The unique solution satisfying a system of initial conditions

$$u^{(j)}(s) = \alpha_j \quad (0 \leq j \leq n-1)$$

is obtained by solving the system of linear equations

$$\lambda_1 u_1^{(j)}(s) + \dots + \lambda_n u_n^{(j)}(s) = \alpha_j \quad (0 \leq j \leq n-1)$$

whose determinant is the Wronskian  $W(s)$ .

Note that a solution  $u$  of (2.5.5) can of course be zero at a point  $s \in I$  without being identically zero; but if at the same time

$$u(s) = u'(s) = \dots = u^{(n-1)}(s) = 0$$

then  $u$  is identically zero in  $I$ , as already follows from the general uniqueness theorem (XI, 4.7).

(2.7) In all the results of this section  $\mathbf{R}^n$  can be replaced by  $\mathbf{C}^n$ , the matrices which occur then being square matrices of order  $n$  with *complex* elements, the latter being functions of the *real* variable  $t \in I$ .

### 3. Linear differential equations with constant coefficients

(3.1) Consider in particular a vector linear differential equation

$$(3.1.1) \quad \mathbf{x}' = A \cdot \mathbf{x} + \mathbf{b}(t)$$

where  $A$  is a *constant* matrix; it is convenient here to consider the case where  $\mathbf{x}$  and  $\mathbf{b}(t)$  are vectors of  $\mathbf{C}^n$ ,  $A$  a matrix with *complex* elements (1.1). The *resolvent matrix* of (3.1.1) has the form

$$(3.1.2) \quad R(t, s) = e^{(t-s)A}$$

(VI, 4.7), for it is clear that this matrix satisfies the conditions

$$U'(t) = AU(t) \quad \text{and} \quad U(s) = I$$

(VI, 8.7.12).

(3.2) The matrix (3.1.2) is easily made explicit when  $A$  is reduced to Jordan canonical form. Recall that there exists an invertible matrix  $P$  (with complex elements) such that

$$PAP^{-1} = \begin{pmatrix} J_1 & 0 & \dots & 0 \\ 0 & J_2 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \dots & J_r \end{pmatrix}$$

where each of the Jordan matrices  $J_h$  is a square matrix of order  $\nu_h$  (with  $\sum_h \nu_h = n$ ) of the form  $\lambda_h I + N$ , with

$$(3.2.1) \quad N = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

the  $\lambda_n$  being the *eigenvalues* of  $A$ , roots (in general complex) of the *characteristic equation*

$$(3.2.2) \quad \det(\lambda I - A) = 0$$

(remember that several matrices  $J_h$  may correspond to the same root). We can thus (1.4) confine our attention (by a linear change of unknowns) to the case where  $A = \lambda I + N$  is already a Jordan matrix of order  $n$ . It is then easily verified that

$$(3.2.3) \quad e^{tA} = e^{\lambda t} \begin{pmatrix} 1 & t & \frac{t^2}{2!} & \cdots & \frac{t^{n-1}}{(n-1)!} \\ 0 & 1 & t & \cdots & \frac{t^{n-2}}{(n-2)!} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

(3.3) In particular this can be applied to the *linear equations of order  $n$  with constant coefficients*

$$(3.3.1) \quad x^{(n)} + a_1 x^{(n-1)} + \cdots + a_n x = b(t) \quad (a_h \in \mathbf{C}).$$

Applying the method of (2.5), the characteristic equation of the matrix (2.5.4) is

$$(3.3.2) \quad \lambda^n + a_1 \lambda^{n-1} + \cdots + a_n = 0.$$

Indeed, writing  $A \cdot x = \lambda x$  for a *non-zero* vector  $x = (x_1, \dots, x_n)$ , the following relations are found:

$$\begin{aligned} x_2 &= \lambda x_1, & x_3 &= \lambda x_2, & \dots, & & x_n &= \lambda x_{n-1} \\ & - a_n x_1 - a_{n-1} x_2 - \cdots - a_1 x_n & & & & & &= \lambda_n \end{aligned}$$

which implies that  $x_1 \neq 0$  and  $(\lambda^n + a_1 \lambda^{n-1} + \cdots + a_{n-1} \lambda + a_n) x_1 = 0$ , hence our assertion. Furthermore, taking into account (2.5), the reasoning of (3.2) shows that if  $\lambda_1, \lambda_2, \dots, \lambda_m$  are the *distinct* roots of (3.3.2), the solutions of the *homogeneous* equation corresponding to (3.3.1) are linear combinations of the functions

$$(3.3.3) \quad t^k e^{\lambda_j t}$$

where  $1 \leq j \leq m$ , and where for each  $j$ ,  $k$  takes at most  $\mu_j$  values  $0, 1, \dots, \mu_j - 1$ , where  $\mu_j$  is the *order of multiplicity* of the root  $\lambda_j$  of (3.3.2). But in fact linear algebra shows that  $k$  must take *all* these values, otherwise the solutions of the homogeneous equation corresponding to (3.3.1) would be linear combinations of *less* than  $n$  functions (since  $\sum_{j=1}^m \mu_j = n$ ). Actually one can easily verify directly that the functions (3.3.3) are solutions for *every*  $k < \mu_j$  (observe that this implies that in the Jordan canonical form of the constant matrix  $A$  of the particular type (2.5.4), two Jordan matrices cannot correspond to the same root, which is of course not the case for any matrix  $A$ ). The  $n$  functions (3.3.3) thus form a fundamental system of solutions to the homogeneous equation corresponding to (3.3.1).

#### 4. Linear differential systems with periodic coefficients

(4.1) Consider a vector homogeneous linear differential equation

$$(4.1.1) \quad \mathbf{x}' = A(t) \cdot \mathbf{x}$$

where  $A(t)$  is a complex matrix of order  $n$  continuous in the whole of  $\mathbf{R}$  and *periodic* of real period  $\omega \neq 0$ , in other words  $A(t + \omega) = A(t)$  for every  $t \in \mathbf{R}$ .

(4.2) (Floquet's theorem). *For every matrix  $V(t)$  whose columns form a fundamental system of solutions of (4.1.1), there exist a constant matrix  $B$  and an invertible matrix  $P(t)$ , continuously differentiable in  $\mathbf{R}$  and periodic of period  $\omega$ , such that*

$$(4.2.1) \quad V(t) = P(t) e^{tB}.$$

Furthermore the matrix  $C = e^{\omega B}$  is determined up to similarity by the equation (4.1.1).

It is clear that the function  $t \rightarrow V(t + \omega)$  is also a solution of the equation  $U'(t) = A(t)U(t)$ , and as  $V(t + \omega)$  is an invertible matrix, then necessarily

$$V(t + \omega) = V(t)C$$

where  $C$  is an invertible constant matrix (2.3). When the chosen fundamental system is replaced by another,  $V(t)$  is replaced by  $V(t)Q$ , where  $Q$  is an invertible constant matrix, and  $C$  is then replaced by  $Q^{-1}CQ$ , and so is determined up to similarity. We know that there exists at least one complex matrix  $B$  such that  $e^{\omega B} = C$  (VIII, 9.9). Putting  $P(t) = V(t)e^{-tB}$

$$P(t + \omega) = V(t + \omega) e^{-\omega B} e^{-tB} = V(t) e^{-tB} = P(t)$$

for any  $t$ , which completes the proof.

(4.3) If the characteristic polynomial of  $B$  is  $\prod_{j=1}^n (\lambda - \beta_j)$ , that of  $e^{\omega B}$  is  $\prod_{j=1}^n (\lambda - \lambda_j)$  with  $\lambda_j = e^{\omega \beta_j}$  and is entirely determined by the equation (4.1.1). The  $\lambda_j$  are said to be the *multipliers* of the equation (4.1.1). Note that

$$\lambda_1 \lambda_2 \dots \lambda_n = \det C = \det(V(\omega)V(0)^{-1})$$

and hence because of (2.3.4)

$$(4.3.1) \quad \lambda_1 \lambda_2 \dots \lambda_n = \exp \left( \int_0^\omega \text{Tr}(A(t)) dt \right).$$

If it is desired to make the matrix  $V(t)$  more explicit, one can reduce to the case where  $B$  has Jordan canonical form. It then follows from (3.2.3) that one obtains a fundamental system of solutions of (4.1.1) of which each has the form

$$e^{t\beta} (t^m \mathbf{p}_0(t) + t^{m-1} \mathbf{p}_1(t) + \dots + \mathbf{p}_m(t))$$

where the  $\mathbf{p}_j(t)$  are periodic functions of period  $\omega$  with values in  $\mathbf{C}^n$ . In particular, for each of the *distinct* eigenvalues  $\beta_j$  of  $B$ , there is a solution of (4.1.1) of the form  $e^{t\beta_j} \mathbf{p}(t)$ , where  $\mathbf{p}$  is periodic of period  $\omega$ . For this solution to have a period which is a *multiple* of  $\omega$ , it is necessary and sufficient that the corresponding multiplier  $\lambda_j = e^{\omega \beta_j}$  be a *root of unity*.

## 5. Linear differential equations in the complex domain

(5.1) Consider now the complex domain, supposing that in the linear equation

$$(5.1.1) \quad \mathbf{w}' = A(z) \cdot \mathbf{w} + \mathbf{b}(z).$$

$A$  and  $\mathbf{b}$  are *analytic* in an open *simply connected* set  $D \subset \mathbf{C}$ ,  $\mathbf{b}$  taking its values in  $\mathbf{C}^n$  and  $A$  in the space of matrices of order  $n$  with complex elements (identified with  $\mathbf{C}^{n^2}$ ). All the statements of no. 2 remain valid replacing  $\mathbf{R}$  by  $\mathbf{C}$ ,  $I$  by  $D$  and the continuous functions by the analytic functions: it simply suffices to invoke (1.3) in place of (1.2). Naturally the linear independence of the solutions is relative to the linear combinations with complex coefficients. Taking  $A$  equal to a *constant* matrix (complex) and  $D = \mathbf{C}$ , one can then extend all the results of no. 3. Finally, concerning linear systems with periodic coefficients, one can always suppose, by a linear change of the variable  $z$ , that the period  $\omega$  is *real*, and this time it is necessary to take for  $D$  a "strip"  $a < \mathcal{I}z < b$  invariant under real translations. Under these conditions Floquet's theorem (4.2) remains valid without change.

(5.2) The study of linear systems in the complex domain becomes much more difficult as soon as open sets  $D$  which are *not simply connected* are considered. Difficulties occur even when considering a disc  $\Delta$  with its centre deleted, which is to say that  $A$  has an *isolated singularity* (VIII, 2.1), which, by translation, may be taken to be the point  $z = 0$ . For example, consider the scalar linear equations of the first order

$$w' = \frac{\alpha}{z} w, \quad w' = -\frac{1}{z^2} w.$$

For the first, there is in general no analytic solution in  $\Delta - \{0\}$ , the solutions  $cz^\alpha$  being only definable in a "cut plane" if  $\alpha$  is not an integer (in other words  $z = 0$  is a "branch point"). For the second, the solutions  $ce^{1/z}$  are defined in  $\Delta - \{0\}$  but have an *essential singularity* at  $z = 0$ , although  $A(z)$  only has a double pole.

The appearance of "branch points" for the resolvent matrix  $R(z)$  at the isolated singularities of  $A(z)$  is a general phenomenon which is directly connected with Floquet's theorem:

(5.3) Let  $\Delta$  be an open disc of centre 0 in  $\mathbf{C}$ , and suppose that  $A(z)$  is analytic in  $\Delta - \{0\}$  (in other words, 0 is an *isolated singularity* of  $A(z)$ ). Let  $\Delta_0$  be the disc  $\Delta$  cut along the negative real axis (VIII, 8.4). Then, in  $\Delta_0$ , every fundamental matrix  $V(z)$  is of the form

$$(5.3.1) \quad V(z) = S(z) e^{B \cdot \log z}$$

where  $S(z)$  is the restriction to  $\Delta_0$  of an invertible matrix analytic in  $\Delta - \{0\}$  and  $B$  is a constant matrix.

It is sufficient to make the change of variable  $z = e^{iu}$ ; if  $r$  is the radius of  $\Delta$ , corresponding to  $\Delta_0$  we have the open set  $L$ :

$$-\pi < \Re u < \pi, \quad -\log r < \Im u < +\infty$$

(VIII, 9.3). To the equation (5.1.1) there corresponds the linear equation in  $L$

$$(5.3.2) \quad \frac{d\mathbf{w}_1}{du} = i e^{iu} A(e^{iu}) \cdot \mathbf{w}_1$$

where  $\mathbf{w}_1(u) = \mathbf{w}(e^{iu})$ .

But the matrix  $A_1(u) = i e^{iu} A(e^{iu})$  is analytic in the whole of the *half-plane*  $D$ :  $-\log r < \mathcal{I}u < +\infty$ , and *periodic* of period  $2\pi$ . Every fundamental matrix of (5.3.2) in  $L$  is thus of the form

$$(5.3.3) \quad V_1(u) = P(u) e^{uB}$$

where  $B$  is a constant matrix and  $P$  is analytic in  $D$  and *periodic* of period  $2\pi$ . Reverting to the variable  $z$ , it is deduced (VIII, 9.8.3) that  $P(u) = S(e^{iu}) = S(z)$ , where  $S(z)$  is analytic in  $\Delta - \{0\}$ ; replacing  $u$  by  $1/i \log z$  in (5.3.3), the theorem is obtained.

It was seen in the examples of (5.2) that essential singular points can occur in a fundamental matrix, even when  $A(z)$  has only a *double* pole. There is again here a general fact, for when  $A(z)$  has only a *simple* pole, theorem (5.3) can be made more precise as follows:

(5.4) *If, in the conditions of (5.3), we have  $A(z) = (1/z)C(z)$ , where  $C(z)$  is analytic in  $\Delta$ , then the point 0 is at most a pole of  $S(z)$ .*

It is enough to prove that there exist an integer  $m > 0$  and a constant  $c > 0$  such that  $\|S(z)\| \leq c|z|^{-m}$  in  $\Delta - \{0\}$  (VIII, 3.3). Now, by a linear change of unknown, we can always suppose that the matrix  $B$  of (5.3.1) is in Jordan canonical form. The expression (3.2.3) for the exponential of a Jordan matrix then immediately shows that if  $\beta_j$  ( $1 \leq j \leq l$ ) are the eigenvalues of  $B$ , and  $N$  an integer  $> 0$  such that  $-N < \Re \beta_j$  for every  $j$ , we have in  $L$ , since  $|\Re u| < \pi$ ,

$$(5.4.1) \quad \|e^{-uB}\| \leq a|u|^n e^{N \cdot \mathcal{I}u} \leq b \cdot e^{(N+1)\mathcal{I}u}$$

where  $a$  and  $b$  are constants. Since from (5.3.3)

$$S(e^{iu}) = P(u) = V_1(u) e^{-uB},$$

it is seen that it is sufficient to prove the existence of a constant  $M > 0$  such that in  $L$

$$(5.4.2) \quad \|V_1(u)\| \leq e^{M \cdot \mathcal{I}u}.$$

Now, by virtue of (5.3.2), for  $u = s + it$  with  $-\pi \leq s \leq \pi$ ,  $t \geq t_0 > -\log r$

$$\mathbf{w}_1(s + it) = \mathbf{w}_1(s + it_0) - \int_{t_0}^t C_1(s + i\xi) \cdot \mathbf{w}_1(s + i\xi) d\xi$$

where the matrix  $C_1(u) = C(e^{iu})$  is *bounded* in  $D$ . We conclude that

$$\|\mathbf{w}_1(s + it)\| \leq \|\mathbf{w}_1(s + it_0)\| + k \int_{t_0}^t \|\mathbf{w}_1(s + i\xi)\| d\xi$$

for some constant  $k > 0$  independent of  $s$  and  $t$ . Gronwall's lemma (XI, 2.3.4) then gives the majorization

$$\|\mathbf{w}_1(s + it)\| \leq \|\mathbf{w}_1(s + it_0)\| e^{k(t-t_0)}$$

for every solution of (5.3.2). But for one such solution,  $\|\mathbf{w}_1(s + it_0)\|$  is bounded when  $s$  varies between  $-\pi$  and  $\pi$ ; applying this to the  $n$  columns of  $V_1(u)$ , a majorization of the type (5.4.2) is obtained as required.

It remains, when  $A(z)$  is given, to obtain explicitly the matrix  $B$  (which is in fact not completely determined, since if we add to it for example a matrix of the form  $2ki\pi \cdot I$

( $k$  integer), this replaces  $S(z)$  by  $z^{-k}S(z)$ , which does not change the type of singularity of  $S(z)$  at the point 0). In the case where  $A(z)$  has a pole of the first order, one can also propose determining the Laurent development of  $S(z)$  (see Chap. XIV in the particular case of linear equations of the second order).

### PROBLEMS

1. Let  $F(X) = a_0X^n + a_1X^{n-1} + \cdots + a_n$  be a polynomial with complex coefficients. Denote by  $F(D)w$  the linear combination  $\sum_{j=0}^n a_j w^{(n-j)}$  for a function  $w$ ,  $n$  times continuously differentiable defined in  $\mathbf{R}$ .

(a) Let  $G, H$  be two relatively prime polynomials such that  $F = GH$ ; show that every solution  $u$  of the equation  $F(D)w = 0$  can be written in one way only in the form  $u_1 + u_2$ , where  $G(D)u_1 = 0$  and  $H(D)u_2 = 0$  (use the fact that there are two polynomials  $P, Q$  such that  $PG + QH = 1$ , and deduce from this that

$$u = P(D)G(D)u_1 + Q(D)H(D)u_2.$$

(b) Deduce from (a) a new proof of the results of (3.3), by decomposing  $F(X)$  into a product of polynomials of the form  $(X - \lambda_j)^{n_j}$ .

(c) Let

$$\frac{1}{F(X)} = \sum_{j=1}^q \sum_{h=1}^{n_j} \frac{\alpha_{jh}}{(X - \lambda_j)^h}$$

be the decomposition of the rational function  $1/F(X)$  into partial fractions. Show that for every continuous function  $b(t)$ , the function

$$\sum_{j=1}^q \sum_{h=1}^{n_j} \alpha_{jh} \int_{t_0}^t \frac{(t-s)^{h-1}}{(h-1)!} e^{\lambda_j(t-s)} b(s) ds$$

is a solution of the equation  $F(D)w = b(t)$ .

2. Let  $A(t)$  be a matrix of order  $n$  whose elements are continuous functions of  $t$  in an open interval  $I \subset \mathbf{R}$ .

(a) Show that the solution  $U(t)$  of the linear equation  $X' = A(t)X$  which satisfies the initial condition  $U(t_0) = I$ , is an invertible matrix in  $I$  and that its inverse is a solution of the equation  $X' = -XA(t)$  (if  $V(t)$  is the solution of the latter equation satisfying  $V(t_0) = I$ , consider the equations satisfied by  $UV$  and  $VU$ ). Deduce that for every square matrix  $C$  of order  $n$ , the solution of  $X' = A(t)X$  taking the value  $C$  at the point  $t_0$  is  $U(t)C$ .

(b) Let  $B(t)$  be a second matrix of order  $n$  continuous in  $I$ , and let  $U(t)$  and  $V(t)$  be the solutions of  $X' = A(t)X$  and  $X' = XB(t)$  taking the value  $I$  at the point  $t_0$ . Show that the solution of the equation

$$X' = A(t)X + XB(t)$$

equal to  $C$  at the point  $t_0$  is equal to  $U(t)CV(t)$ .

(c) Let  $A(t), B(t), C(t), D(t)$  be four matrices of order  $n$  continuous in  $I$ , and let  $(U(t), V(t))$  be a solution of the system of two linear matrix equations

$$X' = A(t)X + B(t)Y, \quad Y' = C(t)X + D(t)Y.$$

Show that if  $V(t)$  is invertible in  $\mathbf{I}$ ,  $W(t) = U(t)V(t)^{-1}$  is a solution of the equation

$$(*) \quad Z' = B(t) + A(t)Z - ZD(t) - ZC(t)Z$$

("Riccati's equation"), and conversely.

(d) Let  $W(t)$  and  $W_1(t)$  be two solutions of the equation (\*); show that if  $W(t) - W_1(t)$  is invertible in  $\mathbf{I}$ ,  $W(t) - W_1(t)$  can be expressed with the help of the solutions of the equations

$$X' = -(D(t) + C(t)W_1(t))X \quad \text{and} \quad X' = X(A(t) - W_1(t)C(t))$$

(consider the equation which  $(W(t) - W_1(t))^{-1}$  satisfies and use (b)).

3. Let  $u_k$  ( $1 \leq k \leq n$ ) be  $n - 1$  times continuously differentiable in an open interval  $\mathbf{I} \subset \mathbf{R}$ .

(a) Show that if the functions  $u_k$  are linearly dependent, the matrix  $(u_k^{(h)}(t))$  ( $0 \leq h \leq n - 1$ ,  $1 \leq k \leq n$ ) is of rank  $< n$  at every point  $t \in \mathbf{I}$ .

(b) Conversely, suppose that for every  $t \in \mathbf{I}$ , the preceding matrix is of rank  $< n$  (which signifies that the Wronskian of the  $u_k$  is identically zero in  $\mathbf{I}$ ). Show that in every non-empty open interval  $\mathbf{J} \subset \mathbf{I}$ , there exists a non-empty open interval  $\mathbf{U} \subset \mathbf{J}$  such that the restrictions of the  $u_k$  to  $\mathbf{U}$  are linearly dependent. (Let  $p$  be the smallest of the numbers  $q < n$  such that the Wronskians of any  $q$  of the functions  $u_k$  are identically zero in  $\mathbf{J}$ . Consider a point  $a \in \mathbf{J}$  where the Wronskian of  $p - 1$  of the functions  $u_k$  is not zero, for example that of  $u_1, \dots, u_{p-1}$ . Show that the  $u_k$  are all solutions of a linear differential equation of order  $p - 1$  in a neighbourhood of  $a$ , the  $u_k$  of indices  $\leq p - 1$  forming a fundamental system of solutions of this equation.)

(c) Let  $u_1(t) = t^2$ , and let  $u_2(t) = t^2$  for  $t \geq 0$ ,  $u_2(t) = -t^2$  for  $t \leq 0$ ; the functions  $u_1$  and  $u_2$  are continuously differentiable in  $\mathbf{R}$  and  $u_1 u_2' - u_2 u_1' = 0$  in  $\mathbf{R}$ , but  $u_1$  and  $u_2$  are not linearly dependent in  $\mathbf{R}$ .

4. Show that the linear system

$$\begin{cases} w_1' = w_2 \\ w_2' = 2z^{-2}w_1 \end{cases}$$

has for fundamental matrix

$$V(z) = \begin{pmatrix} z^2 & z^{-1} \\ 2z & -z^{-2} \end{pmatrix}$$

which has a pole at the point  $z = 0$ , although the matrix  $A(z)$  of the system has a *double* pole at the point  $z = 0$ .

# Perturbations of linear differential systems

## 1. Stability of a solution of a differential equation

(1.1) Let us consider a vector differential equation

$$(1.1.1) \quad \mathbf{x}' = \mathbf{f}(t, \mathbf{x})$$

where  $\mathbf{x} \in \mathbf{R}^n$ ,  $\mathbf{f}$  is a function with values in  $\mathbf{R}^n$ , defined, continuous and locally Lipschitz in an open set  $D \subset \mathbf{R} \times \mathbf{R}^n$  containing the set

$$(1.1.2) \quad t_0 \leq t < +\infty, \quad \|\mathbf{x}\| \leq b.$$

Suppose that there exists a solution  $\mathbf{u}$  of (1.1.1) defined for every  $t \geq t_0$ . This solution is said to be *stable in the interval*  $[t_0, +\infty[$  if it satisfies the following condition:

(1.1.3) For each  $\varepsilon > 0$ , there exists  $\delta \in ]0, b]$  such that for each point  $\mathbf{x}_0 \in \mathbf{R}^n$  satisfying the condition  $\|\mathbf{x}_0 - \mathbf{u}(t_0)\| \leq \delta$ , the unique solution  $\mathbf{v}$  of (1.1.1) such that  $\mathbf{v}(t_0) = \mathbf{x}_0$  is defined in the whole interval  $[t_0, +\infty[$  and satisfies in this interval the inequality

$$(1.1.4) \quad \|\mathbf{v}(t) - \mathbf{u}(t)\| \leq \varepsilon.$$

It is said that  $\mathbf{u}$  is *asymptotically stable* in  $[t_0, +\infty[$  if it satisfies the condition (1.1.3) and if, furthermore, there exists a  $\delta_0 \in ]0, b]$  such that, for each  $\mathbf{x}_0$  satisfying the condition  $\|\mathbf{x}_0 - \mathbf{u}(t_0)\| \leq \delta_0$ , the unique solution  $\mathbf{v}$  of (1.1.1) such that  $\mathbf{v}(t_0) = \mathbf{x}_0$  is defined in the whole interval  $[t_0, +\infty[$  and satisfies the condition

$$(1.1.5) \quad \lim_{t \rightarrow +\infty} \|\mathbf{u}(t) - \mathbf{v}(t)\| = 0.$$

A solution of (1.1.1) defined in  $[t_0, +\infty[$  is said to be *unstable* if it is not stable in this interval.

(1.2) *The example of homogeneous linear equations with constant coefficients.*

The problem of the stability of the solutions of a homogeneous vector equation

$$(1.2.1) \quad \mathbf{x}' = A \cdot \mathbf{x}$$

where  $A$  is a constant matrix of order  $n$  with complex elements and  $\mathbf{x} \in \mathbf{C}^n$ , is solved immediately by examination of the *eigenvalues* of the matrix  $A$ . Indeed, since the difference of two solutions of (1.2.1) is still a solution, one need only study the stability of

the solution 0. Clearly it can be assumed that  $A$  has been reduced to Jordan canonical form. Now the formula (3.2.3) of Chap. XII shows that, for a Jordan matrix  $A = \lambda I + N$ ,  $e^{tA}$  remains *bounded* as  $t$  tends to  $+\infty$  only if:

—either  $\Re \lambda < 0$ ;

—or  $\Re \lambda = 0$  and  $A$  is of order 1.

Moreover,  $e^{tA}$  tends to 0 as  $t$  tends to  $+\infty$  only in the first of these two cases. Consequently:

(1.3) *For the solutions of (1.2.1) to be stable in an interval  $[t_0, +\infty[$ , it is necessary and sufficient that the eigenvalues of  $A$  have their real parts  $\leq 0$ , and that for those whose real part is 0, the corresponding Jordan matrices be of order 1. For the solutions of (1.2.1) to be asymptotically stable, it is necessary and sufficient that the eigenvalues of  $A$  have their real parts  $< 0$ .*

When there are solutions of (1.2.1) stable, but not asymptotically stable, they “oscillate” in the neighbourhood of  $t = +\infty$ , a typical example being that of the scalar equation of the fourth order with real coefficients

$$x^{(iv)} + ax''' + bx'' + cx' + dx = 0$$

with eigenvalues which are simple and purely imaginary conjugates in pairs (the solutions being *not* periodic in general, contrary to the case of the analogous equation of the second order).

## 2. Stability of solutions of equations near linear equations

General results on the stability of solutions of differential equations are known in only a small number of cases. We shall only consider equations of the type

$$(2.1) \quad \mathbf{x}' = A \cdot \mathbf{x} + \mathbf{f}(t, \mathbf{x})$$

obtained from a homogeneous linear equation (1.2.1) by a “small” perturbation, in a sense which it is of course necessary to make precise. We shall consider two cases, according to whether (1.2.1) has stable or asymptotically stable solutions (it will be necessary to make stricter hypotheses on  $\mathbf{f}$  in the first case than in the second case).

(2.2) *Suppose that the solutions of (1.2.1) are stable (in other words, all the eigenvalues of  $A$  have their real parts  $\leq 0$ , and for those which are purely imaginary, the corresponding Jordan matrices are of order 1). Let  $\mathbf{f}$  be a function defined in an open set  $D \subset \mathbf{R} \times \mathbf{C}^n$ , containing the set of points  $(t, \mathbf{x})$  such that*

$$(2.2.1) \quad t_0 \leq t < +\infty, \quad \|\mathbf{x}\| \leq b,$$

*and taking its values in  $\mathbf{C}^n$ ; suppose that  $\mathbf{f}$  is continuous and locally Lipschitz in  $D$  and such that*

$$(2.2.2) \quad \|\mathbf{f}(t, \mathbf{x})\| \leq \gamma(t) \|\mathbf{x}\|$$

*where  $\gamma$  is a function continuous and  $\geq 0$  such that the integral  $\int_{t_0}^{+\infty} \gamma(t) dt$  is convergent. Then there exists a constant  $L > 0$  such that, for every  $t_1 > t_0$ , and every vector  $\mathbf{x}_1 \in \mathbf{C}^n$  satisfying*

$\|\mathbf{x}_1\| \leq L^{-1}b$ , the solution  $\mathbf{u}$  of (2.1) such that  $\mathbf{u}(t_1) = \mathbf{x}_1$  is defined in the whole interval  $[t_1, +\infty[$  and satisfies

$$(2.2.3) \quad \|\mathbf{u}(t)\| \leq L\|\mathbf{x}_1\| \quad \text{for every } t \geq t_1.$$

Furthermore, for each  $\varepsilon > 0$ , there exists a solution  $\mathbf{v}_\varepsilon$  of (1.2.1) and a number  $t_\varepsilon > t_1$  such that, for all  $t \geq t_\varepsilon$

$$(2.2.4) \quad \|\mathbf{u}(t) - \mathbf{v}_\varepsilon(t)\| \leq \varepsilon.$$

To show that  $\mathbf{u}$  is defined in the whole of the interval  $[t_1, +\infty[$ , it is sufficient to show that if  $\mathbf{u}$  is defined in an interval  $[t_1, t_2]$  with  $t_2 > t_1$ , we have  $\|\mathbf{u}(t_2)\| \leq b$  (XI, 4.7). Now, since

$$\mathbf{u}'(t) = A \cdot \mathbf{u}(t) + \mathbf{f}(t, \mathbf{u}(t))$$

for  $t_1 \leq t \leq t_2$ , formula (XII, 2.4.1) gives here

$$(2.2.5) \quad \mathbf{u}(t) = e^{(t-t_1)A} \cdot \mathbf{u}(t_1) + \int_{t_1}^t e^{(t-s)A} \cdot \mathbf{f}(s, \mathbf{u}(s)) \, ds.$$

But by hypothesis, there exists a constant  $K > 0$  such that  $\|e^{tA}\| \leq K$  for every  $t \in \mathbf{R}_+$ . Therefore, for  $t_1 \leq t \leq t_2$ ,

$$\|\mathbf{u}(t)\| \leq nK\|\mathbf{u}(t_1)\| + nK \int_{t_1}^t \gamma(s)\|\mathbf{u}(s)\| \, ds$$

and hence, by virtue of Gronwall's lemma (XI, 2.3.4),

$$\|\mathbf{u}(t)\| \leq L\|\mathbf{u}(t_1)\|$$

with  $L = nK(1 + nKB e^B)$ , where  $B = \int_{t_0}^{+\infty} \gamma(s) \, ds$ , which is finite by hypothesis.

To prove the last assertion, we note that since  $\mathbf{u}$  is bounded, there exists a  $t_\varepsilon > t_1$  such that

$$(2.2.6) \quad \int_{t_\varepsilon}^{+\infty} \gamma(s)\|\mathbf{u}(s)\| \, ds \leq \frac{\varepsilon}{nK}$$

and hence, by virtue of (2.2.2), for every  $t \geq t_\varepsilon$

$$\left\| e^{(t-t_\varepsilon)A} \cdot \int_{t_\varepsilon}^t e^{(t_\varepsilon-s)A} \cdot \mathbf{f}(s, \mathbf{u}(s)) \, ds \right\| = \left\| \int_{t_\varepsilon}^t e^{(t-s)A} \cdot \mathbf{f}(s, \mathbf{u}(s)) \, ds \right\| \leq \varepsilon.$$

Taking into account (2.2.5), it is thus sufficient to take

$$\mathbf{v}_\varepsilon(t) = e^{(t-t_1)A} \cdot \mathbf{x}_\varepsilon$$

with

$$\mathbf{x}_\varepsilon = \mathbf{u}(t_1) + \int_{t_1}^{t_\varepsilon} e^{(t_1-s)A} \cdot \mathbf{f}(s, \mathbf{u}(s)) \, ds.$$

(2.3) Suppose that the solutions of (1.2.1) are asymptotically stable (in other words, all the eigenvalues of  $A$  have their real parts  $< 0$ ). Suppose  $\mathbf{f}$  is defined in an open set  $D \subset \mathbf{R} \times \mathbf{C}^n$  containing the set of points  $(t, \mathbf{x})$  satisfying (2.2.1),  $\mathbf{f}$  being continuous and locally Lipschitz in  $D$ ;

suppose further that for each  $\varepsilon > 0$ , there exists  $\delta > 0$  such that the relations  $t \geq t_0$ ,  $\|\mathbf{x}\| \leq \delta$  imply

$$(2.3.1) \quad \|\mathbf{f}(t, \mathbf{x})\| \leq \varepsilon \|\mathbf{x}\|.$$

Then 0 is an asymptotically stable solution of the equation (2.1); more precisely, if  $\sigma > 0$  is such that all the eigenvalues of  $A$  have their real parts  $< -\sigma$ , then there exists  $\alpha > 0$  such that, for every  $\mathbf{x}_0 \in \mathbf{C}^n$  satisfying  $\|\mathbf{x}_0\| \leq \alpha$ , the solution  $\mathbf{u}$  of (2.1) such that  $\mathbf{u}(t_0) = \mathbf{x}_0$  is defined for  $t \geq t_0$ , and is such that

$$(2.3.2) \quad \lim_{t \rightarrow +\infty} e^{\sigma t} \|\mathbf{u}(t)\| = 0.$$

Choose  $\varepsilon > 0$  such that the real parts of the eigenvalues of  $A$  are  $< -\sigma - 2\varepsilon$ ; then there exists a constant  $k > 0$  such that, for  $t \geq 0$ ,

$$(2.3.3) \quad \|e^{tA}\| \leq k e^{-(\sigma + 2\varepsilon)t}.$$

$\delta > 0$  is now determined such that, for  $t \geq t_0$  and  $\|\mathbf{x}\| \leq \delta$

$$(2.3.4) \quad \|\mathbf{f}(t, \mathbf{x})\| \leq \frac{\varepsilon}{nk} \|\mathbf{x}\|.$$

This being so, let  $\mathbf{u}$  be a solution of (2.1) defined in an interval  $[t_0, t_1]$  such that  $\mathbf{u}(t_0) = \mathbf{x}_0$  with  $\|\mathbf{x}_0\| < \delta$ . To prove that  $\mathbf{u}$  is defined in the whole interval  $[t_0, +\infty[$  (as soon as  $\|\mathbf{x}_0\|$  is sufficiently small), it is sufficient, by virtue of (XI, 4.7), to show that  $\|\mathbf{u}(t)\| < \delta$  for  $t_0 \leq t \leq t_1$ . Assume the contrary, and let  $t_2$  be the smallest value of  $t$  in  $[t_0, t_1]$  such that

$$\|\mathbf{u}(t_2)\| = \delta.$$

Applying the formula (XII, 2.4.1), for  $t_0 \leq t \leq t_2$

$$\mathbf{u}(t) = e^{(t-t_0)A} \cdot \mathbf{x}_0 + \int_{t_0}^t e^{(t-s)A} \cdot \mathbf{f}(s, \mathbf{u}(s)) \, ds.$$

Hence, by virtue of the hypothesis and (2.3.3) and (2.3.4)

$$\|\mathbf{u}(t)\| \leq nk e^{-(\sigma + 2\varepsilon)(t-t_0)} \|\mathbf{x}_0\| + \varepsilon \int_{t_0}^t e^{-(\sigma + 2\varepsilon)(t-s)} \|\mathbf{u}(s)\| \, ds$$

which can also be written, putting  $w(t) = e^{(\sigma + 2\varepsilon)(t-t_0)} \|\mathbf{u}(t)\|$

$$w(t) \leq kn \|\mathbf{x}_0\| + \varepsilon \int_{t_0}^t w(s) \, ds.$$

By Gronwall's lemma (XI, 2.3.4), this gives the majorization

$$\|w(t)\| \leq kn \|\mathbf{x}_0\| e^{\varepsilon(t-t_0)},$$

hence finally

$$(2.3.5) \quad e^{\sigma(t-t_0)} \|\mathbf{u}(t)\| \leq kn \|\mathbf{x}_0\| e^{-\varepsilon(t-t_0)}.$$

This shows first of all that  $\|\mathbf{u}(t)\| \leq kn \|\mathbf{x}_0\|$  for  $t_0 \leq t \leq t_2$ . If we have taken  $\|\mathbf{x}_0\| \leq \alpha = \delta/2kn$ , it is seen that for  $t = t_2$  this contradicts the hypothesis  $\|\mathbf{u}(t_2)\| = \delta$ ; thus, with this

value of  $\alpha$ , we have proved that  $\mathbf{u}$  is defined for *every*  $t \geq t_0$ . Moreover the relation (2.3.2) follows from (2.3.5). Q.E.D.

*Remarks (2.4)* The conclusions of (2.3) are no longer valid if it is only supposed that the real parts of the eigenvalues of  $A$  are  $\leq 0$ . For example, the solution 0 of the scalar equation  $x' = x^2$  is unstable, none of the solutions  $t \rightarrow c/(1 - ct)$  for  $c > 0$  being defined for *every*  $t \geq 0$ .

On the other hand, note that in (2.3) it is not possible to suppose only that  $\|\mathbf{f}(t, \mathbf{x})\| \leq k\|\mathbf{x}\|$  for some constant  $k$ , as is shown by the example of the linear equations with constant coefficients: in the neighbourhood of  $\mathbf{x} = 0$ ,  $\mathbf{f}$  must be "negligible" compared to  $\|\mathbf{x}\|$ . Furthermore, one can never draw conclusions about the behaviour of the solutions of (2.1) when the initial value  $\|\mathbf{x}_0\|$  is too large. For example, for the scalar equation  $x' = -x + x^2$ , where the conditions of (2.3) are satisfied for every  $t_0$ , the solutions

$$u(t) = \frac{c}{c - (c - 1)e^t}$$

are not defined for every  $t \geq 0$  as soon as  $u(0) = c > 1$ .

### 3. Conditional stability

(3.1) When the matrix  $A$  does not have all its eigenvalues such that  $\Re \lambda \leq 0$ , it has been seen in (1.3) that the solution 0 of (1.2.1) is no longer stable. But if *certain* of these eigenvalues have their real parts  $< 0$ , there are nevertheless solutions of (1.2.1) which *tend to 0 with  $1/t$* ; they form a vector space whose dimension  $k$  is the *sum of the orders of multiplicity* of the proper values of  $A$  whose real part is  $< 0$ . We shall see that subject to restrictive hypotheses on the "perturbation"  $\mathbf{f}(t, \mathbf{x})$ , we also obtain *certain* solutions of (2.1) defined for every  $t \geq t_0$  and tending to 0 with  $1/t$ , forming a "family with  $k$  parameters".

By a linear change of unknown function it may evidently be supposed that  $A$  has been reduced to Jordan canonical form. By grouping on the one hand the Jordan matrices corresponding to the proper values with real parts  $< 0$ , and on the other hand the remaining ones, one can write

$$A = \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix}$$

$B$  is a matrix of order  $k$  and the real parts of its eigenvalues are all  $< 0$ ;  $C$  is of order  $n - k$ , the real parts of its eigenvalues being all  $\geq 0$ . It is assumed further that for those which are *purely imaginary*, the corresponding Jordan matrices are *all of order 1*. Then

$$e^{tA} = \begin{pmatrix} e^{tB} & 0 \\ 0 & e^{tC} \end{pmatrix}$$

and the preceding hypotheses imply the existence of constants  $\sigma > 0$ ,  $K > 0$  such that

$$(3.1.1) \quad \|e^{tB}\| \leq K e^{-2\sigma t} \quad \text{for } t \geq 0$$

$$(3.1.2) \quad \|e^{-tC}\| \leq K \quad \text{for } t \geq 0.$$

We shall write  $e^{tA} = U(t) + V(t)$  with

$$(3.1.3) \quad U(t) = \begin{pmatrix} e^{tB} & 0 \\ 0 & 0 \end{pmatrix}, \quad V(t) = \begin{pmatrix} 0 & 0 \\ 0 & e^{tC} \end{pmatrix}$$

so that

$$(3.1.4) \quad \begin{aligned} U(t+s) &= U(t)U(s), & V(t+s) &= V(t)V(s) \\ U(t)V(s) &= V(s)U(t) = 0 \end{aligned}$$

for any  $s$  and  $t$ , and

$$(3.1.5) \quad U'(t) = AU(t), \quad V'(t) = AV(t).$$

(3.2) Suppose that the hypotheses of (3.1) on  $A$  are satisfied; suppose that  $f$  is continuous and locally Lipschitz in an open set  $D \subset \mathbf{R} \times \mathbf{C}^n$  containing the set defined by (2.2.1). Suppose further that  $f(t, 0) = 0$  and that, for each  $\varepsilon > 0$ , there exists  $\delta \in ]0, b[$  such that the relations  $t \geq t_0$ ,  $\|x_1\| \leq \delta$ ,  $\|x_2\| \leq \delta$  imply

$$(3.2.1) \quad \|f(t, x_1) - f(t, x_2)\| \leq \varepsilon \|x_1 - x_2\|.$$

Then there exists  $\alpha > 0$  having the following property: for every vector  $x_0 \in \mathbf{C}^n$  such that  $J \cdot x_0 = 0$ , where

$$J = \begin{pmatrix} 0 & 0 \\ 0 & I_{n-k} \end{pmatrix},$$

and such that  $\|x_0\| \leq \alpha$ , there exists a solution  $u$  of (2.1) defined for  $t \geq t_0$  and satisfying the integral equation

$$(3.2.2) \quad \begin{aligned} u(t) &= U(t - t_0) \cdot x_0 + \int_{t_0}^t U(t - s) \cdot f(s, u(s)) \, ds \\ &\quad - \int_t^{+\infty} V(t - s) \cdot f(s, u(s)) \, ds \end{aligned}$$

(note that  $x_0$  is not equal to  $u(t_0)$  in general). Furthermore we have

$$\|u(t)\| \leq 2K\|x_0\| \exp(-\sigma(t - t_0)) \quad \text{for } t \geq t_0.$$

Suppose further that  $A$  has no purely imaginary eigenvalues. Then, conversely, there exists  $\beta > 0$  such that every solution  $u$  of (2.1), defined for  $t \geq t_0$  and satisfying  $\|u(t)\| \leq \beta$  for  $t \geq t_0$ , satisfies the equation (3.2.2) for one and only one  $x_0$  such that  $J \cdot x_0 = 0$ .

1. Let us prove that for a suitable choice of  $\alpha$ , (3.2.2) possesses a solution  $u$  defined for  $t \geq t_0$  and tending to 0 with  $1/t$ ; an immediate differentiation, using (3.1.4), (3.1.5) and the hypothesis  $J \cdot x_0 = 0$ , then proves that  $u$  is a solution of (2.1). We shall prove that it is

possible to choose  $\alpha$  sufficiently small for the functions  $\mathbf{u}_m(t)$  ( $m \geq 0$ ) to be defined by successive approximations

$$(3.2.3) \quad \begin{cases} \mathbf{u}_0(t) = 0 \\ \mathbf{u}_{m+1}(t) = U(t - t_0) \cdot \mathbf{x}_0 + \int_{t_0}^t U(t - s) \cdot \mathbf{f}(s, \mathbf{u}_m(s)) ds \\ \quad - \int_t^{+\infty} V(t - s) \cdot \mathbf{f}(s, \mathbf{u}_m(s)) ds \end{cases}$$

and satisfy the inequality

$$(3.2.4) \quad \|\mathbf{u}_{m+1}(t) - \mathbf{u}_m(t)\| \leq \frac{K}{2^m} \|\mathbf{x}_0\| e^{-\sigma(t-t_0)} \quad \text{for } t \geq t_0.$$

Note that the relation (3.2.4) implies

$$(3.2.5) \quad \|\mathbf{u}_{m+1}(t)\| \leq 2K \|\mathbf{x}_0\| e^{-\sigma(t-t_0)} \leq b e^{-\sigma(t-t_0)}$$

if  $2K \|\mathbf{x}_0\| \leq b$ . The induction will then go through, and the sequence  $(\mathbf{u}_m)$  converge uniformly in  $[t_0, +\infty[$  to a solution  $\mathbf{u}$  of (2.1) satisfying  $\|\mathbf{u}(t)\| \leq 2K \|\mathbf{x}_0\| e^{-\sigma(t-t_0)}$ .

Note first that by virtue of the hypothesis  $\mathbf{f}(t, 0) = 0$

$$\mathbf{u}_1(t) = U(t - t_0) \cdot \mathbf{x}_0,$$

hence by (3.1.1)

$$\|\mathbf{u}_1(t)\| \leq K \|\mathbf{x}_0\| e^{-2\sigma(t-t_0)}.$$

Suppose  $\mathbf{u}_1, \dots, \mathbf{u}_m$  defined for  $t \geq t_0$  and satisfying (3.2.4) where  $m$  is replaced by  $0, \dots, m-1$ . Take an  $\varepsilon > 0$ , which will be fixed later, and a corresponding value of  $\delta$  such that we have (3.2.1) for  $\|\mathbf{x}_1\| \leq \delta$ ,  $\|\mathbf{x}_2\| \leq \delta$ . If  $2K \|\mathbf{x}_0\| \leq \delta$ , the relation (3.2.1) gives in particular

$$\|\mathbf{f}(t, \mathbf{u}_m(t))\| \leq \varepsilon \|\mathbf{u}_m(t)\| \leq 2\varepsilon K \|\mathbf{x}_0\| e^{-\sigma(t-t_0)}$$

for  $t \geq t_0$ . Since on the other hand, by virtue of (3.1.2),  $\|V(t-s)\| \leq K$  for  $s \geq t$ , we see that the improper integral of the second member of (3.2.3) is *absolutely convergent*, and the formula (3.2.3) thus defines  $\mathbf{u}_{m+1}(t)$  for  $t \geq t_0$ . Now

$$\begin{aligned} \mathbf{u}_{m+1}(t) - \mathbf{u}_m(t) &= \int_{t_0}^t U(t-s) \cdot (\mathbf{f}(s, \mathbf{u}_m(s)) - \mathbf{f}(s, \mathbf{u}_{m-1}(s))) ds \\ &\quad - \int_t^{+\infty} V(t-s) \cdot (\mathbf{f}(s, \mathbf{u}_m(s)) - \mathbf{f}(s, \mathbf{u}_{m-1}(s))) ds \end{aligned}$$

and the relation (3.2.1) gives again, for every  $t \geq t_0$

$$\|\mathbf{f}(t, \mathbf{u}_m(t)) - \mathbf{f}(t, \mathbf{u}_{m-1}(t))\| \leq \varepsilon \|\mathbf{u}_m(t) - \mathbf{u}_{m-1}(t)\|.$$

Hence, by virtue of (3.1.1) and (3.1.2)

$$\begin{aligned} \|\mathbf{u}_{m+1}(t) - \mathbf{u}_m(t)\| &\leq n\varepsilon K \int_{t_0}^t e^{-2\sigma(t-s)} \|\mathbf{u}_m(s) - \mathbf{u}_{m-1}(s)\| ds \\ &\quad + n\varepsilon K \int_t^{+\infty} \|\mathbf{u}_m(s) - \mathbf{u}_{m-1}(s)\| ds. \end{aligned}$$

Using (3.2.4) where  $m$  is replaced by  $m-1$ ,

$$\begin{aligned} \|\mathbf{u}_{m+1}(t) - \mathbf{u}_m(t)\| &\leq \frac{n\varepsilon K^2}{2^{m-1}} \|\mathbf{x}_0\| \left( e^{-\sigma(2t-t_0)} \int_{t_0}^t e^{\sigma s} ds + \int_t^{+\infty} e^{-\sigma(s-t_0)} ds \right) \\ &\leq \frac{n\varepsilon K^2}{2^{m-2}\sigma} \|\mathbf{x}_0\| e^{-\sigma(t-t_0)}. \end{aligned}$$

The inequality (3.2.4) will thus be satisfied by taking

$$(3.2.6) \quad \varepsilon \leq \frac{1}{4nK\sigma}$$

then by choosing  $\delta$  so that (3.2.1) is satisfied for vectors of norm  $\leq \delta$ . Finally we take

$$(3.2.7) \quad \|\mathbf{x}_0\| \leq \alpha = \frac{\delta}{2K}$$

and under these conditions the existence of the solution  $\mathbf{u}$  of (3.2.2) is proved as well as the inequality

$$(3.2.8) \quad \|\mathbf{u}(t)\| \leq 2K\|\mathbf{x}_0\| e^{-\sigma(t-t_0)} \quad \text{for } t \geq t_0.$$

Moreover the initial value of this solution satisfies the relation

$$(3.2.9) \quad \mathbf{u}(t_0) = \mathbf{x}_0 - \int_{t_0}^{+\infty} V(t_0 - s) \cdot \mathbf{f}(s, \mathbf{u}(s)) \, ds.$$

Because of the uniqueness of a solution of (2.1) taking a given value at the point  $t_0$ , there is *only one* possible value of  $\mathbf{x}_0$  giving a solution of (2.1) for which  $\mathbf{u}(t_0)$  is given; moreover

$$(3.2.10) \quad \|\mathbf{u}(t_0)\| \leq 2K\|\mathbf{x}_0\|$$

by (3.2.8).

2. Let us now make the additional hypothesis that  $A$  has no purely imaginary eigenvalues; it is then possible (changing if necessary the value of  $\sigma$ ) to replace (3.1.2) by

$$(3.2.11) \quad \|e^{-tC}\| \leq K e^{-\sigma t} \quad \text{for } t \geq 0.$$

The numbers  $\varepsilon$  and  $\delta$  being chosen as above, suppose that  $\mathbf{u}$  is a solution of (2.1) defined for  $t \geq t_0$  and such that  $\|\mathbf{u}(t)\| \leq \delta$  for  $t \geq t_0$ . Since then  $\|\mathbf{f}(t, \mathbf{u}(t))\| \leq \varepsilon\|\mathbf{u}(t)\| \leq \varepsilon\delta$ , and, by virtue of (3.2.11)

$$\|V(t-s)\| \leq K e^{-\sigma(s-t)} \quad \text{for } s \geq t,$$

it is seen that the integral  $\int_t^{+\infty} V(t-s) \cdot \mathbf{f}(s, \mathbf{u}(s)) \, ds$  is *absolutely convergent* for every  $t \geq t_0$ . In addition, by (XII, 2.4.1)

$$\begin{aligned} \mathbf{u}(t) &= U(t-t_0) \cdot \mathbf{u}(t_0) + V(t-t_0) \cdot \mathbf{u}(t_0) \\ &\quad + \int_{t_0}^t U(t-s) \cdot \mathbf{f}(s, \mathbf{u}(s)) \, ds + \int_{t_0}^t V(t-s) \cdot \mathbf{f}(s, \mathbf{u}(s)) \, ds \end{aligned}$$

or again, from the preceding

$$(3.2.12) \quad \begin{aligned} \mathbf{u}(t) &= U(t-t_0) \cdot \mathbf{u}(t_0) + V(t-t_0) \cdot \mathbf{x}_0 \\ &\quad + \int_{t_0}^t U(t-s) \cdot \mathbf{f}(s, \mathbf{u}(s)) \, ds - \int_t^{+\infty} V(t-s) \cdot \mathbf{f}(s, \mathbf{u}(s)) \, ds \end{aligned}$$

where  $\mathbf{x}_0$  is given by (3.2.9). It is therefore sufficient to prove that we necessarily have  $J \cdot \mathbf{x}_0 = 0$ . Now, since  $\|\mathbf{f}(s, \mathbf{u}(s))\| \leq \varepsilon\|\mathbf{u}(s)\|$  for every  $s \geq t_0$ , it follows from (3.2.11) and (3.1.1) that in the second member of (3.2.12) the two integrals are respectively majorized in norm by  $n\varepsilon K/2\sigma$  and  $n\varepsilon K/\sigma$  for  $t \geq t_0$ . In other words, in (3.2.12), all the terms are *bounded* for  $t \geq t_0$ , except  $V(t-t_0) \cdot \mathbf{x}_0$ , and it is concluded that this last term must be bounded too. But the

formula (XII, 3.2.2) shows that if  $\mathbf{x}_0 = (x_{0j})_{1 \leq j \leq n}$ , the components of  $V(t - t_0) \cdot \mathbf{x}_0$  of index  $j < k$  are zero and those of index  $\geq k$  are of the form

$$e^{\lambda(t-t_0)} \left( x_{0h} + tx_{0,h+1} + \frac{t^2}{2!} x_{0,h+2} + \cdots + \frac{t^{r-1}}{(r-1)!} x_{0,h+r-1} \right)$$

with  $h \geq k$  and  $\Re \lambda > 0$ , and each of the components  $x_{0l}$  with  $k \leq l \leq n$  occurs at least once in these expressions. The hypothesis can only be satisfied if  $x_{0l} = 0$  for  $k \leq l \leq n$ , which completes the proof.

(3.3) Another type of hypothesis on the "perturbation"  $\mathbf{f}(t, \mathbf{x})$  leads to an *asymptotic development* of the solutions of (2.1) remaining bounded in the neighbourhood of  $+\infty$ . Suppose this time that  $A$  has  $k$  *purely imaginary* eigenvalues (counted with their order of multiplicity), the corresponding Jordan matrices being of order 1, and that the other eigenvalues of  $A$  have their real parts  $> 0$ . Then (after linear change of the unknowns)

$$A = \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix}$$

where  $B$  is a matrix of order  $k$ ,  $C$  a matrix of order  $n - k$ , and where, for two constants  $K > 0$ ,  $\sigma > 0$ ,

$$(3.3.1) \quad \|e^{tB}\| \leq K \quad \text{for every } t \in \mathbf{R}$$

$$(3.3.2) \quad \|e^{-tC}\| \leq K e^{-\sigma t} \quad \text{for } t \geq 0.$$

$U$  and  $V$  are again defined by the formulae (3.1.3).

In this case:

(3.4) Suppose satisfied the hypotheses of (3.3) on  $A$ ; suppose that  $\mathbf{f}$  is continuous in  $D \subset \mathbf{R} \times \mathbf{C}^n$  containing the set (2.2.1) with  $t_0 > 0$ ; suppose further that  $\mathbf{f}(t, 0) = 0$  and that there exists a constant  $\rho > 0$  such that, for  $t \geq t_0$ ,  $\|\mathbf{x}_1\| \leq b$ ,  $\|\mathbf{x}_2\| \leq b$

$$(3.4.1) \quad \|\mathbf{f}(t, \mathbf{x}_1) - \mathbf{f}(t, \mathbf{x}_2)\| \leq ct^{-1-\rho} \|\mathbf{x}_1 - \mathbf{x}_2\| \quad (c \text{ constant } > 0).$$

Then there exists  $\alpha > 0$  such that, for every vector  $\mathbf{x}_0 \in \mathbf{C}^n$  satisfying  $J \cdot \mathbf{x}_0 = 0$  and  $\|\mathbf{x}_0\| \leq \alpha$ , there exists a bounded solution  $\mathbf{u}$  of (2.1) defined for  $t \geq t_0$  and satisfying the integral equation

$$(3.4.2) \quad \mathbf{u}(t) = U(t - t_0) \cdot \mathbf{x}_0 - \int_{t_0}^{+\infty} e^{(t-s)A} \cdot \mathbf{f}(s, \mathbf{u}(s)) ds.$$

Furthermore, if the  $\mathbf{u}_m(t)$  are defined by the process of successive approximations

$$(3.4.3) \quad \begin{cases} \mathbf{u}_0(t) = 0 \\ \mathbf{u}_{m+1}(t) = U(t - t_0) \cdot \mathbf{x}_0 - \int_{t_0}^{+\infty} e^{(t-s)A} \cdot \mathbf{f}(s, \mathbf{u}_m(s)) ds \end{cases}$$

the sequence  $(\mathbf{u}_m)$  converges uniformly to  $\mathbf{u}$  for  $t \geq t_0$  and

$$(3.4.4) \quad \mathbf{u}_{m+1}(t) - \mathbf{u}_m(t) = O(t^{-m\rho})$$

as  $t$  tends to  $+\infty$ . Finally, every solution  $\mathbf{u}$  of (2.1), defined for  $t \geq t_0$  and such that  $\|\mathbf{u}(t)\| \leq b$  for  $t \geq t_0$ , satisfies the equation (3.4.2) for one and only one  $\mathbf{x}_0$  such that  $J \cdot \mathbf{x}_0 = 0$ .

To prove the first two assertions, it will be sufficient to show by induction that the  $\mathbf{u}_m$  are defined and satisfy

$$(3.4.5) \quad \|\mathbf{u}_{m+1}(t) - \mathbf{u}_m(t)\| \leq \|\mathbf{x}_0\| \frac{1}{m!} \left( \frac{nKc}{\rho t^\rho} \right)^m \quad \text{for } t \geq t_0.$$

It can then be concluded from this that, putting  $a = \exp(cnK/\rho t_0^\rho)/c$

$$(3.4.6) \quad \|\mathbf{u}_{m+1}(t)\| \leq a\|\mathbf{x}_0\| \leq b$$

as soon as  $\|\mathbf{x}_0\| \leq \alpha = b/a$ , and the induction can be continued and indeed gives a sequence uniformly convergent to a solution of (3.4.2) such that  $\|\mathbf{u}(t)\| \leq b$  for  $t \geq t_0$ .

By virtue of (3.3.1), we first have  $\|\mathbf{u}_1(t)\| \leq nK\|\mathbf{x}_0\|$ . Then, by virtue of (3.3.2) and (3.4.1)

$$\begin{aligned} \|\mathbf{u}_{m+1}(t) - \mathbf{u}_m(t)\| &\leq nKc \int_t^{+\infty} \frac{\|\mathbf{u}_m(s) - \mathbf{u}_{m-1}(s)\|}{s^{1+\rho}} ds \\ &\leq \frac{(nKc)^m \|\mathbf{x}_0\|}{c(m-1)! \rho^{m-1}} \int_t^{+\infty} \frac{ds}{s^{1+m\rho}} = \frac{\|\mathbf{x}_0\|}{cm!} \left( \frac{nKc}{\rho t^\rho} \right)^m \end{aligned}$$

hence the desired conclusion. The uniqueness of  $\mathbf{x}_0$  for given  $\mathbf{u}(t_0)$  is proved as in (3.2).

Finally, if  $\mathbf{u}$  is a solution of (2.1) such that  $\|\mathbf{u}(t)\| \leq b$  for  $t \geq t_0$ , the integral  $\int_t^{+\infty} e^{(t-s)A} \cdot \mathbf{f}(s, \mathbf{u}(s)) ds$  is absolutely convergent by virtue of the fact that  $\|e^{(t-s)A}\| \leq K$  for  $s \geq t$  and  $\|\mathbf{f}(s, \mathbf{u}(s))\| \leq bcs^{-1-\rho}$  by (3.4.1). Then

$$\begin{aligned} \mathbf{u}(t) &= U(t - t_0) \cdot \mathbf{u}(t_0) + V(t - t_0) \cdot \mathbf{u}(t_0) + \int_{t_0}^t e^{(t-s)A} \cdot \mathbf{f}(s, \mathbf{u}(s)) ds \\ &= U(t - t_0) \cdot \mathbf{x}_0 + V(t - t_0) \cdot \mathbf{x}_0 - \int_t^{+\infty} e^{(t-s)A} \cdot \mathbf{f}(s, \mathbf{u}(s)) ds \end{aligned}$$

where  $\mathbf{x}_0 = \mathbf{u}(t_0) + \int_{t_0}^t e^{(t-s)A} \cdot \mathbf{f}(s, \mathbf{u}(s)) ds$ . It is then seen that in the second member the first and third terms are *bounded* for  $t \geq t_0$ , and so this must be the case for the second term, which is possible only if  $J \cdot \mathbf{x}_0 = 0$ , as is shown by the same reasoning as in (3.2).

(3.5) The inequalities (3.4.5) also show that, for  $t$  tending to  $+\infty$

$$(3.5.1) \quad \mathbf{u}(t) - \mathbf{u}_m(t) = O(t^{-m\rho})$$

and in particular for  $m = 1$ , since  $\mathbf{f}(t, 0) = 0$

$$(3.5.2) \quad \mathbf{u}(t) - U(t - t_0) \cdot \mathbf{x}_0 = O(t^{-\rho}).$$

The hypothesis on  $A$  implies that the components of  $U(t - t_0) \cdot \mathbf{x}_0$  are of the form  $e^{i\omega_j(t-t_0)} \mathbf{x}_{0j}$ , with  $\omega_j$  real for  $1 \leq j \leq k$  and are not all zero if  $\mathbf{x}_0 \neq 0$ . The formula (3.5.2) thus gives a *generalized principal part* (III, 7.6) of  $\mathbf{u}(t)$  in the neighbourhood of  $+\infty$ ; if we can obtain asymptotic developments of the  $\mathbf{u}_m$  for  $m > 1$ , (3.5.1) will similarly give generalized asymptotic developments of  $\mathbf{u}(t)$ .

*Remarks* (3.6) Note that in (3.4) we have not really used (except in the last assertion) the hypothesis that the equation (2.1) is a ‘‘perturbation’’ of an equation with *constant*

coefficients. The reasoning of (3.4) proving the existence of the  $u_m$  and of  $u$  can be applied *without modification* to an equation of the form

$$(3.6.1) \quad \mathbf{x}' = A(t) \cdot \mathbf{x} + \mathbf{f}(t, \mathbf{x})$$

where  $\mathbf{f}$  satisfies the same conditions as in (3.4), and where it is supposed that the resolvent matrix (XII, 2.1)  $R(t, s)$  of the linear equation  $\mathbf{x}' = A(t) \cdot \mathbf{x}$  can be written

$$R(t, s) = \begin{pmatrix} U(t, s) & W(t, s) \\ 0 & V(t, s) \end{pmatrix}$$

where  $U(t, s)$  is a matrix of order  $k$  bounded for all  $s$  and  $t$ ,  $V(t, s)$  and  $W(t, s)$  matrices bounded for  $t \leq s$ .

(3.7) We can even again weaken this last condition, by supposing only that for  $s \geq t \geq t_0$  ( $t_0$  sufficiently large), the norms of  $V(t, s)$  and  $W(t, s)$  are bounded by  $c \cdot t^\alpha$  for some  $\alpha > 0$ , on condition that it is supposed that  $\rho > \alpha$  in (3.4.1).

#### 4. Critical points of autonomous systems in two variables

A differential equation is called *autonomous* if it has the form

$$(4.1) \quad \mathbf{x}' = \mathbf{f}(\mathbf{x})$$

where the real variable  $t$  does not occur,  $\mathbf{x} \in \mathbf{R}^n$  and  $\mathbf{f}$  is defined in an open set  $D \subset \mathbf{R}^n$ . A vector  $\mathbf{f}(\mathbf{x})$  is thus associated with every point  $\mathbf{x} \in D$ . The pair  $(\mathbf{x}, \mathbf{f}(\mathbf{x}))$  is what was formerly called a "bound vector of initial point  $\mathbf{x}$ "† and is occasionally represented by an arrow with initial point  $\mathbf{x}$  and terminal point  $\mathbf{x} + \mathbf{f}(\mathbf{x})$  (Fig. 75). It is also said that  $\mathbf{x} \rightarrow (\mathbf{x}, \mathbf{f}(\mathbf{x}))$  is a "vector field" in  $D$ ; in applications,  $t$  most often represents time, the vectors of the field forces or velocities. If  $u$  is a solution of (4.1) defined in an interval  $I = ]a, b[$  of  $\mathbf{R}$ , the function

$$t \rightarrow u_h(t) = u(t + h)$$

is also a solution defined in the interval

$I_h = ]a - h, b - h[$ , and the images  $u(I)$  and  $u_h(I)$  are the same in  $\mathbf{R}^n$ . These images are of particular interest and are called *trajectories* of the equation (4.1). In particular it turns out to be important to examine the case where a solution  $u$  of (4.1) tends to a finite limit  $c \in D$  as  $t$  tends to  $+\infty$  or to  $-\infty$ . Then  $\mathbf{f}(u(t))$  tends to the vector  $\mathbf{f}(c) = (s_j)_{1 \leq j \leq n}$  and if  $s_j \neq 0$ , it is deduced from the relation  $u'_j(t) \sim s_j$  that  $u_j(t) \sim s_j t$  in the neighbourhood of  $+\infty$  or of  $-\infty$  (III, 10.2), contradicting the hypothesis that  $u(t)$  has a finite limit.

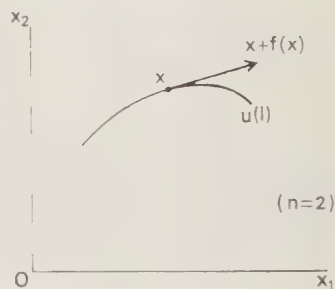


FIGURE 75

† Modern linear algebra is not concerned with the ridiculous notions of "free vector", "bound vector", "polar vector" etc., accumulated by generations of pedants; there is now only one notion of vector (or point of a vector space).

The limit  $\mathbf{c}$  must therefore satisfy the relation

$$(4.1.1) \quad \mathbf{f}(\mathbf{c}) = 0.$$

We say that the solutions of this equation are the *critical points* of (4.1); they play an essential role in the study of the solutions in the neighbourhood of  $\pm\infty$ .

(4.2) We shall confine our attention to the case  $n = 2$  and to studying summarily the possible forms of the trajectories in the neighbourhood of an *isolated* critical point, which may be supposed to be  $\mathbf{c} = 0$ . Suppose further that the system (4.1) is of the form

$$(4.2.1) \quad \begin{cases} x'_1 = ax_1 + bx_2 + f_1(x_1, x_2) \\ x'_2 = cx_1 + dx_2 + f_2(x_1, x_2) \end{cases}$$

where  $a, b, c, d$  are *real* constants and where the matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is invertible (in other words,  $ad - bc \neq 0$ ). Assume  $f_1$  and  $f_2$  to be *zero* for  $\mathbf{x} = 0$  and such that the *partial derivatives* satisfy in the neighbourhood of 0 inequalities of the form

$$(4.2.2) \quad \left| \frac{\partial f_i}{\partial x_k}(\mathbf{x}) \right| \leq K \|\mathbf{x}\|^\rho$$

for an exponent  $\rho > 0$ . (In the simplest cases, the second members of (4.2.1) will be polynomials in  $x_1, x_2$  with no constant term,  $f_1$  and  $f_2$  being the sums of terms of total degree  $\geq 2$  in these polynomials.) Such a critical point is said to be *non-degenerate*.

First of all the matrix  $A$  can be reduced to a canonical form by a linear change of unknown function; but to remain in the real domain, the Jordan canonical form can be used only if the eigenvalues of  $A$ , which are roots of the characteristic equation

$$(4.2.3) \quad \lambda^2 - (a + d)\lambda + ad - bc = 0,$$

are real. There are then three possible forms for the canonical form of  $A$ :

$$\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} (\lambda \neq \mu), \quad \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}, \quad \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

where  $\lambda$  and  $\mu$  in the first case,  $\lambda$  in the other two cases, are the (real) eigenvalues of  $A$ , necessarily  $\neq 0$  since  $A$  is invertible. When the roots of (4.2.3) are imaginary conjugates  $\alpha \pm i\beta$  ( $\alpha$  and  $\beta$  real,  $\beta \neq 0$ ), it is shown in Algebra that by a linear change of variables (replacing  $A$  by  $PAP^{-1}$  for a suitable invertible matrix  $P$ ) it may be supposed that  $A$  has the form

$$\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}.$$

The form of the trajectories in these different cases will be examined ( $A$  being always assumed to have been put in one of the preceding forms). Since we are interested in the trajectories rather than the solutions of (4.2.1) as a function of  $t$ , and since replacing  $t$  by  $-t$  changes the sign of the second members, the study of the solutions for  $t$  tending to  $-\infty$  reduces to that of the solutions for  $t$  tending to  $+\infty$  by changing the sign of  $A$ .

(4.3) *General cases*: these are those where *no relation of equality* is imposed on the elements of the matrix  $A$ .

$$(I) \quad A = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \quad \text{with } \lambda < 0 < \mu.$$

By virtue of (4.2.2), this is the case of *conditional stability* treated in (3.2). As  $t$  tends to  $+\infty$ , the only solutions of (4.2.1) which remain defined and bounded in the neighbourhood of  $+\infty$  are those for which, with the notations of (3.2), the integral equation (3.2.2) is satisfied with  $J \cdot \mathbf{x}_0 = 0$ , which here means that  $\mathbf{x}_0 = (x_{01}, 0)$ . Making the linear change of unknown  $\mathbf{x} = e^{\lambda t} \mathbf{y}$

$$\mathbf{v}(t) = e^{-\lambda t} \mathbf{u}(t)$$

is a solution of the system

$$(4.3.1) \quad \begin{cases} y_1' = g_1(t, y_1, y_2) \\ y_2' = (\mu - \lambda)y_2 + g_2(t, y_1, y_2) \end{cases}$$

where  $g_j(t, y_1, y_2) = e^{-\lambda t} f_j(e^{\lambda t} y_1, e^{\lambda t} y_2)$ . Thus, by virtue of (4.2.2) and Taylor's formula, for  $\mathbf{g} = (g_1, g_2)$ , there is a majorization of the form

$$\|\mathbf{g}(t, \mathbf{z}_1) - \mathbf{g}(t, \mathbf{z}_2)\| \leq c e^{\rho \lambda t} \|\mathbf{z}_1 - \mathbf{z}_2\|$$

for  $\|\mathbf{z}_1\|$  and  $\|\mathbf{z}_2\|$  sufficiently small. The result of (3.4) can therefore be applied to the system (4.3.1) and since  $e^{\lambda t} \leq 1$  for  $t \geq 0$ , the solutions of (4.2.1) bounded for  $t \geq 0$  are of the form

$$(4.3.2) \quad \begin{cases} u_1(t) = r e^{\lambda t} + o(e^{\lambda t}) \\ u_2(t) = o(e^{\lambda t}) \end{cases}$$

in the neighbourhood of  $+\infty$ , two distinct solutions corresponding to distinct values of the constant  $r$ . But since  $t \rightarrow \mathbf{u}(t + t_0)$  is also a solution of (4.2.1) for  $t \geq 0$  and  $u_1(t + t_0) \sim (r e^{\lambda t_0}) e^{\lambda t}$ , there exist only *two trajectories* of (4.2.1) (corresponding to  $r = \pm 1$ ) along which  $\mathbf{u}(t)$  tends to 0 with  $1/t$ . Along one of these  $u_1(t)$  decreases to 0 through values  $> 0$ , along the other  $u_1(t)$  increases to 0 through values  $\leq 0$ , and for both these trajectories  $u_2(t)/u_1(t)$  tends to 0. By replacing  $t$  by  $-t$ , it is similarly seen that there exist two trajectories on which  $\mathbf{u}(t)$  tends to 0 for  $t$  tending to  $-\infty$ ,  $u_1(t)/u_2(t)$  tending this time to 0. For every other trajectory having a point in a sufficiently small neighbourhood  $V$  of 0, the point  $\mathbf{u}(t)$  cannot remain in  $V$  when  $|t|$  becomes sufficiently large. In this case it is said that the critical point 0 is a *saddle* for the system (4.2.1) (Fig. 76, where the arrows correspond to  $t$  increasing).

$$(II) \quad A = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \quad \text{with } \lambda < \mu < 0$$

(the case where  $0 < \lambda < \mu$  is deduced by changing  $t$  into  $-t$ ). Here we make the linear change of unknown  $\mathbf{x} = e^{\mu t} \mathbf{y}$ , which gives the system

$$(4.3.3) \quad \begin{cases} y_1' = (\lambda - \mu)y_1 + g_1(t, y_1, y_2) \\ y_2' = g_2(t, y_1, y_2) \end{cases}$$

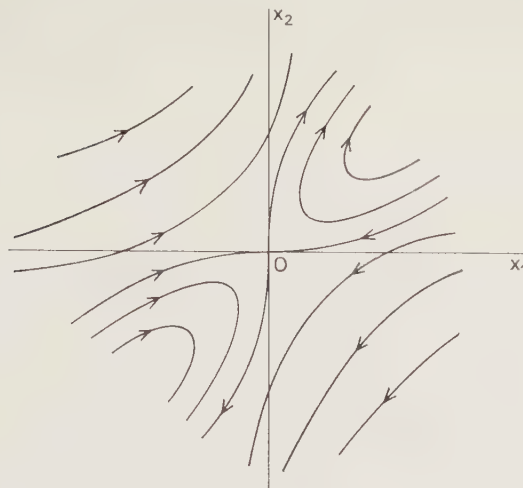


FIGURE 76

with  $\|g(t, z_1) - g(t, z_2)\| \leq c e^{\rho \mu t} \|z_1 - z_2\|$  for  $\|z_1\|$  and  $\|z_2\|$  sufficiently small. Since  $\mu < 0$ , this is the case of *stability* treated in (2.2) and *all* the solutions of (4.3.3), taking for  $t = 0$  a value belonging to a sufficiently small neighbourhood of 0, remain bounded as  $t$  tends to  $+\infty$ . More precisely, there exists one well-determined solution  $e^{tB} \cdot y_0$  of the linear equation

$$y' = B \cdot y \quad \text{with } B = \begin{pmatrix} \lambda - \mu & 0 \\ 0 & 0 \end{pmatrix}$$

such that

$$v(t) = e^{tB} \cdot y_0 + o(1).$$

Returning to the equation (4.2.1), put  $y_0 = (r, s)$  and suppose first that  $s \neq 0$ ; then the corresponding solutions of (4.2.1) are such that

$$(4.3.4) \quad \begin{cases} u_1(t) = o(e^{\mu t}) \\ u_2(t) = s e^{\mu t} + o(e^{\mu t}) \end{cases}$$

in other words,  $u_1(t)/u_2(t)$  tends to 0 along all these trajectories. Taking into account the fact that  $u(t + t_0)$  is also a solution, there are only *two* exceptional trajectories, corresponding to  $r = \pm 1, s = 0$ . These are studied by making here the change of unknown  $x = e^{\lambda t} y$ , which reduces to the case considered in (3.4); then (as in I)) along these trajectories

$$(4.3.5) \quad \begin{cases} u_1(t) = \pm e^{\lambda t} + o(e^{\lambda t}) \\ u_2(t) = o(e^{\lambda t}) \end{cases}$$

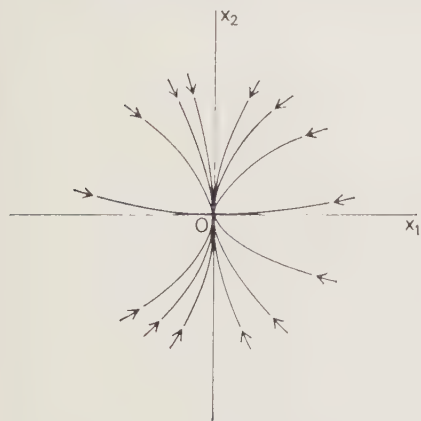


FIGURE 77

so  $u_2(t)/u_1(t)$  tends this time to 0. The critical point 0 is then called an *improper node of the first kind* (Fig. 77).

$$(III) \quad A = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \quad \text{with } \alpha \neq 0 \text{ and } \beta \neq 0;$$

by changing  $t$  to  $-t$ , it may be supposed that  $\alpha < 0$ . Make the change of unknown

$$\mathbf{x} = e^{\alpha t} \mathbf{y},$$

which again reduces to the case treated in (2.2); all the solutions of (4.2.1) taking for  $t = 0$  a value belonging to a sufficiently small neighbourhood of 0 tend to 0 with  $1/t$  and can be written

$$(4.3.6) \quad \begin{cases} u_1(t) = e^{\alpha t}(r \cos \beta t + s \sin \beta t) + o(e^{\alpha t}) \\ u_2(t) = e^{\alpha t}(-r \sin \beta t + s \cos \beta t) + o(e^{\alpha t}) \end{cases}$$

$r$  and  $s$  being two constants not both zero.

The origin is an "asymptotic point" of all the trajectories. The critical point 0 is called a *focus* (Fig. 78).

(4.4) *Special cases*: here there are relations of *equality* between the elements of  $A$ .

$$(IV) \quad A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \quad \text{with } \lambda \neq 0;$$

changing  $t$  to  $-t$ , suppose that  $\lambda < 0$ . The change of unknown  $\mathbf{x} = e^{\lambda t} \mathbf{y}$  again reduces to the case (2.2); here for all the solutions near to 0

$$(4.4.1) \quad \begin{cases} u_1(t) = r e^{\lambda t} + o(e^{\lambda t}) \\ u_2(t) = s e^{\lambda t} + o(e^{\lambda t}) \end{cases}$$

the constants  $r$  and  $s$  being not both zero. The trajectories each have here a half-line tangent at the point 0, two distinct trajectories corresponding to distinct half-lines (Fig. 79): 0 is said to be a *proper node*.

$$(V) \quad A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \quad \text{with } \lambda \neq 0,$$

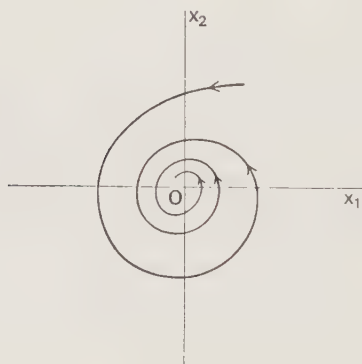


FIGURE 78

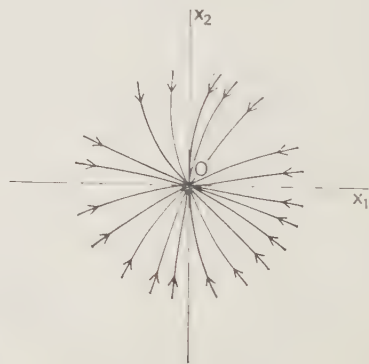


FIGURE 79

and let us again confine our attention to the case  $\lambda < 0$ . Here we can no longer make the previous change of unknown, for a Jordan matrix of order 2 would be obtained to which (2.2) could no longer be applied. However it is sufficient to make the change of unknown  $\mathbf{x} = e^{\nu t} \mathbf{y}$  with

$$\lambda < \nu < 0$$

such that  $(1 + \rho)\nu < \lambda$ . Now consider the system

$$(4.4.2) \quad \begin{cases} y_1' = (\lambda - \nu)y_1 + g_1(t, y_1, y_2) \\ y_2' = y_1 + (\lambda - \nu)y_2 + g_2(t, y_1, y_2) \end{cases}$$

with  $\|g(t, \mathbf{z}_1) - g(t, \mathbf{z}_2)\| \leq c e^{\rho \nu t} \|\mathbf{z}_1 - \mathbf{z}_2\|$  in the neighbourhood of 0. A solution of this system satisfies the integral equation

$$\mathbf{v}(t) = e^{tB} \cdot \mathbf{y}_0 + \int_0^t e^{(t-s)B} \cdot \mathbf{g}(s, \mathbf{v}(s)) ds$$

where  $B = A - \nu I$  and there exists a constant  $L$  (independent of the solution under consideration) such that  $\|\mathbf{v}(t)\| \leq L \|\mathbf{v}(0)\|$ . Then

$$\|\mathbf{v}(t) - e^{tB} \cdot \mathbf{y}_0\| \leq K \|\mathbf{v}(0)\| \int_0^t e^{(\lambda - \nu + \varepsilon)(t-s)} e^{\rho \nu s} ds$$

where  $\varepsilon > 0$  can be taken arbitrarily small and  $K$  is a constant depending on  $\varepsilon$ . Returning to the equation (4.2.1), for all the solutions near to 0,

$$(4.4.3) \quad \begin{cases} v_1(t) = r e^{\lambda t} + O(e^{(1+\rho)\nu t}) \\ v_2(t) = (rt + s) e^{\lambda t} + O(e^{(1+\rho)\nu t}). \end{cases}$$

All the trajectories have here the same tangent at the point 0; we say that we have an *improper node of the second kind* (Fig. 80).

$$(VI) \quad A = \begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix} \quad \text{with } \beta \neq 0.$$

It can be shown here (problem 12) that there are two possible alternative cases:

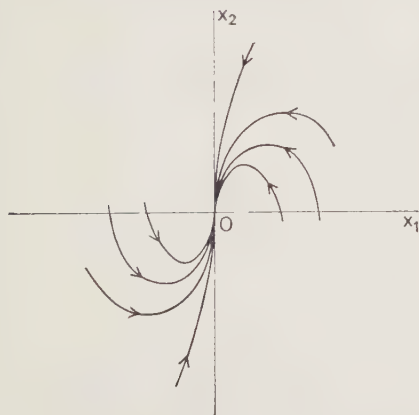


FIGURE 80

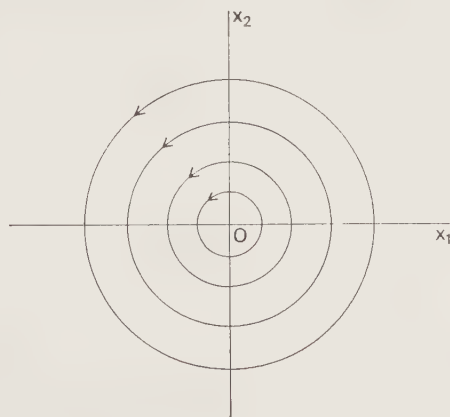


FIGURE 81

1. 0 is a *focus* (cf. III).
2. There are infinitely many *closed* trajectories (i.e. such that the corresponding solutions  $\mathbf{u}(t)$  are *periodic*). It is then said that the critical point 0 is a *centre*. For example, in the case of the linear system  $\mathbf{x}' = A.\mathbf{x}$ , all the trajectories are closed, since the solutions are

$$\begin{cases} u_1(t) = r \cos \beta t + s \sin \beta t \\ u_2(t) = -r \sin \beta t + s \cos \beta t \end{cases}$$

circles of centre 0 (Fig. 81).

### PROBLEMS

1. Show that the conclusion of (2.2) is still valid when (2.1) is replaced by  $\mathbf{x}' = A(t).\mathbf{x} + \mathbf{f}(t, \mathbf{x})$ , and when it is supposed that the resolvent matrix  $R(s, t)$  is bounded for  $t \geq s \geq t_0$ , the hypotheses on  $\mathbf{f}$  being unchanged.
2. The equation  $x'' - (2/t)x' + x = 0$  has for fundamental system of solutions,  $\sin t - t \cos t$ ,  $\cos t + t \sin t$ , which are not bounded. Deduce that in (2.2) the hypothesis that  $\int_{t_0}^{+\infty} \gamma(t) dt$  converges cannot be replaced by the weaker hypothesis that  $\lim_{t \rightarrow +\infty} \gamma(t) = 0$ .
3. The equation  $x'' + (2/t)x' + x = 0$  has for fundamental system of solutions,  $t^{-1} \sin t$  and  $t^{-1} \cos t$ , which tend to 0 with  $1/t$ . Deduce from this fact and from problem 2 that in (2.3), the equation  $\mathbf{x}' = A.\mathbf{x} + \mathbf{f}(t, \mathbf{x})$  cannot be replaced by  $\mathbf{x}' = A(t).\mathbf{x} + \mathbf{f}(t, \mathbf{x})$  assuming only that all the solutions of  $\mathbf{x}' = A(t).\mathbf{x}$  are asymptotically stable.
4. Let

$$A(t) = \begin{pmatrix} -a & 0 \\ 0 & \sin \log t + \cos \log t - 2a \end{pmatrix}$$

with  $1 < 2a < 1 + e^{-\pi}$ . The solutions of  $\mathbf{x}' = A(t).\mathbf{x}$  are

$$\begin{cases} x_1 = c_1 e^{-at} \\ x_2 = c_2 \exp(t \sin \log t - 2at) \end{cases}$$

and are therefore asymptotically stable. If we take  $\mathbf{f}(t, \mathbf{x}) = B(t).\mathbf{x}$  where

$$B(t) = \begin{pmatrix} 0 & 0 \\ e^{-at} & 0 \end{pmatrix}$$

the solutions of  $\mathbf{x}' = A(t).\mathbf{x} + \mathbf{f}(t, \mathbf{x})$  are

$$\begin{cases} x_1 = c_1 e^{-at} \\ x_2 = \exp(t \sin \log t - 2at) \left( c_2 + c_1 \int_0^t \exp(-s \sin \log s) ds \right). \end{cases}$$

Show that for  $c_1 \neq 0$  these solutions are not bounded in the neighbourhood of  $+\infty$ . Deduce that in (2.2) one cannot replace (2.1) by  $\mathbf{x}' = A(t).\mathbf{x} + \mathbf{f}(t, \mathbf{x})$  assuming only that the solutions of  $\mathbf{x}' = A(t).\mathbf{x}$  are asymptotically stable, the hypotheses on  $\mathbf{f}$  remaining unchanged.

5. Let  $R(t, s)$  be the resolvent matrix of a homogeneous linear equation  $\mathbf{x}' = A(t) \cdot \mathbf{x}$ . Suppose that for  $t \geq t_0$  the function

$$\int_{t_0}^t \|R(t, s)\| ds$$

is bounded.

(a) Let  $V(t)$  be a fundamental matrix of  $\mathbf{x}' = A(t) \cdot \mathbf{x}$  so that

$$R(t, s) = V(t)V(s)^{-1}.$$

If  $\int_{t_0}^t \|R(t, s)\| ds \leq K$ , show that there exists a constant  $C$  such that

$$\|V(t)\| \leq C e^{-t/Kn}$$

for  $t \geq t_0$ . (Observe that

$$V(t) \int_{t_0}^t \frac{ds}{\|V(s)\|} = \int_{t_0}^t V(t)V(s)^{-1} \cdot \frac{V(s)}{\|V(s)\|} ds.)$$

(b) Show that if  $\int_{t_0}^t \|R(t, s)\| \leq K$  for  $t \geq t_0$  and if  $\|f(t, \mathbf{x})\| \leq \gamma \|\mathbf{x}\|$  for  $\|\mathbf{x}\| \leq a$  and  $t \geq t_0$ , with  $n\gamma K < 1$ , then the solution  $\mathbf{x} = 0$  of

$$\mathbf{x}' = A(t) \cdot \mathbf{x} + f(t, \mathbf{x})$$

is asymptotically stable. (Observe that for  $t_0 < t_1 < t$ , for every solution  $\mathbf{u}$

$$\|\mathbf{u}(t)\| \leq n\|V(t)\| \cdot \|\mathbf{x}_0\| + n\|V(t)\| \cdot \left\| \int_{t_0}^{t_1} V(s)^{-1} \cdot f(s, \mathbf{u}(s)) ds \right\| + n\gamma K \sup_{t_1 \leq s \leq t} \|\mathbf{u}(s)\|.)$$

6. In (2.3) leave unchanged the hypotheses on  $A$  and replace the hypotheses on  $f$  by the following weaker hypotheses:

(1) There exist constants  $\alpha > 0$ ,  $a > 0$ ,  $K > 0$  and  $b$  real such that the relations  $\|\mathbf{x}\| < \alpha$ ,  $t \geq t_0$  imply

$$\|f(t, \mathbf{x})\| \leq K\|\mathbf{x}\| + \|\mathbf{x}\|^{1+a}t^b.$$

(2) For each  $\varepsilon > 0$ , there exists  $\delta > 0$  and  $T > t_0$  such that the relations  $\|\mathbf{x}\| \leq \delta$  and  $t \geq T$  imply

$$\|f(t, \mathbf{x})\| \leq \varepsilon\|\mathbf{x}\| + \|\mathbf{x}\|^{1+a}t^b.$$

Show then that 0 is again a stable asymptotic solution of (2.1). (Reason as in (2.3), decomposing the interval of integration  $[t_0, t]$  with the help of the point  $T$  and remarking that once  $T$  is fixed,  $\mathbf{u}(T)$  is arbitrarily small with  $\mathbf{u}(t_0) = \mathbf{x}_0$ .)

7. Suppose that the eigenvalues of  $A$  have their real parts  $< 0$ , and that  $f$  satisfies the conditions of (3.2); on the other hand, let  $g(t)$  be a function defined and continuous for  $t \geq t_0$ , with values in  $\mathbf{C}^n$ , and such that  $\lim_{t \rightarrow +\infty} g(t) = 0$ . Show that, for each  $\varepsilon > 0$ , there exists  $T > t_0$  such that every solution  $\mathbf{u}$  of

$$\mathbf{x}' = A \cdot \mathbf{x} + f(t, \mathbf{x}) + g(t)$$

satisfying  $\|\mathbf{u}(T)\| \leq \delta$  has limit 0 as  $t$  tends to  $+\infty$ . (Use successive approximations as in (3.2).)

8. Suppose on the one hand that the square matrix  $A$  can be written

$$A = \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix}$$

where  $B$  is a matrix of order  $k$ , the real parts of its eigenvalues all being  $\leq 0$ , and for those

which are purely imaginary, the corresponding Jordan matrices being all of order 1.  $C$  is of order  $n - k$ , and the real parts of its eigenvalues are all  $> 0$ . There is then a constant  $K$  such that

$$\begin{aligned}\|e^{tB}\| &\leq K \quad \text{for } t \geq 0 \\ \|e^{-tC}\| &\leq K \quad \text{for } t \geq 0.\end{aligned}$$

Define  $U(t)$  and  $V(t)$  as in (3.1.3). Suppose on the other hand that  $\mathbf{f}$  is continuous in  $I \times \mathbf{C}^n$  (where  $I = [t_0, +\infty[$ ), and satisfies the following conditions:

- (1) The integral  $\int_{t_0}^{+\infty} \|\mathbf{f}(t, 0)\| dt$  is convergent.
- (2) For any  $\mathbf{z}_1, \mathbf{z}_2$  in  $\mathbf{C}^n$  and  $t \in I$

$$\|\mathbf{f}(t, \mathbf{z}_1) - \mathbf{f}(t, \mathbf{z}_2)\| \leq \gamma(t) \|\mathbf{z}_1 - \mathbf{z}_2\|$$

where  $\gamma(t)$  is continuous and  $\geq 0$  in  $I$  and such that the integral  $\int_{t_0}^{+\infty} \gamma(t) dt$  is convergent. Lastly, let  $\mathbf{b}(t)$  be a function continuous in  $I$ , with values in  $\mathbf{C}^n$ .

Show that if  $t_1 > t_0$  is such that  $\int_{t_1}^{+\infty} \gamma(t) dt < 1$ , to every bounded solution  $\mathbf{v}(t)$  of

$$(1) \quad \mathbf{x}' = A\mathbf{x} + \mathbf{b}(t)$$

defined in  $I$ , there corresponds a bounded solution  $\mathbf{u}(t)$  of

$$(2) \quad \mathbf{x}' = A\mathbf{x} + \mathbf{b}(t) + \mathbf{f}(t, \mathbf{x})$$

such that

$$\mathbf{u}(t) = \mathbf{v}(t) + \int_{t_1}^t U(t-s) \cdot \mathbf{f}(s, \mathbf{u}(s)) ds - \int_t^{+\infty} V(t-s) \cdot \mathbf{f}(s, \mathbf{u}(s)) ds$$

and the difference  $\mathbf{u}(t) - \mathbf{v}(t)$  tends to 0 with  $1/t$ . (Proceed by successive approximations as in (3.2).) Conversely, every bounded solution  $\mathbf{u}(t)$  of equation (2) arises in this way from a bounded solution  $\mathbf{v}(t)$  of (1).

9. The hypotheses of (3.4) are fulfilled for the linear equation

$$x'' + \frac{1}{4t^2} x = 0$$

except that  $A$  is a Jordan matrix of order 2. No solution of this equation is bounded as  $t$  tends to  $+\infty$ .

10. Show that for the system

$$\begin{cases} x_1' = -x_1 - \frac{2x_2}{\log(x_1^2 + x_2^2)} \\ x_2' = -x_2 + \frac{2x_1}{\log(x_1^2 + x_2^2)} \end{cases}$$

the point 0 is a focus, although it is a proper node for the system

$$\begin{cases} x_1' = -x_1 \\ x_2' = -x_2 \end{cases}$$

(Pass to polar coordinates.)

11. Show that the point 0 is a focus for the system

$$\begin{cases} x_1' = -x_2 - x_1(x_1^2 + x_2^2)^{1/2} \\ x_2' = x_1 - x_2(x_1^2 + x_2^2)^{1/2} \end{cases}$$

and is a centre for the system

$$\begin{cases} x'_1 = -x_2 \\ x'_2 = x_1 \end{cases}$$

(same method).

12. With the notations of no. 4, suppose that the case (VI) occurs; the trajectories have the differential equation in polar coordinates

$$(*) \quad \frac{dr}{d\theta} = F(r, \theta),$$

where

$$F(r, \theta) = \frac{\cos \theta f_1(r \cos \theta, r \sin \theta) + \sin \theta f_2(r \cos \theta, r \sin \theta)}{\beta + \frac{\cos \theta}{r} f_2(r \cos \theta, r \sin \theta) - \frac{\sin \theta}{r} f_1(r \cos \theta, r \sin \theta)}$$

so that the denominator tends to  $\beta$  and the numerator is  $o(r)$  as  $x = re^{i\theta}$  tends to 0. Choose  $r_1 > 0$  such that  $F$  is defined and continuous for

$$0 \leq r < r_1 \quad \text{and} \quad \theta \in \mathbf{R}$$

( $F(r, \theta + 2\pi) = F(r, \theta)$ ). There exists a number  $r_2$  such that  $0 < r_2 < r_1$  and such that, if  $M$  is the least upper bound of  $|F(r, \theta)|$  for  $r \leq r_2$  and  $\theta \in \mathbf{R}$ , then  $r_2/2M \geq 3\pi$ . There exists  $r_3$  such that  $0 < r_3 \leq \frac{1}{2}r_2$  and such that, for every  $(r_0, \theta_0)$  satisfying  $0 < r_0 \leq r_3$ , the equation (\*) has a solution  $\rho(\theta)$  defined in every interval  $|\theta - \theta_0| < 3\pi$  and satisfying the condition  $\rho(\theta) < r_2$ . For such a solution suppose (by changing if necessary the sign of  $\beta$ ) that  $\rho(\theta_0 + 2\pi) \leq \rho(\theta_0)$ , equality corresponding to a *periodic* solution of (\*). If there exists no periodic solution of (\*) passing through a point  $(r, \theta)$  such that  $r < r_2$ , the solution  $\rho(\theta)$  can be continued into the interval  $[\theta_0 + 2\pi, \theta_0 + 4\pi]$  and  $\rho(\theta + 2\pi) < \rho(\theta)$  for  $\theta_0 \leq \theta \leq \theta_0 + 2\pi$ . Show by induction that  $\rho$  can be continued to the whole interval  $[\theta_0, +\infty[$ , and that for each  $\theta$  satisfying  $\theta_0 \leq \theta \leq \theta_0 + 2\pi$ , the sequence  $(\rho(\theta + 2n\pi))$  is strictly decreasing; if  $\sigma(\theta)$  is its limit,  $\sigma(\theta + 2\pi) = \sigma(\theta)$ . Show that  $\sigma$  is continuous in the interval  $[\theta_0, \theta_0 + 2\pi]$  (use problem 2 of Chap. V) and satisfies the equation (\*) (consider the equivalent integral equation). If we do not have  $\sigma(\theta) = 0$  identically, we will therefore have a periodic solution of (\*). Deduce from this the conclusion obtained in the case (VI) of no. 4.

Determine the periodic orbits for the system

$$\begin{cases} x'_1 = -x_2 + x_1(x_1^2 + x_2^2) \sin \frac{\pi}{(x_1^2 + x_2^2)^{1/2}} \\ x'_2 = x_1 + x_2(x_1^2 + x_2^2) \sin \frac{\pi}{(x_1^2 + x_2^2)^{1/2}} \end{cases}$$

13. For every homogeneous polynomial  $P(x, y)$  of degree  $m$ , the curves of the equation  $P(x, y) = c$  ( $c$  constant) are the trajectories of the system

$$\begin{cases} x'_1 = \frac{\partial P}{\partial y}(x_1, x_2) \\ x'_2 = -\frac{\partial P}{\partial x}(x_1, x_2). \end{cases}$$

Give examples of possible forms of these trajectories in the neighbourhood of 0 for  $m \geq 3$ .

# Linear differential equations of the second order

## 1. Principal problems

Linear differential equations of the second order are (together with systems of two linear equations of the first order) the simplest linear differential equations that do not in general reduce to “quadratures”; they occur furthermore in many problems of Mechanics and mathematical Physics.

We shall mostly be concerned with a *homogeneous* linear differential equation

$$(1.1) \quad x'' + p(t)x' + q(t)x = 0$$

where  $p$  and  $q$  are continuous *complex* functions of the *real* variable  $t$  in an *open* interval  $I = ]a, b[$  (bounded or not) of  $\mathbf{R}$  and where we study the complex solutions of (1.1) in  $I$  (see however no. 5 for extension to the complex domain). Besides the numerical calculation in  $I$  of a solution of (1.1) satisfying given numerical initial conditions, which is in principle solved by the general methods of Chap. XI,<sup>†</sup> the principal problem which occurs is:

(A) *The study of the solutions of (1.1) in the neighbourhood of the endpoints of  $I$ .*

We first concern ourselves with obtaining *asymptotic developments* or *generalized asymptotic developments* (III, 7.6). By a change of the variable  $t$ , one can always reduce to the case where  $b = +\infty$  and where the solutions of (1.1) are studied in the neighbourhood of  $+\infty$ . In the case where the solutions are *real* and “*oscillating*” in the neighbourhood of  $+\infty$ , we are also interested in the *zeros* of a solution in an interval  $]a, T[$  as  $T$  tends to  $+\infty$ , and if possible in an asymptotic evaluation of the number of these zeros as  $T$  tends to  $+\infty$ .

In applications differential equations of the form

$$(1.2) \quad x'' + \lambda^2 q(t, \lambda)x = 0$$

are met in which there appears a complex parameter  $\lambda$  which tends in absolute value to  $+\infty$ ; it is important to determine:

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<sup>†</sup> The reader is referred to works on Numerical Calculus for exposition of the various techniques to improve these general methods insofar as the rapidity of the convergence is concerned.

(B) *The asymptotic behaviour of the solution of (1.2) as a function of  $\lambda$  in the neighbourhood of  $|\lambda| = +\infty$ .*

Finally, differential equations (1.1) frequently occur where  $p$  and  $q$  are *real* and *periodic* and moreover depend on a parameter  $\lambda$ . It is then important to know if:

(C) *The equation (1.1) possesses periodic solutions for certain values of the parameter  $\lambda$ .*

## 2. Generalities

In view of the general theory of (XII, 2.5 and 2.6), the solutions of the equation (1.1) form a *vector space of dimension 2*, a basis of this space constituting what is called a *fundamental system of solutions* ( $u_1, u_2$ ) (i.e. two *non-proportional* solutions). The *Wronskian* of two solutions  $u_1, u_2$  of (1.1)

$$(2.1) \quad W(t) = u_1(t)u_2'(t) - u_2(t)u_1'(t)$$

is given by the formula

$$(2.2) \quad W(t) = W(t_0) \exp \left( - \int_{t_0}^t p(s) ds \right).$$

This relation, together with the fact that the linear equation of the first order can be integrated “by quadratures”, show that knowledge of *just one* solution  $u_1(t)$  of (1.1), not identically zero, gives by means of two quadratures a second solution

$$(2.3) \quad u_2(t) = u_1(t) \int_{t_0}^t \frac{W(s)}{u_1^2(s)} ds$$

not proportional to  $u_1(t)$ .

Another important aspect of the linear equation of the second order is the possibility of reducing to the case where  $p = 0$  by a linear change of unknown. If we put  $x = zy$  where  $z$  and  $y$  are two functions of  $t$ , the equation (1.1) becomes a differential equation in  $y$

$$zy'' + (2z' + pz)y' + (z'' + pz' + qz)y = 0$$

and it is sufficient to choose the function  $z$  so that  $2z' + pz = 0$ , that is

$$(2.4) \quad z(t) = \exp \left( - \frac{1}{2} \int_{t_0}^t p(s) ds \right).$$

Finally, we recall Lagrange’s formula for the integration of the equation with second member

$$(2.5) \quad x'' + p(t)x' + q(t)x = f(t)$$

when a fundamental system  $u_1, u_2$  of solutions of (1.1) is known. Starting from (XII, 2.4.1), or by direct verification, a solution of (2.5) is obtained by the formula

$$(2.6) \quad v(t) = \int_{t_0}^t \frac{u_1(s)u_2(t) - u_2(s)u_1(t)}{W(s)} f(s) ds.$$

### 3. Liouville's transformation

Consider a linear equation of the second order of the form

$$(3.1) \quad x'' \pm a^2(t)h(t)x = 0$$

where it is supposed that  $a(t)$  is a *real* function which is  $> 0$ , defined and continuous in a neighbourhood of  $+\infty$  and such that the function

$$(3.2) \quad s = \int_{t_0}^t a(\xi) d\xi$$

tends to  $+\infty$  with  $t$ ;  $h$  is a continuous complex function in a neighbourhood of  $+\infty$ . Liouville's transformation, which, as will be seen, often enables one to obtain an asymptotic development of the solutions of (3.1) in the neighbourhood of  $+\infty$ , consists first of all in making the change of variable (3.2); then

$$\frac{dx}{dt} = \frac{dx}{ds} a(t), \quad \frac{d^2x}{dt^2} = \frac{d^2x}{ds^2} a^2(t) + \frac{dx}{ds} a'(t).$$

Designating by  $\psi(s)$  the inverse function of (3.2) and putting  $b(s) = a(\psi(s))$  (so that  $b'(s) = a'(\psi(s))/a(\psi(s))$ ) the equation (3.1) thus becomes (putting  $q(s) = h(\psi(s))$ )

$$(3.3) \quad \frac{d^2x}{ds^2} + \frac{b'(s)}{b(s)} \frac{dx}{ds} \pm q(s)x = 0.$$

The linear change of unknown (2.4) which makes the term in  $x'$  vanish is here

$$(3.4) \quad x = \frac{1}{\sqrt{b(s)}} y$$

and gives finally the equation

$$(3.5) \quad y'' + \left( \pm q(s) + \frac{1}{4} \frac{b'^2(s)}{b^2(s)} - \frac{1}{2} \frac{b''(s)}{b(s)} \right) y = 0.$$

Note that

$$(3.6) \quad \frac{1}{2} \left( \frac{b''}{b} - \frac{b'^2}{2b^2} \right) = \frac{1}{2} \left( \left( \frac{b'}{b} \right)' + \frac{1}{2} \frac{b'^2}{b^2} \right) = \frac{1}{2a^2} \left( \left( \frac{a'}{a} \right)' - \frac{a'^2}{2a^2} \right)$$

where, in the third expression, the derivatives are taken with respect to  $t$  and  $t$  is then replaced by  $\psi(s)$ .

*Examples* (3.7) If  $a(t) = t^\lambda$  with  $\lambda > -1$

$$\frac{1}{2a^2} \left( \left( \frac{a'}{a} \right)' - \frac{a'^2}{2a^2} \right) = \frac{-\lambda(\lambda+2)}{4t^{2(\lambda+1)}}$$

and

$$s = \frac{t^{\lambda+1}}{\lambda+1}$$

hence

$$\frac{1}{2} \left( \frac{b''}{b} - \frac{b'^2}{2b^2} \right) = \frac{-\lambda(\lambda+2)}{4(\lambda+1)^2 s^2}.$$

Similarly, for  $\lambda = -1$

$$\frac{1}{2a^2} \left( \left( \frac{a'}{a} \right)' - \frac{a'^2}{2a^2} \right) = \frac{1}{4}$$

and

$$s = \log t$$

hence

$$\frac{1}{2} \left( \frac{b''}{b} - \frac{b'^2}{2b^2} \right) = \frac{1}{4}.$$

(3.8) If  $a(t) = (\log t)^\mu$  ( $\mu$  real)

$$\frac{1}{2a^2} \left( \left( \frac{a'}{a} \right)' - \frac{a'^2}{2a^2} \right) \sim -\frac{\mu}{2t^2(\log t)^{2\mu+1}}$$

and

$$s \sim t(\log t)^\mu$$

hence

$$\frac{1}{2} \left( \frac{b''}{b} - \frac{b'^2}{2b^2} \right) \sim -\frac{\mu^2}{2s^2 \log s}.$$

(3.9) If  $a(t) = e^{t^\alpha}$  ( $\alpha > 0$ )

$$\frac{1}{2a^2} \left( \left( \frac{a'}{a} \right)' - \frac{a'^2}{2a^2} \right) \sim -\frac{\alpha^2}{4} t^{2\alpha-2} e^{-2t^\alpha}$$

and

$$s \sim \frac{e^{t^\alpha}}{\alpha t^{\alpha-1}}$$

hence

$$\frac{1}{2} \left( \frac{b''}{b} - \frac{b'^2}{2b^2} \right) \sim -\frac{1}{4s^2}.$$

#### 4. Asymptotic developments of the solutions

(4.1) The study of the solutions of (1.1) in the neighbourhood of the endpoints of  $\mathbf{I}$  can always be reduced, by a change of variable of the type  $t' = 1/(t-a)$  or  $t' = -t$ , to the case where one of these endpoints is  $+\infty$ , and in this section it is assumed that this has been done. To simplify, assume that  $p = 0$  (as can always be done (2.4)) and *furthermore* that in the neighbourhood of  $+\infty$  the function  $q(t)$  admits an asymptotic development

$$(4.1.1) \quad q(t) = q_0 + \frac{q_1}{t} + \frac{q_2}{t^2} + \cdots + \frac{q_n}{t^n} + o\left(\frac{1}{t^n}\right) \quad \text{with } n \geq 2,$$

the  $q_j$  being arbitrary *complex* numbers.

Note that for any complex numbers  $\omega$  and  $\rho$ , the function

$$u(t) = e^{\omega t} t^{-\rho}$$

(with  $t^{-\rho} = e^{-\rho \log t}$ ) is a solution of the second order equation

$$(4.1.2) \quad x'' - \left( \omega^2 - \frac{2\rho\omega}{t} + \frac{\rho(\rho+1)}{t^2} \right) x = 0$$

where the first two terms of the development of the coefficient of  $x$  can be taken to be *any* quantities by a suitable choice of  $\omega$  and of  $\rho$ . We shall *compare* the solutions of (1.1), with  $q$  given by (4.1.1), to the solutions of (4.1.2) and obtain in this way the required asymptotic developments.

(4.2) *Suppose that in (4.1.1)  $q_0 \neq 0$ , and determine the complex numbers  $\omega$  and  $\rho$  by the conditions*

$$(4.2.1) \quad \omega^2 + q_0 = 0, \quad -2\omega\rho + q_1 = 0$$

*and by the condition that the function  $e^{\omega t} t^{-\rho}$  remains bounded as  $t$  tends to  $+\infty$ : if  $q_0$  is not real and negative, we may take  $\Re \omega < 0$ , and in the contrary case  $\omega = \pm i\lambda$  with  $\lambda > 0$ , and the sign is chosen so that  $\Re(\rho) = \Re(\pm q_1/2i\lambda)$  is  $\geq 0$ . Then there exists a solution of the differential equation  $x'' + q(t)x = 0$  which admits in the neighbourhood of  $+\infty$  a generalized asymptotic development of the form*

$$(4.2.2) \quad u(t) = e^{\omega t} t^{-\rho} \left( 1 + \frac{c_1}{t} + \cdots + \frac{c_n}{t^n} + o\left(\frac{1}{t^n}\right) \right)$$

*and whose derivatives  $u'(t)$  and  $u''(t)$  have asymptotic developments of equal precision which are obtained by differentiating (4.2.2) term by term.*

Let us make the linear change of unknown

$$(4.2.3) \quad x = e^{\omega t} t^{-\rho} y.$$

The function  $y$  is a solution of the second order equation

$$(4.2.4) \quad y'' + 2\left(\omega - \frac{\rho}{t}\right)y' + \frac{F(t)}{t^2}y = 0$$

with

$$(4.2.5) \quad \begin{aligned} F(t) &= t^2 \left( q(t) - q_0 - \frac{q_1}{t} \right) + \rho(\rho+1) \\ &= (q_2 + \rho(\rho+1)) + \frac{q_3}{t} + \cdots + \frac{q_n}{t^{n-2}} + o\left(\frac{1}{t^{n-2}}\right). \end{aligned}$$

We shall consider (4.2.4) as a "perturbation" of the second order equation

$$(4.2.6) \quad y'' + \left( 2\left(\omega - \frac{\rho}{t}\right) - \frac{\rho}{t^2\left(\omega - \frac{\rho}{t}\right)} \right) y' = 0$$

of which a fundamental system of solutions is given by

$$v(t) = e^{-2\omega t} t^{2\rho}, \quad w(t) = 1.$$

Transforming in the usual way the equation (4.2.4) to a system of two equations of the first order by putting  $y_1 = y$ ,  $y_2 = y'$ , one obtains the vector equation

$$\mathbf{y}' = A(t) \cdot \mathbf{y} + B(t) \cdot \mathbf{y}$$

with

$$A(t) = \begin{pmatrix} 0 & 1 \\ 0 & r(t) \end{pmatrix} \quad B(t) = \begin{pmatrix} 0 & 0 \\ 0 & \frac{G(t)}{t^2} \end{pmatrix}$$

$$(4.2.7) \quad r(t) = -2 \left( \omega - \frac{\rho}{t} \right) + \frac{\rho}{t^2 \left( \omega - \frac{\rho}{t} \right)}$$

$$(4.2.8) \quad G(t) = b_0 + \frac{b_1}{t} + \dots + \frac{b_{n-2}}{t^{n-2}} + o\left(\frac{1}{t^{n-2}}\right).$$

The fundamental property of this equation is that it satisfies the conditions of (XIII, 3.6). In fact:

1. The equation  $\mathbf{y}' = A(t) \cdot \mathbf{y}$  possesses a solution  $(w, 0)$  bounded in  $\mathbf{R}$ .
2. The resolvent matrix  $V(t)V(s)^{-1}$  of  $\mathbf{y}' = A(t) \cdot \mathbf{y}$ , with

$$V(t) = \begin{pmatrix} 1 & v(t) \\ 0 & v'(t) \end{pmatrix}$$

is bounded for  $s \geq t \geq t_0$  ( $t_0$  fixed sufficiently large); it can be written

$$\begin{pmatrix} 1 & \frac{1}{2\left(\frac{\rho}{s} - \omega\right)} \left( e^{2\omega(s-t)} \left(\frac{t}{s}\right)^{2\rho} - 1 \right) \\ 0 & \frac{\omega - \frac{\rho}{s}}{\omega - \frac{\rho}{t}} e^{2\omega(s-t)} \left(\frac{t}{s}\right)^{2\rho} \end{pmatrix}$$

and our assertion follows from the relations

$$\begin{aligned} \frac{1}{2}|\omega| &\leq \left| \omega - \frac{\rho}{s} \right| \leq 2|\omega| \quad \text{for } s \geq t_0 \geq 2 \left| \frac{\omega}{\rho} \right| \\ \left| e^{2\omega(s-t)} \left(\frac{t}{s}\right)^{2\rho} \right| &\leq M \quad \text{for } s \geq t, \end{aligned}$$

in view of the choices of  $\omega$  and  $\rho$ .

3. In the neighbourhood of  $+\infty$ , and for any two vectors  $\mathbf{z}_1$  and  $\mathbf{z}_2$

$$\|B(t) \cdot \mathbf{z}_1 - B(t) \cdot \mathbf{z}_2\| \leq \frac{k}{t^2} \|\mathbf{z}_1 - \mathbf{z}_2\|$$

where  $k$  is a constant.

We can thus apply the process of successive approximations described in (XIII, 3.4) (replacing the constant  $\rho > 0$  which occurs there by 1). To complete the proof it is sufficient to show that the successive terms  $u_{m+1} - u_m$  of the development obtained have effectively *principal parts* of the form  $c_m t^{-m-h}$  ( $h$  fixed integer  $\geq 0$ , which is the exponent of the first non-zero term of the development (4.2.8) of  $G(t)$ ). It follows at once that we need only show that the integral

$$e^{-2\omega t} t^{2\rho} \int_t^{+\infty} \frac{e^{2\omega s} ds}{s^{2\rho+m+h+1}} = O\left(\frac{1}{s^{m+h+1}}\right)$$

which results immediately by an integration by parts.

Note that in this way not only is an asymptotic development (4.2.2) obtained, but an explicit *majorization* of the remainder when there is a corresponding majorization for the remainder of (4.1.1).

(4.3) The result of (4.2) is immediately generalized to the case where (4.1.1) is replaced by a development of the form

$$(4.3.1) \quad q(t) = a + bt^{-1} + q_1 t^{-1-1/k} + q_2 t^{-1-2/k} + \dots + q_n t^{-1-n/k} + o(t^{-1-n/k})$$

where  $k$  is an integer  $> 0$ . An asymptotic development of  $G(t)$  in powers of  $t^{-1/k}$  is then obtained in place of (4.2.8), and (XIII, 3.6) is applied as above.

(4.4) Having obtained a solution  $u_1(t)$  having the development (4.2.2), a second solution  $u_2(t)$  is obtained, forming with the first a fundamental system, by applying the formula (2.3). Taking into account the generalized asymptotic development

$$\begin{aligned} \int_{t_0}^t e^{-2\omega s} s^{2\rho} ds &= -e^{-2\omega t} \left( \frac{t^{2\rho}}{2\omega} + \frac{2\rho t^{2\rho-1}}{(2\omega)^2} + \dots \right. \\ &\quad \left. + \frac{2\rho(2\rho-1)\dots(2\rho-p+1)}{(2\omega)^{p+1}} t^{2\rho-p} + o(t^{2\rho-p}) \right) \end{aligned}$$

in the neighbourhood of  $t = +\infty$ , we obtain for  $u_2(t)$  a development of the form (4.2.2) but where  $\omega$  and  $\rho$  are simply *changed in sign*.

(4.5) Starting from the fundamental result (4.2), we can treat completely the case where there is an asymptotic development

$$(4.5.1) \quad q(t) = t^k \left( q_0 + \frac{q_1}{t} + \dots + \frac{q_n}{t^n} + o\left(\frac{1}{t^n}\right) \right) \quad \text{with } q_0 \neq 0$$

$k$  being a positive or negative *integer*, by applying one or more times Liouville's transformation in a suitable way. We shall only consider the case  $k < 0$  (cf. problem 2):

(A) *Case where  $k \leq -3$ .* We can forcefully apply the method of (4.2) replacing  $r(t)$  by 0 and  $v(t)$  by  $t$ : although this last solution is *not bounded* in the neighbourhood of  $+\infty$ ,  $t^{k+2}v(t)$  is bounded and remark (XIII, 3.7) can be applied: thus a solution is obtained having an asymptotic development

$$(4.5.2) \quad u_1(t) = 1 + t^{k+2} \left( c_1 + \frac{c_2}{t} + \dots + \frac{c_n}{t^{n-1}} + o\left(\frac{1}{t^{n-1}}\right) \right)$$

from which is deduced a second solution having a development of the form

$$(4.5.3) \quad u_2(t) = t + t^{k+2} \left( c'_1 + \frac{c'_2}{t} + \cdots + \frac{c'_n}{t^{n-1}} + o\left(\frac{1}{t^{n-1}}\right) \right).$$

(B) *Case where  $k = -2$ .* Liouville's transformation is made with  $a(t) = 1/t$  (cf. (3.7)), which corresponds to  $\psi(s) = e^s$  and  $b(s) = e^{-s}$  and gives a differential equation of the form

$$(4.5.4) \quad y'' + (q_0 - \frac{1}{4} + q_1 e^{-s} + \cdots + q_n e^{-ns} + o(e^{-ns}))y = 0.$$

The method of (XIII, 3.4) can be applied to this equation, even when  $q_0 = \frac{1}{4}$ , by virtue of remark (XIII, 3.7); the asymptotic development for a bounded solution  $v_1(s)$  proceeds here in powers of  $e^{-s}$ . Returning to the initial equation with the help of the formula (3.4), a solution is obtained having an asymptotic development

$$(4.5.5) \quad u_1(t) = t^\sigma \left( 1 + \frac{c_1}{t} + \frac{c_2}{t^2} + \cdots + \frac{c_n}{t^n} + o\left(\frac{1}{t^n}\right) \right)$$

where  $\sigma = \omega + \frac{1}{2}$  is the root of *smallest real part* of the equation

$$(4.5.6) \quad \sigma^2 - \sigma + q_0 = 0$$

if  $\omega$  is not purely imaginary, one of the two roots of (4.5.6) in the contrary case. When formula (2.3) is applied to calculate an asymptotic development of a second solution the primitives of the powers  $t^{-2\sigma-k}$  must be taken, and it is thus necessary to distinguish two cases, according to whether  $k + 2\sigma$  can take the value  $+1$  or not. Since  $1 - \sigma$  is the second root of (4.5.6), these two cases can also be stated as follows:

(B1) *The difference of the two roots of (4.5.6) is not an integer:* in this case the second solution is given by

$$(4.5.7) \quad u_2(t) = t^{1-\sigma} \left( 1 + \frac{c'_1}{t} + \cdots + \frac{c'_n}{t^n} + o\left(\frac{1}{t^n}\right) \right).$$

(B2) *The difference of the roots of (4.5.6) is an integer  $h$*  (which may be supposed  $\geq 0$ ); then  $\sigma = \frac{1}{2}(1 - h)$ ,  $1 - \sigma = \frac{1}{2}(1 + h)$ , and the second solution has in general the form

$$(4.5.8) \quad u_2(t) = u_1(t) \log t + t^{1-\sigma} \left( c'_0 + \frac{c'_1}{t} + \cdots + \frac{c'_n}{t^n} + o\left(\frac{1}{t^n}\right) \right).$$

However the term in  $\log t$  may sometimes be missing.

(C) *Case where  $k = -1$ .* A Liouville transformation is made with

$$a(t) = \frac{1}{\sqrt{t}}$$

(cf. (3.7)) which corresponds to  $\psi(s) = \frac{1}{4}s^2$  and  $b(s) = 2/s$ . A differential equation of the form

$$(4.5.9) \quad y'' + \left( q_0 + \frac{16q_1 - 3}{4s^2} + \cdots + \frac{4^n q_n}{s^{2n}} + o\left(\frac{1}{s^{2n}}\right) \right) y = 0$$

is then obtained to which (4.2) and (4.4) are applicable. Thus for asymptotic developments of two solutions forming a fundamental system, there are expressions of the form

$$(4.5.10) \quad t^{1/4} e^{2\omega\sqrt{t}} \left( 1 + \frac{c_1}{t^{1/2}} + \cdots + \frac{c_n}{t^{n/2}} + o\left(\frac{1}{t^{n/2}}\right) \right)$$

where  $\omega$  is one or the other of the roots of  $\omega^2 + q_0 = 0$ .

(4.6) In practice, in the cases treated in (4.2) or (4.5), to determine the asymptotic developments of the solutions of  $x'' + q(t)x = 0$ , we *substitute* for  $x$  in this equation the asymptotic development of the corresponding form, but where the coefficients are undetermined. For  $x''$  is substituted the development obtained by *differentiating twice term by term* the development of  $x$ , and finally for  $q(t)$  its development (4.1.1) is substituted. The coefficients of the development of  $x'' + q(t)x$  thus obtained are then equated to zero, which gives a recurrent system of linear equations enabling one to determine the coefficients in turn.

## 5. Extension to the complex domain

The preceding methods can be applied to the study of the solutions of a second order differential equation

$$(5.1) \quad w'' + p(z)w' + q(z)w = 0$$

in the neighbourhood of an *isolated singularity* of the analytic functions  $p$  and  $q$ . By a change of variable  $z' = 1/(z - a)$ , one can reduce to the case where  $p$  and  $q$  are analytic in the *exterior*  $E$  of a disc:  $|z| > R$ . By the same change of unknown as in no. 2, it can be assumed that  $p = 0$ . We confine ourselves to the case which corresponds to that studied in (4.2) and (4.5), i.e. where  $q(1/z)$  is *analytic at the point*  $z = 0$ . Then in  $E$

$$(5.2) \quad q(z) = q_0 + \frac{q_1}{z} + \cdots + \frac{q_n}{z^n} + \cdots$$

where the series  $\sum_{n=0}^{\infty} q_n z^n$  is just the *Taylor series* of the analytic function  $q(1/z)$  at the point  $z = 0$ , and *converges* therefore for  $|z| > R$ .

(5.3) Suppose first  $q_0 \neq 0$ , and let us try to proceed as in (4.2). By a change of variable  $z \rightarrow z e^{i\alpha}$  our attention can first be confined to the case where  $q_0 = -\omega^2$ , with  $\omega$  real and  $> 0$ . We then determine  $\rho$  by the equation  $2\omega\rho - q_1 = 0$ , and study first the solutions in the *intersection*  $E_+$  of  $E$  and the *open half-plane*  $\Re z > 0$ . For each  $\delta > 0$ , let  $S_\delta$  be the angular sector

$$(5.3.1) \quad -\frac{\pi}{2} + \delta < \arg z < \frac{\pi}{2} - \delta;$$

clearly in each  $S_\delta$

$$(5.3.2) \quad |e^{-\omega z} z^{-\rho}| \leq c e^{-\omega|z|\cos\delta} |z|^\mu$$

for suitable constants  $c > 0$  and  $\mu$  real. For each  $z \neq 0$  in  $\mathbf{C}$ , denote by  $L_z$  the half-line  $t \rightarrow zt$  with  $1 \leq t < +\infty$ . By virtue of (5.3.2), the integral

$$z \rightarrow \int_{L_z} e^{-2\omega(\zeta-z)} \left(\frac{z}{\zeta}\right)^{2\rho} F(\zeta) d\zeta$$

has a meaning and is an analytic function of  $z$  in  $E_+$  for every function  $F$  analytic in  $E_+$  and bounded in each sector  $S_\delta$  (VII, 10.4). The process of successive approximations of (4.2) can then be repeated without change, the integrals always being taken along  $L_z$ ; the successive terms  $u_m(z)$  are analytic functions in  $E_+$  and bounded in each  $S_\delta$ . Again write, in  $S_\delta$ ,  $v(z) = o(w(z))$  (resp.  $v(z) \sim w(z)$ ) if the ratio  $v(z)/w(z)$  tends to 0 (resp. to 1) as  $|z|$  tends to  $+\infty$ ,  $z$  remaining in  $S_\delta$ . A majorization calculation similar to that of (XIII, 3.4) proves then that in  $S_\delta$

$$(5.3.3) \quad u_{m+1}(z) - u_m(z) \sim c_m z^{-m-h}$$

for constants  $c_m$  and  $h$  not depending on  $\delta$ . Taking into account (VII, 10.1), a solution  $u_1^+(z)$  is thus obtained analytic in  $E_+$  with, for each  $\delta > 0$ , an asymptotic development valid in  $S_\delta$

$$(5.3.4) \quad u_1^+(z) = e^{-\omega z} z^{-\rho} \left( 1 + \frac{c_1}{z} + \cdots + \frac{c_n}{z^n} + o\left(\frac{1}{z^n}\right) \right)$$

with arbitrary precision, the  $c_j$  not depending on  $\delta$ . Take care not to assume that these asymptotic developments arise from a convergent power series in  $1/z$  (cf. problem 3); the  $c_j$  are determined by recurrence as in (4.6). One can then determine a second solution  $u_2^+(z)$  in  $E_+$  by application of formula (2.3), the integral being taken along a path in  $E_+$ ; a second analytic solution of the same form as (5.3.4) is found, but where  $\omega$  and  $\rho$  are replaced by  $-\omega$  and  $-\rho$  respectively (so that this solution tends in absolute value to  $+\infty$  with  $|z|$  in each  $S_\delta$ ).

(5.4) It is possible to proceed similarly in the intersection  $E_-$  of  $E$  and the open half-plane  $\Re z < 0$ , but here it is necessary to replace  $\omega$  by  $-\omega$  and  $\rho$  by  $-\rho$  to obtain a process of successive approximations convergent to a solution  $u_1^-(z)$  which tends to 0 with  $1/|z|$  in every angular sector  $-S_\delta$ . A second solution  $u_2^-(z)$  is deduced by (2.3), a solution which in absolute value tends to  $+\infty$  with  $|z|$  in every sector  $-S_\delta$ .

Finally, by restricting  $z$  to taking purely imaginary values  $z = it$ , we obtain by the procedure of (4.2), two solutions  $u_1^{0+}(it)$ ,  $u_2^{0+}(it)$  with asymptotic developments (4.2.2) for  $t$  tending to  $+\infty$ , and two solutions  $u_1^{0-}(it)$ ,  $u_2^{0-}(it)$  for  $t$  tending to  $-\infty$ .

By virtue of (XII, 1.3), the solutions  $u_1^+(z)$  and  $u_2^+(z)$ , analytic in  $E_+$ , can be continued analytically to two solutions  $u_1, u_2$  in the complement  $E_0$  in  $E$  of the half-line  $L_{-\mathbf{R}}$ , which is simply-connected. In  $E_-$  (resp. on the imaginary axis), they are therefore expressed as linear combinations of the solutions  $u_1^-, u_2^-$  (resp.  $u_1^{0+}, u_2^{0+}$ , or  $u_1^{0-}, u_2^{0-}$ ). The determination of these linear combinations requires a detailed study of the analytic continuation of  $u_1^+$  and  $u_2^+$ , a problem to which no general treatment will be given (cf. XV, 3).

(5.5) There is however a case in which these relations are obtained at once, that where  $q_0 \neq 0$ ,  $q_1 = 0$ , for then  $\rho = 0$  and the function  $e^{-\omega z}$  which is used in the process of

successive approximations is *bounded in the closed half-plane*  $\Re z \geq 0$ . Immediately the asymptotic development (5.3.4) (with  $\rho = 0$ ) is also *valid in*  $\bar{E}_+$ , *the intersection of*  $E$  *and the closed half-plane*  $\Re z \geq 0$ , and in particular *on the imaginary axis*. Comparison of the asymptotic developments obtained on the imaginary axis for the various solutions described above then immediately resolves the problem of analytic continuation raised in (5.4).

It is easily seen that the case  $q_0 \neq 0$ ,  $q_1 \neq 0$  can always be reduced to the case where  $q_1 = 0$  by a Liouville transformation with  $a(t) = 1 + (\lambda/t)$  for a suitable choice of the constant  $\lambda$ .

(5.6) Consider secondly the case where, in (5.2),  $q_0 = 0$ ,  $q_1 \neq 0$  which corresponds to the case (C) of (4.5). This leads to making the change of variable  $z = \frac{1}{4}s^2$  and the change of unknown  $w = \sqrt{s/2}y$  (with a choice of the determination of  $\sqrt{s}$ ), which gives an equation of the form (4.5.9), so falling into the case studied in (5.3). One must therefore consider a root  $\omega = |\omega| e^{i\alpha}$  of  $\omega^2 + q_1 = 0$ ; returning to the variable  $z$ , the open set in which asymptotic developments are obtained is the set  $S'_{\theta, \delta}$  of points  $z = |z| e^{i\theta}$  such that

$$(5.6.1) \quad -2\alpha - \pi + \delta < \theta < -2\alpha + \pi - \delta.$$

These developments are of the form

$$(5.6.2) \quad z^{1/4} e^{2\omega z^{1/2}} \left( c_0 + \frac{c_1}{z^{1/2}} + \cdots + \frac{c_n}{z^{n/2}} + o\left(\frac{1}{z^{n/2}}\right) \right)$$

where  $\omega$  is one or the other of the roots of  $\omega^2 + q_1 = 0$ .

(5.7) In order to examine the case where  $q_0 = q_1 = 0$  in (5.2), it is convenient to return to a neighbourhood of 0 by the change of variable  $z \rightarrow 1/z$ . Consider now more generally a differential equation of the form

$$(5.7.1) \quad w'' + \frac{a(z)}{z} w' + \frac{b(z)}{z^2} w = 0$$

in the neighbourhood of  $z = 0$ , the functions  $a(z)$  and  $b(z)$  being *analytic at the point*  $z = 0$  (in other words, the equations (5.1) where  $p$  has at the point 0 a *pole of order*  $\leq 1$  and  $q$  a *pole of order*  $\leq 2$ ). Transform (5.7.1) to a system of two equations of the first order in a slightly different way than in (XI, 4.8.2) by here putting

$$w_1 = w, \quad w_2 = zw'$$

which gives

$$w' = \frac{1}{z} A(z) \cdot w$$

with

$$A(z) = \begin{pmatrix} 0 & 1 \\ -b(z) & 1 - a(z) \end{pmatrix}.$$

Then apply (XII, 5.4) which (taking into account the Jordan canonical form of a matrix) proves the existence, in the plane  $\mathbf{C}$  cut along a half-line of endpoint 0, of at least one solution  $u_1(z)$  of (5.7.1) of the form  $z^{\rho} v_1(z)$  where  $v_1$  is *analytic at the point* 0

and  $v_1(0) = 1$ . Substituting an expression of this form in (5.7.1) and making the principal part vanish we obtain the equation in  $\rho$  (called *characteristic equation*)

$$(5.7.2) \quad \rho(\rho - 1) + a(0)\rho + b(0) = 0.$$

Restricting the variable  $z$  to the real axis and returning to the neighbourhood of  $+\infty$ , the preceding solution corresponds to a solution of principal part  $t^{-\rho}$ ; from case (B) of (4.5), it is therefore necessary to take for  $\rho$  a root  $\rho'$  of *largest real part* of (5.7.2). If the difference of two roots of (5.7.2) is *not an integer*, the second root  $\rho''$  of (5.7.2) gives a second solution  $z^{\rho'} v_2(z)$  of (5.7.1) with  $v_2(0) = 1$  and  $v_2$  *analytic at the point* 0. If on the other hand  $\rho' - \rho'' = h$  is an integer  $\geq 0$ , a second solution of (5.7.1) is in general of the form  $u_1(z) \log z + z^{\rho''} v_3(z)$  with  $v_3(0) = 1$  and  $v_3$  analytic at 0.

Of course, as in (4.6), the coefficients of the Taylor series of  $v_1$  are obtained by substituting for  $w$  the product of  $z^{\rho'}$  and a power series  $1 + c_1 z + \dots + c_n z^n + \dots$  in (5.7.1) and determining the  $c_n$  by recurrence. We know *in advance* (providing we have taken a root  $\rho'$  of largest real part) that the series obtained is *convergent* in the neighbourhood of 0 (and in fact in every disc of centre 0 not containing any singular point of the coefficients  $a(z)$  or  $b(z)$  of (5.7.1)).

When in (5.7.1)  $a$  and  $b$  are analytic at 0, it is said that the point  $z = 0$  (which is in general *singular* for the coefficients of (5.7.1)) is *regular for the differential equation*; for the equation deduced from (5.7.1) by the change of variable  $z \rightarrow 1/z$  (resp.  $z \rightarrow z - \alpha$ ), the *point at infinity* (resp. *the point*  $\alpha$ ) is said to be *regular*.

## 6. Equations of the second order depending on a parameter

Consider equations of the type (1.2) where  $\lambda$  is *complex* and where for large values of  $|\lambda|$ ,  $q(t, \lambda)$  is “approximately” independent of  $\lambda$ . To be precise, assume here  $t$  *real*,  $q(t, \lambda)$  being a complex function defined for  $t$  belonging to an open interval  $I = ]a, b[$  of  $\mathbf{R}$ , *bounded or not*, and for  $\lambda$  complex such that  $|\lambda| \geq R$ . Assume that we can write

$$(6.1) \quad \lambda^2 q(t, \lambda) = \lambda^2 + r(t, \lambda)$$

where for every  $\lambda$  satisfying  $|\lambda| > R$ ,  $t \rightarrow r(t, \lambda)$  is continuous in  $I$ , and where for  $|\lambda| \geq R$  and  $t \in I$ ,

$$(6.2) \quad |r(t, \lambda)| \leq F(t)$$

$F$  being a function continuous and  $\geq 0$  in  $I$  such that the improper integral  $\int_a^b F(t) dt$  is *convergent*.

The method for studying the solutions of the equation

$$(6.3) \quad x'' + (\lambda^2 + r(t, \lambda))x = 0$$

as  $|\lambda|$  tends to  $+\infty$  consists in “comparing” them to those of the equation with constant coefficients

$$(6.4) \quad x'' + \lambda^2 x = 0,$$

that is to the functions  $e^{\pm \lambda t}$ , by writing the equation (6.3) in the form

$$x'' + \lambda^2 x = -r(t, \lambda)x$$

and transforming the equation into an equivalent integral equation by means of (2.6). To be precise

(6.5) Suppose that  $I = ]0, b[$ . There exists  $R' > R$  such that, for every  $\lambda$  satisfying  $|\lambda| \geq R'$  and  $\mathcal{J}\lambda \geq 0$ , there exists a solution  $u(t, \lambda)$  of (6.3) satisfying the integral equation

$$(6.5.1) \quad u(t, \lambda) = e^{i\lambda t} + \frac{1}{\lambda} \int_t^b \sin \lambda(t-s) r(s, \lambda) u(s, \lambda) ds.$$

Furthermore, if the  $u_m(t, \lambda)$  are defined by the process of successive approximations

$$(6.5.2) \quad \begin{cases} u_0(t, \lambda) = e^{i\lambda t} \\ u_{m+1}(t, \lambda) = e^{i\lambda t} + \frac{1}{\lambda} \int_t^b \sin \lambda(t-s) r(s, \lambda) u_m(s, \lambda) ds \end{cases}$$

the sequence  $(u_m)$  converges uniformly to  $u$  for  $t \in I$  and  $|\lambda| \geq R'$ , and if  $\lambda = \sigma + i\tau$ , there exists a number  $M > 0$  independent of  $t$  and of  $\lambda$  such that

$$(6.5.3) \quad |u_{m+1}(t, \lambda) - u_m(t, \lambda)| \leq \frac{M^{m+1}}{|\lambda|^{m+1}} e^{-\tau t}.$$

It is immediately verified that every solution of (6.5.1) is twice continuously differentiable and satisfies (6.3). Everything reduces to proving, by induction on  $m$ , that the  $u_m$  are defined in  $I$ , satisfy (6.5.3) (which implies the existence of the limit  $u$  (for  $|\lambda|$  sufficiently large) and its continuity) and that the integrals

$$\int_t^b \sin \lambda(t-s) r(s, \lambda) u_m(s, \lambda) ds$$

tend to

$$\int_t^b \sin \lambda(t-s) r(s, \lambda) u(s, \lambda) ds.$$

Now, for  $s \geq t$  and  $\tau \geq 0$ , evidently

$$|\sin \lambda(t-s)| \leq e^{\tau(s-t)}.$$

Thus first

$$\begin{aligned} |u_1(t, \lambda) - u_0(t, \lambda)| &= \frac{1}{|\lambda|} \left| \int_t^b \sin \lambda(t-s) r(s, \lambda) e^{i\lambda s} ds \right| \\ &\leq \frac{1}{|\lambda|} \left| \int_t^b e^{\tau(s-t)} F(s) e^{-\tau s} ds \right| = \frac{M}{|\lambda|} e^{-\tau t} \end{aligned}$$

putting  $M = \int_0^b F(s) ds$ . If (6.5.3) is proved replacing  $m$  by  $m-1$

$$\begin{aligned} |u_{m+1}(t, \lambda) - u_m(t, \lambda)| &= \frac{1}{|\lambda|} \left| \int_t^b \sin \lambda(t-s) r(s, \lambda) (u_m(s, \lambda) - u_{m-1}(s, \lambda)) ds \right| \\ &\leq \frac{M^m}{|\lambda|^{m+1}} \int_t^b e^{\tau(s-t)} F(s) e^{-\tau s} ds = \frac{M^{m+1}}{|\lambda|^{m+1}} e^{-\tau t} \end{aligned}$$

which proves the existence of the  $u_m$  and the relation (6.5.3) for every  $m$ . Suppose now that  $|\lambda| > 2M$ ; then it follows from (6.5.3), for  $\lambda$  fixed, that the series  $(u_{m+1} - u_m)$  is normally convergent in  $I$ , and hence its sum  $u - u_0$  is continuous in  $I$  and

$$(6.5.4) \quad |u(t, \lambda) - u_m(t, \lambda)| \leq \frac{M^{m+1}}{|\lambda|^{m+1}} \frac{1}{1 - (M/|\lambda|)} e^{-\tau t} \leq \frac{2M^{m+1}}{|\lambda|^{m+1}} e^{-\tau t}$$

for every  $t \in I$ . We deduce as above that the integral

$$\int_t^b \sin \lambda(t-s) r(s, \lambda) (u(s, \lambda) - u_m(s, \lambda)) ds$$

exists and satisfies the relation

$$\left| \int_t^b \sin \lambda(t-s) r(s, \lambda) (u(s, \lambda) - u_m(s, \lambda)) ds \right| \leq \frac{2M^{m+2}}{|\lambda|^{m+1}} e^{-\tau t}$$

which completes the proof.

(6.6) When in (6.2)  $F(t)$  is bounded in  $I$ ,  $q(t, \lambda)$  does not vanish in  $I$  for  $|\lambda|$  sufficiently large. In applications cases are also met where  $q(t_0, \lambda) = 0$  for every  $\lambda$ , for certain values of  $t_0 \in I$ . The typical example is the equation

$$x'' - \lambda t x = 0$$

which will be studied in Chap. XV. This equation serves as a “model” for “comparison” of its solutions to those of equations of the form

$$x'' - \lambda(t + r(t, \lambda))x = 0$$

where  $r$  is “small” when  $|\lambda|$  is “large”. This problem is not discussed here.

## 7. Oscillation and comparison theorems

We return to equations (1.1) of the second order where the variable is *real* and also the coefficients  $p(t)$  and  $q(t)$ . We shall be interested in the *zeros* of the *real* solutions of (1.1). Note first that if  $u$  is a solution of (1.1), it can only have *simple* zeros in  $]a, b[$  (that is zeros  $t_0$  where  $u'(t_0) \neq 0$ ) if  $u$  is not identically zero: this follows from the uniqueness of the solution of (1.1) satisfying given initial conditions at a point  $t_0 \in I$  (XI, 3.6). In particular,  $u$  changes sign when  $t$  passes through a zero  $t_0$  of  $u$ . This also proves that the zeros of  $u$  are *isolated*, for otherwise a zero  $t_0$  of  $u$  would be the limit of a sequence  $(t_n)_{n \geq 1}$  of distinct zeros of  $u$ , so we would have

$$u'(t_0) = \lim_{n \rightarrow \infty} \frac{u(t_n) - u(t_0)}{t_n - t_0} = 0,$$

a contradiction. Our study of the zeros of  $u$  is based on the following two *comparison* theorems:

(7.1) (Oscillation theorem). *Let  $q_1, q_2$  be two continuous real functions in  $]a, b[$  such that  $q_2(t) \geq q_1(t)$  in  $]a, b[$ . Let  $u$  be a real solution of*

$$(7.1.1) \quad x'' + q_1(t)x = 0$$

and  $v$  a real solution of

$$(7.1.2) \quad x'' + q_2(t)x = 0$$

in  $]a, b[$ . Suppose that for an interval  $[\alpha, \beta] \subset ]a, b[$ , with  $\alpha < \beta$ ,  $u(\alpha) = u(\beta) = 0$  and  $u(t) > 0$  for  $\alpha < t < \beta$ . Then, unless  $v/u$  is constant in  $[\alpha, \beta]$  (which is possible only if  $q_2(t)$  and  $q_1(t)$  coincide in  $[\alpha, \beta]$ ), there exists  $\gamma$  satisfying  $\alpha < \gamma < \beta$  such that  $v(\gamma) = 0$ .

Suppose that the contrary is the case and assume that  $q_1$  and  $q_2$  do not coincide in  $[\alpha, \beta]$ ; by changing the sign of  $v$ , if necessary, it may be supposed that  $v(t) > 0$  for  $\alpha < t < \beta$ . From the relations

$$u''(t) + q_1(t)u(t) = 0, \quad v''(t) + q_2(t)v(t) = 0,$$

we deduce

$$(7.1.3) \quad u''(t)v(t) - v''(t)u(t) = (q_2(t) - q_1(t))u(t)v(t)$$

where the second member is continuous,  $\geq 0$  and non-zero at at least one point of  $[\alpha, \beta]$ . Hence the integral of the second member of (7.1.3) taken between  $\alpha$  and  $\beta$  is  $> 0$  (I, 3.1), so

$$(u'(t)v(t) - v'(t)u(t))|_{\alpha}^{\beta} > 0.$$

Since  $u(\alpha) = u(\beta) = 0$ , one finally obtains

$$(7.1.4) \quad u'(\beta)v(\beta) - u'(\alpha)v(\alpha) > 0.$$

But since  $u(\alpha) = 0$  and  $u(t) > 0$  for  $\alpha < t < \beta$ , necessarily  $u'(\alpha) > 0$  (since it was seen that  $u'(\alpha) \neq 0$ ); similarly  $u'(\beta) < 0$ . Since by continuity  $v(\alpha) \geq 0$  and  $v(\beta) \geq 0$ , the first member of (7.1.4) is  $\leq 0$ , a contradiction which proves (7.1).

Applying (7.1) to the particular case where  $q_1 = q_2$ :

(7.2) If  $u$  and  $v$  are two real solutions of

$$(7.2.1) \quad x'' + q(t)x = 0$$

forming a fundamental system, and if  $u(\alpha) = u(\beta) = 0$  and  $u(t) > 0$  for  $\alpha < t < \beta$ , there exists  $\gamma$  satisfying  $\alpha < \gamma < \beta$  such that  $v(\gamma) = 0$ .

(7.3) To enunciate the second comparison theorem, it is convenient to transform first an equation (7.2.1) to a system of two equations of the first order by the method of (XI, 4.8.2) putting  $x_1 = x$ ,  $x_2 = x'$ , which gives

$$(7.3.1) \quad \begin{aligned} x'_1 &= x_2 \\ x'_2 &= -q(t)x_1 \end{aligned}$$

and then to make the non-linear change of unknowns

$$(7.3.2) \quad x_1 = r \sin \theta, \quad x_2 = r \cos \theta$$

which gives the non-linear system

$$(7.3.3) \quad r' = (1 - q(t))r \sin \theta \cos \theta$$

$$(7.3.4) \quad \theta' = \cos^2 \theta + q(t) \sin^2 \theta.$$

Every solution  $(\rho(t), \omega(t))$  of this system formed of real functions will thus give a

solution of (7.2.1) by the formulae  $u(t) = \rho(t) \sin \omega(t)$ ,  $u'(t) = \rho(t) \cos \omega(t)$ . Moreover, since the second member of (7.3.4) as well as its derivative with respect to  $\theta$  is bounded for every real  $\theta$  and every  $t$  belonging to a bounded closed interval contained in  $]a, b[$ , the solutions of (7.3.4) are defined in  $]a, b[$  (XI, 3.7). But once a solution  $\theta$  of (7.3.4) is known, by putting it into (7.3.3),  $r$  is obtained by quadrature, and hence there is a solution  $(r, \theta)$  of the system of these two equations defined *throughout*  $]a, b[$ . Conversely, for each solution  $u(t)$  of (7.2.1), not identically zero, consider a point  $t_0 \in I$  and determine two real numbers  $r_0 \neq 0$  and  $\theta_0$  by the conditions  $r_0 \sin \theta_0 = u(t_0)$ ,  $r_0 \cos \theta_0 = u'(t_0)$ . Then the unique solution  $(\rho(t), \omega(t))$  of the system of the two equations (7.3.3), (7.3.4) such that  $\rho(t_0) = r_0$ ,  $\omega(t_0) = \theta_0$ , is necessarily such that

$$u(t) = \rho(t) \sin \omega(t), \quad u'(t) = \rho(t) \cos \omega(t)$$

for every  $t \in ]a, b[$ , by virtue of the uniqueness of the solutions of (7.2.1). One is thus led to studying the solutions of (7.3.3) and (7.3.4); note that if  $(\rho(t), \omega(t))$  is one such solution,  $\rho(t)$  cannot vanish at a point of  $]a, b[$  unless it is identically zero; it may therefore be assumed that  $\rho(t) > 0$  for every  $t \in ]a, b[$ . The zeros of the solutions of (7.2.1) which are not identically zero then correspond to the values of  $t$  for which the corresponding solution of (7.3.4) takes one of the values  $n\pi$  ( $n$  positive or negative integer).

(7.4) Let  $q_1, q_2$  be two continuous real functions in  $]a, b[$  such that  $q_2(t) \geq q_1(t)$  for all  $t$ . Let  $\varphi_1$  be a real solution of

$$(7.4.1) \quad \theta' = \cos^2 \theta + q_1(t) \sin^2 \theta$$

and  $\varphi_2$  a real solution of

$$(7.4.2) \quad \theta' = \cos^2 \theta + q_2(t) \sin^2 \theta$$

in  $]a, b[$ . Then, if  $\varphi_1(\alpha) \leq \varphi_2(\alpha)$  for an  $\alpha \in ]a, b[$ , we have  $\varphi_1(t) \leq \varphi_2(t)$  for  $\alpha \leq t < b$ . If further  $q_2(t) > q_1(t)$  for every  $t \in ]a, b[$ , we have  $\varphi_1(t) < \varphi_2(t)$  for  $\alpha < t < b$ .

From (7.4.1) and (7.4.2)

$$\varphi_2'(t) - \varphi_1'(t) = (q_1(t) - 1)(\sin^2 \varphi_2(t) - \sin^2 \varphi_1(t)) + (q_2(t) - q_1(t)) \sin^2 \varphi_2(t).$$

If  $w(t) = \varphi_2(t) - \varphi_1(t)$ ,

$$w'(t) = f(t)w(t) + h(t)$$

where

$$f(t) = (q_1(t) - 1)(\sin \varphi_2(t) + \sin \varphi_1(t)) \left( \frac{\sin \varphi_2(t) - \sin \varphi_1(t)}{\varphi_2(t) - \varphi_1(t)} \right)$$

(the last parenthesis being replaced by  $\cos \varphi_1(t)$  when  $\varphi_1(t) = \varphi_2(t)$ ) and

$$h(t) = (q_2(t) - q_1(t)) \sin^2 \varphi_2(t) \geq 0.$$

The function  $f$  is continuous and bounded in every bounded closed interval contained in  $]a, b[$  and therefore

$$w'(t) - f(t)w(t) \geq 0$$

in  $]a, b[$ . Let  $F(t)$  be a primitive of  $-f(t)$  in  $]a, b[$ ; multiplying the preceding inequality by  $e^{F(t)}$  one has

$$\frac{d}{dt} (e^{F(t)} w(t)) \geq 0$$

in  $]a, b[$ , hence for  $\alpha \leq t < b$

$$e^{F(t)}w(t) \geq e^{F(\alpha)}w(\alpha) \geq 0$$

which proves the first assertion. Moreover, if  $w(t) = 0$  for a  $t > \alpha$ , this implies in the first place that  $w(\alpha) = 0$  and that  $e^{F(s)}w(s)$  is constant for  $\alpha \leq s \leq t$ . Because of the expression for the function  $h$ , if it is supposed further that  $q_2(t) > q_1(t)$  for all  $t \in ]a, b[$ , this is possible only if  $\sin^2 \varphi_2(s) = 0$  for  $\alpha \leq s \leq t$ , and since  $\varphi_2$  is continuous, then  $\varphi_2(s) = n\pi$  for a fixed integer  $n$ , for any  $s$  in  $[\alpha, t]$ . But this contradicts the equation (7.4.2) satisfied by  $\varphi_2$ , since  $\varphi_2'(s) = 1$  is obtained in this interval.

From (7.4) restrictions are immediately deduced on the *number of zeros* of a solution of (7.2.1) in  $]a, b[$  when majorizations or minorizations for  $q(t)$  are known:

(7.5) Suppose that in (7.2.1)  $0 < \lambda^2 \leq q(t) \leq \mu^2$  for every  $t \in ]a, b[$ . Then, for every solution  $u(t)$  of (7.2.1), every closed interval of length  $\pi/\lambda$  contains at least one zero of  $u$ , and if  $u(\alpha) = 0$ , there is no zero of  $u$  in the interval  $]\alpha, \alpha + \pi/\mu[$ .

The solution of  $x'' + \lambda^2 x = 0$  are periodic of period  $T = 2\pi/\lambda$  and the change of variables (7.3.2) then gives for the solutions  $\theta$  of the corresponding equation (7.4.3)

$$\tan \theta = \frac{T}{2\pi} \tan \left( \frac{2\pi}{T} t - \gamma \right)$$

in other words, these solutions are such that  $\theta(t + (T/2)) = \theta(t) + \pi$ . The conclusion of (7.5) follows at once from the comparison of (7.2.1) to the two equations  $x'' + \lambda^2 x = 0$ ,  $x'' + \mu^2 x = 0$  by means of (7.4), and from the fact that at a point  $\alpha$  of  $]a, b[$ , there is always a solution of one or the other of these equations which satisfies the same initial conditions as a given solution of (7.2.1).

## 8. Problems of boundary values

Beside the *initial conditions*  $x(t_0) = x_0, x'(t_0) = x'_0$  for a differential equation of the second order (XI, 1.1), there occur in numerous applications other types of conditions imposed on the solutions of such an equation. Consider the *real* solutions of a linear equation of the second order

$$(8.1) \quad x'' + q(t)x = 0$$

where  $t$  is a *real* variable,  $q(t)$  being continuous in an open interval containing a *bounded closed* interval  $[a, b]$  with  $a < b$ . The values of a solution and its derivative are of interest not only at the single point  $a$ , but at *both* the points  $a$  and  $b$ . Two relations of the form

$$(8.2) \quad \begin{aligned} \alpha_1 u(a) + \beta_1 u'(a) + \gamma_1 u(b) + \delta_1 u'(b) &= 0 \\ \alpha_2 u(a) + \beta_2 u'(a) + \gamma_2 u(b) + \delta_2 u'(b) &= 0 \end{aligned}$$

will be referred to as *boundary conditions at  $a$  and  $b$* , provided that these two linear forms in four variables are linearly independent. The situation here is different to that for the Cauchy problem (XI, 1.1); for a given system of conditions (8.2) there are in general

no solutions other than 0 of (8.1) which satisfy these two conditions: this is seen immediately by considering the equation  $x'' - \lambda^2 x = 0$  with  $\lambda > 0$ , where the conditions at the limits are  $u(a) = 0$ ,  $u(b) = 0$ . None of the solutions  $c e^{\lambda t} + d e^{-\lambda t}$  of this equation can vanish at two distinct points, unless  $c = d = 0$ . Similarly for the equation  $x'' + \lambda^2 x = 0$  with  $\lambda > 0$ , the only solutions  $u$ , not identically zero, which satisfy  $u(a) = u(b) = 0$ , occur when  $\lambda(b-a)/\pi$  is an integer. This last problem corresponds in Physics to the problem of the small vibrations of a stretched homogeneous string, fixed at its endpoints, and the various "harmonic vibrations" of the string correspond to the solutions of this problem at the limits for the particular values of  $\lambda$  previously found, also called the "eigenvalues" of the problem.

It will be seen that these examples are typical of what occurs under quite general hypotheses: consider a second order linear equation depending on a real parameter  $\lambda$

$$(8.3) \quad x'' + (\lambda g(t) - f(t))x = 0$$

where  $f$  and  $g$  are continuous in  $[a, b]$  and where furthermore  $g(t) > 0$  for all  $t \in [a, b]$ . The problem will not be treated under the most general type of conditions (8.2), but confined to certain particular cases. It will be seen that the general phenomenon occurring in these cases is the existence of an increasing sequence

$$\lambda_0 < \lambda_1 < \lambda_2 < \dots < \lambda_n < \dots$$

of values of  $\lambda$  tending to  $+\infty$  and for each such value the problem has a solution other than 0, whereas there is no such solution for the remaining values of  $\lambda$ . The  $\lambda_n$  are called the *eigenvalues* of the equation (8.3) for the given boundary conditions.

(8.4) We begin with the case where in each of the conditions (8.2) the values of  $u$  and  $u'$  occur at just one of the two points  $a, b$ . In other words, it is supposed that the conditions can be written in the form

$$(8.4.1) \quad \begin{cases} u(a) \cos \alpha - u'(a) \sin \alpha = 0 \\ u(b) \cos \beta - u'(b) \sin \beta = 0 \end{cases}$$

("Sturm-Liouville problem") and it is further supposed, without loss of generality, that

$$0 \leq \alpha < \pi \quad \text{and} \quad 0 < \beta \leq \pi$$

(but not necessarily  $\alpha \leq \beta$ ).

Note that two solutions  $u_1, u_2$  of (8.3) satisfying (8.4.1) (for the same value of  $\lambda$ ) are necessarily *proportional*, for otherwise *every* solution of (8.3) would be of the form  $c_1 u_1 + c_2 u_2$  for suitable constants  $c_1, c_2$ , so would also satisfy the relations (8.4.1), which is absurd, since for such a solution  $u(a)$  and  $u'(a)$  can be taken *arbitrary*.

(8.5) *There exists a strictly increasing infinite sequence tending to  $+\infty$*

$$(8.5.1) \quad \lambda_0 < \lambda_1 < \lambda_2 < \dots < \lambda_n < \dots$$

*of eigenvalues of the equation (8.3) for the boundary conditions (8.4.1). Furthermore a solution  $u_n$  (other than 0) of (8.3) satisfying (8.4.1) for  $\lambda = \lambda_n$  (determined up to a constant factor) has exactly  $n$  zeros in the open interval  $]a, b[$ .*

The change of variables (7.3.2) in (8.3) gives the equation in  $\theta$

$$(8.5.2) \quad \theta' = \cos^2 \theta + (\lambda g(t) - f(t)) \sin^2 \theta$$

and  $\omega(t, \lambda)$  designates the unique solution of this equation (defined in the whole of  $[a, b]$ ) such that

$$(8.5.3) \quad \omega(a, \lambda) = \alpha.$$

$\omega$  corresponds to the solutions of (8.3) satisfying the *first* of the conditions (8.4.1) (and is proportional to one of these). The eigenvalues are the values of  $\lambda$  for which

$$(8.5.4) \quad \omega(b, \lambda) = \beta + n\pi \quad \text{for an integer } n \in \mathbf{Z}.$$

It is known (XI, 6.4) that

1. The function  $(t, \lambda) \rightarrow \omega(t, \lambda)$  and its partial derivatives  $\partial^k \omega / \partial \lambda^k$ ,  $\partial^{k+1} \omega / \partial t \partial \lambda^k$  ( $k \geq 0$ ), are continuous for  $a \leq t \leq b$  and  $\lambda \in \mathbf{R}$ .

Secondly, it follows from (7.4) and the hypothesis  $g(t) > 0$  in  $[a, b]$ , that:

2. For  $\lambda < \lambda'$  we have  $\omega(t, \lambda) < \omega(t, \lambda')$  for every  $t$  such that  $a < t \leq b$ .

Finally, by virtue of (8.5.2):

3. If  $t \in [a, b]$  satisfies  $\omega(t, \lambda) = k\pi$  ( $k$  integer),  $\frac{\partial \omega}{\partial t}(t, \lambda) = 1$ .

These properties imply the following:

(8.5.5) For an integer  $k$  the equation  $\omega(t, \lambda) = k\pi$  cannot have a solution in the semi-open interval  $]a, b]$  unless  $k \geq 1$ . If this solution exists, it is unique; each of the equations  $\omega(t, \lambda) = h\pi$  then has a unique solution  $t_h(\lambda)$  in  $]a, b]$  for  $1 \leq h \leq k$ , and

$$(8.5.6) \quad t_1(\lambda) < t_2(\lambda) < \dots < t_k(\lambda).$$

Moreover, for every  $\lambda' > \lambda$ ,  $\omega(t, \lambda') = h\pi$  has a solution  $t_h(\lambda')$  in  $]a, b]$  for  $1 \leq h \leq k$ , and

$$(8.5.7) \quad t_h(\lambda') < t_h(\lambda) \quad (1 \leq h \leq k).$$

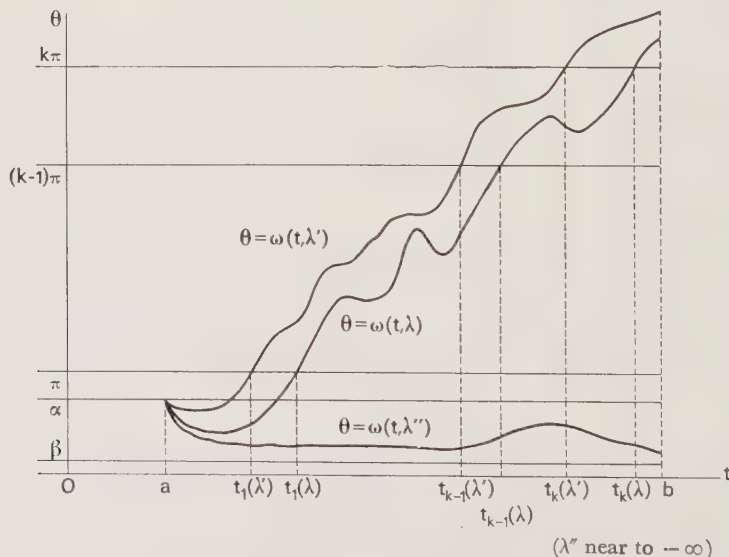


FIGURE 82

Property (3) shows that, even if  $\alpha = 0$ , we have  $\omega(t, \lambda) > 0$  in a non-empty open interval  $]a, a + \delta[$ ; if there is a solution of  $\omega(t, \lambda) = 0$  in  $]a, b]$ , there is a smallest solution  $t_0 > a$  (0, 2.3) and necessarily  $\frac{\partial \omega}{\partial t}(t_0, \lambda) \leq 0$ , which contradicts (3). The same reasoning shows that if there exists a solution  $t_k(\lambda)$  of the equation

$$\omega(t, \lambda) = k\pi$$

in  $]a, b]$ , it is unique. Indeed if  $t'$  is the smallest of these solutions, by (3),  $\frac{\partial \omega}{\partial t}(t', \lambda) > 0$ , so  $\omega(t', \lambda) > k\pi$  for  $t' < t < t' + \delta$  with  $\delta > 0$ . If  $t''$  is the smallest of the solutions  $> t'$ , again  $\frac{\partial \omega}{\partial t}(t'', \lambda) \leq 0$ , contradicting (3). The relations (8.5.6) and (8.5.7) are then immediate consequences of (I, 3.1) and the property (2) (Fig. 82). Let us now prove the following property:

(8.5.8) *For every  $t_0$  satisfying  $a < t_0 \leq b$ ,  $\omega(t_0, \lambda)$  tends to 0 as  $\lambda$  tends to  $-\infty$ .*

Since  $g(t) \geq c > 0$  and  $|f(t)| \leq m$  in  $[a, b]$ , then  $\lambda g(t) - f(t) \leq \lambda c + m$  for  $\lambda < 0$ . Thus, for each  $A > 0$ , there exists  $M > 0$  such that if  $\lambda < -M$ ,  $\lambda g(t) - f(t) \leq -A^2$ . If  $\rho(t, A)$  is the solution of

$$\theta' = \cos^2 \theta - A^2 \sin^2 \theta$$

such that  $\rho(a, A) = \alpha$ , then by virtue of (7.4)

$$(8.5.9) \quad \omega(t_0, \lambda) \leq \rho(t_0, A)$$

for  $\lambda \leq -M$ , and it is enough to show that  $\rho(t_0, A)$  is arbitrarily small for  $A$  sufficiently large (it is already known that  $\rho(t_0, A) > 0$  by (8.5.5)). Now

$$\tan \rho(t_0, A) = \frac{1 + \mu e^{-2A(t_0 - a)}}{A(1 - \mu e^{-2A(t_0 - a)})}$$

with

$$\mu = \frac{A \sin \alpha - \cos \alpha}{A \sin \alpha + \cos \alpha}$$

( $A \sin \alpha \neq -\cos \alpha$  as soon as  $A$  is sufficiently large). Since  $\mu$  remains bounded as  $A$  tends to  $+\infty$ , it is seen that  $\tan \rho(t_0, A)$  tends to 0 with  $1/A$ .

Finally:

(8.5.10) *For each  $t_0$  satisfying  $a < t_0 \leq b$ ,  $\omega(t_0, \lambda)$  tends to  $+\infty$  as  $\lambda$  tends to  $+\infty$ .*

With the same notations,  $\lambda g(t) - f(t) \geq \lambda c - m$  for  $\lambda > 0$ , so there exists  $M > 0$  such that  $\lambda g(t) - f(t) \geq A^2$  for  $\lambda > M$ . If  $\sigma(t, A)$  is the solution of

$$\theta' = \cos^2 \theta + A^2 \sin^2 \theta$$

thus  $\omega(t_0, \lambda) \geq \sigma(t_0, A)$  for  $\lambda \geq M$  and as already seen (7.5)

$$\sigma(t_0, A) \geq A(t_0 - a) - \pi.$$

The completion of the proof of (8.5) is now immediate. Since  $\lambda \rightarrow \omega(b, \lambda)$  is strictly increasing by virtue of (2), for each integer  $n \geq 0$ , there exists one and only one value  $\lambda_n$  of  $\lambda$  satisfying the relation  $\omega(b, \lambda) = \beta + n\pi$  since  $\beta > 0$ , taking into account (8.5.8) and (8.5.10). Since  $n\pi < \beta + n\pi \leq (n+1)\pi$ , the equation  $\omega(t, \lambda_n) = (n+1)\pi$  cannot have roots in  $]a, b[$ , for if  $t'$  were such a root,  $\omega(t, \lambda_n) > (n+1)\pi$  for  $t > t'$  by

(8.5.5), and in particular  $\omega(b, \lambda_n) > (n+1)\pi$ , which is absurd. It follows then from (8.5.5) that for  $1 \leq k \leq n$ , the equation  $\omega(t, \lambda_n) = k\pi$  has exactly one root in  $]a, b[$  and that it has no roots for  $k \geq n+1$  in  $]a, b[$ . Q.E.D.

## 9. Linear equations of the second order with periodic coefficients

It has already been seen (XII, 4.2) that if

$$x'' + q(t)x = 0$$

is a linear equation where  $q(t)$  is real, continuous in  $\mathbf{R}$  and *periodic* of period  $T$ , there is not necessarily any non-trivial solution of this equation which is periodic with period a *multiple* of  $T$ . The typical example is  $x'' - \lambda^2 x = 0$ , which does not have any periodic non-trivial solution for  $\lambda > 0$ . Consider again an equation depending on a real parameter

$$(9.1) \quad x'' + (\lambda g(t) - f(t))x = 0$$

where  $g(t)$  and  $f(t)$  are continuous and *periodic* of period  $T$  and further  $g(t) > 0$  for every  $t \in \mathbf{R}$ . A linear change of the variable reduces to the case where  $T = 1$ . This problem, seeking periodic solutions, is equivalent to a problem with boundary conditions of the type (8.2), although different to the one considered in no. 8.

Consider first a non-trivial solution  $u(t)$  of *period* 1. Then necessarily

$$(9.2) \quad u(0) = u(1), \quad u'(0) = u'(1).$$

Conversely, if a solution  $u$  of (9.1) satisfies the boundary conditions (9.2), the function  $u_1(t) = u(t+1)$  also satisfies equation (9.1) by virtue of the periodicity of  $f$  and  $g$ . Moreover  $u_1(0) = u(1)$  and  $u'_1(0) = u'(1)$ , therefore  $u_1(0) = u(0)$  and  $u'_1(0) = u'(0)$ , which implies that  $u_1 = u$  because of the uniqueness of a solution of Cauchy's problem (XI, 3.6). This shows that  $u$  is *periodic of period* 1.

If instead of (9.2) we consider the boundary conditions

$$(9.3) \quad u(1) = -u(0), \quad u'(1) = -u'(0),$$

the same reasoning shows that a solution of (9.1) satisfying these conditions is such that  $u(t+1) = -u(t)$  (a solution sometimes called *antiperiodic* of period 1); this is thus a solution of (9.1) of *period* 2.

(9.4) *Let*

$$(9.4.1) \quad \mu_0 < \mu_1 < \mu_2 < \dots < \mu_n < \dots$$

*be the sequence of eigenvalues of equation (9.1) for the boundary conditions*

$$(9.4.2) \quad u(0) = 0, \quad u(1) = 0.$$

*On the other hand let  $\nu_0$  be the smallest eigenvalue of (9.1) for the boundary conditions*

$$(9.4.3) \quad u'(0) = 0, \quad u'(1) = 0.$$

1.  $\nu_0 < \mu_0$ ; *the eigenvalues of (9.1) for the conditions (9.2) form an infinite increasing sequence*

$(\lambda_n)$  ( $n \geq 0$ ) and the eigenvalues of (9.1) for the conditions (9.3) an infinite sequence  $(\lambda'_n)$  ( $n \geq 1$ ) such that

$$(9.4.4) \quad \nu_0 \leq \lambda_0 < \lambda'_1 \leq \mu_0 \leq \lambda'_2 < \lambda_1 \leq \mu_1 \leq \dots \leq \mu_{2j} \\ \leq \lambda'_{2j+2} < \lambda_{2j+1} \leq \mu_{2j+1} \leq \lambda_{2j+2} < \lambda'_{2j+3} \leq \mu_{2j+2} \leq \lambda'_{2j+4} \leq \dots$$

2. For  $\lambda = \lambda_0$ , the solutions satisfying (9.2) are multiples of one of them  $v_0$ ; if  $\lambda_{2j+1} < \lambda_{2j+2}$ , the solutions of (9.1) satisfying (9.2) for  $\lambda = \lambda_{2j+1}$  (resp.  $\lambda = \lambda_{2j+2}$ ) are again multiples of one of them  $v_{2j+1}$  (resp.  $v_{2j+2}$ ). On the other hand, if

$$\lambda_{2j+1} = \lambda_{2j+2} = \mu_{2j+1},$$

all the solutions of (9.1) for  $\lambda = \mu_{2j+1}$  satisfy (9.2) (denote then  $v_{2j+1}$  and  $v_{2j+2}$  a fundamental system of solutions).

3. If  $\lambda'_{2j+1} < \lambda'_{2j+2}$ , the solutions of (9.1) satisfying (9.3) for  $\lambda = \lambda'_{2j+1}$  (resp.  $\lambda = \lambda'_{2j+2}$ ) are all multiples of one of them  $w_{2j+1}$  (resp.  $w_{2j+2}$ ). On the other hand, if  $\lambda'_{2j+1} = \lambda'_{2j+2} = \mu_{2j}$ , all the solutions of (9.1) for  $\lambda = \mu_{2j}$  satisfy (9.3) (denote then  $w_{2j+1}$  and  $w_{2j+2}$  a fundamental system of solutions).
4. The function  $v_0$  does not vanish in  $[0, 1]$ ; the functions  $v_{2j+1}$  and  $v_{2j+2}$  have exactly  $2j + 2$  zeros in  $[0, 1[$ ; the functions  $w_{2j+1}$  and  $w_{2j+2}$  have exactly  $2j + 1$  zeros in  $[0, 1[$ .

Let  $F(t, \lambda)$  and  $G(t, \lambda)$  denote the solutions of (9.1) satisfying respectively the initial conditions (where  $F'$ ,  $G'$  signify  $\partial F/\partial t$ ,  $\partial G/\partial t$ )

$$(9.4.5) \quad F(0, \lambda) = 1, \quad F'(0, \lambda) = 0.$$

$$(9.4.6) \quad G(0, \lambda) = 0, \quad G'(0, \lambda) = 1.$$

Then for any  $t$  and  $\lambda$  (2.2)

$$(9.4.7) \quad F(t, \lambda)G'(t, \lambda) - F'(t, \lambda)G(t, \lambda) = 1,$$

and  $F$  and  $G$  together with their partial derivatives of the first order are continuous in  $\mathbf{R}^2$  (XI, 6.4).

If a solution  $u(t, \lambda) = AF(t, \lambda) + BG(t, \lambda)$  satisfies (9.2) ( $A$  and  $B$  constants), two equations are obtained

$$(9.4.8) \quad \begin{cases} (F(1, \lambda) - 1)A + G(1, \lambda)B = 0 \\ F'(1, \lambda)A + (G'(1, \lambda) - 1)B = 0. \end{cases}$$

For such a solution to be non-trivial, it is necessary and sufficient that the determinant of this system be zero; taking into account (9.4.7), on putting

$$(9.4.9) \quad \Phi(\lambda) = F(1, \lambda) + G'(1, \lambda)$$

this amounts to

$$(9.4.10) \quad \Phi(\lambda) = 2.$$

Similarly, if  $u(t, \lambda)$  satisfies (9.3), the following two equations are obtained:

$$(9.4.11) \quad \begin{cases} (F(1, \lambda) + 1)A + G(1, \lambda)B = 0 \\ F'(1, \lambda)A + (G'(1, \lambda) + 1)B = 0 \end{cases}$$

which gives here the condition

$$(9.4.12) \quad \Phi(\lambda) = -2.$$

This being so, it is clear that  $F(t, \nu_0)$  is up to a constant factor the only solution of (9.1) for  $\lambda = \nu_0$  which satisfies the conditions (9.4.3); clearly this function cannot vanish at the points 0 and 1, and by (8.5) it cannot vanish in  $]0, 1[$  either. Again,  $G(t, \mu_0)$  is up to a constant factor the only solution of (9.1) for  $\lambda = \mu_0$  which satisfies (9.4.2). By the oscillation theorem (7.1), we cannot have  $\mu_0 \leq \nu_0$  as otherwise  $F(t, \nu_0)$  would vanish in  $]0, 1[$ . This proves the first assertion of (1). Since  $F(0, \nu_0) = 1$  and since  $F$  does not vanish,  $F(1, \nu_0) > 0$ . On the other hand  $F'(1, \nu_0) = 0$  and therefore from (9.4.7)  $F(1, \nu_0)G'(1, \nu_0) = 1$ . Hence

$$(9.4.13) \quad \Phi(\nu_0) = F(1, \nu_0) + (F(1, \nu_0))^{-1} \geq 2.$$

Then as above, up to a factor,  $G(t, \mu_j)$  is the only solution of (9.1) for  $\lambda = \mu_j$  which satisfies (9.4.2); thus  $G(1, \mu_j) = 0$  and the equation  $G(t, \mu_j) = 0$  has exactly  $j$  zeros in  $]0, 1[$  (8.5). Since  $G'(0, \mu_j) = 1$ , then  $G'(1, \mu_j) > 0$  if  $j$  is *odd* and  $G'(1, \mu_j) < 0$  if  $j$  is *even*. It follows from (9.4.7) that  $F(1, \mu_j)G'(1, \mu_j) = 1$ , hence

$$(9.4.14) \quad \Phi(\mu_j) = G'(1, \mu_j) + (G'(1, \mu_j))^{-1} \begin{cases} \geq 2 & \text{if } j \text{ is odd} \\ \leq -2 & \text{if } j \text{ is even.} \end{cases}$$

The existence of *at least one* eigenvalue  $\lambda_j$  (resp.  $\lambda'_j$ ) for the conditions (9.2) (resp. (9.3)) in each of the closed intervals with endpoints  $\nu_0$  and  $\mu_0$ , or two consecutive  $\mu_k$ , thus follows immediately from the relations (9.4.13) and (9.4.14), given the equations (9.4.10) and (9.4.12) which define these eigenvalues.

To complete the proof of assertions 1-3, it will be sufficient to establish the following lemma:

(9.4.15) (i) *If  $-2 \leq \Phi(\lambda) \leq 2$ , then  $\Phi'(\lambda) = 0$  only if both  $G(1, \lambda) = F'(1, \lambda) = 0$  and  $F(1, \lambda) = G'(1, \lambda) = \pm 1$  (hence  $\Phi(\lambda) = \pm 2$  and  $\lambda = \mu_j$  for an index  $j$ ). If this is not the case,*

$$\Phi'(\lambda) < 0 \quad \text{for } \lambda \leq \mu_0 \quad \text{and} \quad -2 \leq \Phi(\lambda) \leq 2,$$

$$(-1)^j \Phi'(\lambda) > 0 \quad \text{for } \mu_j \leq \lambda \leq \mu_{j+1} \quad \text{and} \quad -2 \leq \Phi(\lambda) \leq 2.$$

(ii) *If  $\Phi(\mu_{2j+1}) = 2$  and  $\Phi'(\mu_{2j+1}) = 0$ , we have  $\Phi''(\mu_{2j}) < 0$ . If  $\Phi(\mu_{2j}) = -2$  and  $\Phi'(\mu_{2j}) = 0$ , we have  $\Phi''(\mu_{2j}) \geq 0$ .*

Suppose in fact that these properties have been established. It follows (from 9.4.15, (i)) that there cannot be two distinct roots of  $\Phi(\lambda) = 2$  or of  $\Phi(\lambda) = -2$  in the interval  $] -\infty, \mu_0[$  nor in any of the open intervals  $]\mu_j, \mu_{j+1}[$ , since for two consecutive roots of these two equations, the derivative  $\Phi'(\lambda)$  cannot have the same sign. Moreover, if  $\Phi(\lambda) = 2$  (resp.  $\Phi(\lambda) = -2$ ) and  $\lambda \neq \mu_{2j+1}$  (resp.  $\lambda \neq \mu_{2j}$ ), it is not possible for  $F(t, \lambda)$  and  $G(t, \lambda)$  to satisfy *both* of the two conditions (9.2) (resp. (9.3)), therefore the solutions of (9.1) satisfying these conditions are multiples of one of them.

If  $\Phi(\mu_{2j+1}) = 2$ , it follows (from 9.4.15, (i) and (ii)) that in a sufficiently small open interval  $]\mu_{2j+1}, \mu_{2j+1} + \delta[$ , we have  $\Phi'(\lambda) < 0$ , therefore  $\Phi(\lambda) < 2$ . The same reasoning as above using (9.4.15, (i)) shows that there can be *no* root of  $\Phi(\lambda) = 2$  in the interval  $]\mu_{2j+1}, \mu_{2j+2}[$ . Moreover, if  $\Phi'(\mu_{2j+1}) \neq 0$ , it is seen as above that all the solutions of (9.1) for  $\lambda = \mu_{2j+1}$  satisfying (9.2), are multiples of one of them. If, on the other hand,  $\Phi'(\mu_{2j+1}) = 0$  and  $\Phi(\mu_{2j+1}) = 2$ , it follows from (9.4.15, (i)) that all the solutions of (9.1) satisfy (9.2). One argues in a similar way for the numbers  $\mu_{2j}$  and the conditions (9.3).

Let us therefore prove (9.4.15). It is clear that  $\frac{\partial F}{\partial \lambda}(0, \lambda) = \frac{\partial^2 F}{\partial t \partial \lambda}(0, \lambda) = 0$ , and that  $\partial F / \partial \lambda$  is, as a function of  $t$ , a solution of the differential equation

$$x'' + (\lambda g(t) - f(t))x = -g(t)F(t, \lambda)$$

hence Lagrange's formula (XII, 2.4.1) gives

$$\frac{\partial F}{\partial \lambda}(t, \lambda) = \int_0^t (F(t, \lambda)G(s, \lambda) - F(s, \lambda)G(t, \lambda))g(s)F(s, \lambda) ds$$

and in particular

$$(9.4.16) \quad \frac{\partial F}{\partial \lambda}(1, \lambda) = \int_0^1 (F(1, \lambda)G(s, \lambda) - F(s, \lambda)G(1, \lambda))g(s)F(s, \lambda) ds.$$

Similarly

$$\frac{\partial G}{\partial \lambda}(t, \lambda) = \int_0^t (F(t, \lambda)G(s, \lambda) - F(s, \lambda)G(t, \lambda))g(s)G(s, \lambda) ds.$$

Hence by differentiation

$$\frac{\partial^2 G}{\partial t \partial \lambda}(1, \lambda) = \int_0^1 (F'(1, \lambda)G(s, \lambda) - F(s, \lambda)G'(1, \lambda))g(s)G(s, \lambda) ds$$

therefore

$$(9.4.17) \quad \Phi'(\lambda) = \frac{\partial F}{\partial \lambda}(1, \lambda) + \frac{\partial^2 G}{\partial t \partial \lambda}(1, \lambda) \\ = \int_0^1 (A(\lambda)G^2(s, \lambda) + B(\lambda)F(s, \lambda)G(s, \lambda) + C(\lambda)F^2(s, \lambda))g(s) ds$$

setting

$$A(\lambda) = F'(1, \lambda), \quad B(\lambda) = F(1, \lambda) - G'(1, \lambda), \quad C(\lambda) = -G(1, \lambda).$$

Since  $g(s) > 0$  in  $[0, 1]$ , if the discriminant  $\Delta(\lambda)$  of the quadratic form  $A(\lambda)\xi^2 + B(\lambda)\xi\eta + C(\lambda)\eta^2$  is  $< 0$ ,  $\Phi'(\lambda)$  has the sign of  $C(\lambda) = -G(1, \lambda)$  since  $F(s, \lambda)$  and  $G(s, \lambda)$  are not identically zero. Now it follows from (8.5) and (8.5.5) applied with  $\alpha = 0, \beta = \pi$  that  $G(t, \lambda) = 0$  has no roots in  $]0, 1[$  for  $\lambda < \mu_0$ , and has exactly  $j$  roots for  $\mu_j < \lambda < \mu_{j+1}$ ; since  $G'(0, \lambda) > 0$ ,  $G(1, \lambda)$  is  $> 0$  for  $\lambda < \mu_0$ , and has the sign of  $(-1)^{j-1}$  for  $\mu_j < \lambda < \mu_{j+1}$ . Because  $\Delta(\lambda) = (\Phi(\lambda))^2 - 4$ , it is seen first that if  $-2 < \Phi(\lambda) < 2$ ,  $\Phi'(\lambda)$  has the sign indicated in (9.4.15, (i)). Suppose secondly that  $\Phi(\lambda) = \pm 2$ , so that  $A(\lambda)\xi^2 + B(\lambda)\xi\eta + C(\lambda)\eta^2$  is, apart from its sign, the square of a linear form: if this form is not identically zero, the fact that  $F(s, \lambda)$  and  $G(s, \lambda)$  are linearly independent implies that the integral of the second member of (9.4.17) cannot be zero. By continuity, then again  $\Phi'(\lambda) < 0$  for  $\lambda = \mu_0$  and  $(-1)^j \Phi'(\lambda) > 0$  for  $\mu_j \leq \lambda \leq \mu_{j+1}$ . If  $\Phi'(\lambda) = 0$  for  $\Phi(\lambda) = \pm 2$ , the form  $A(\lambda)\xi^2 + B(\lambda)\xi\eta + C(\lambda)\eta^2$  is therefore identically zero; since the converse is evident, this completes the proof of (9.4.15, (i)).

We shall now prove that if  $\Phi(\mu_{2j+1}) = 2$  and  $\Phi'(\mu_{2j+1}) = 0$ , then  $\Phi''(\mu_{2j+1}) < 0$ . Because of the preceding

$$(9.4.18) \quad G(1, \mu_{2j+1}) = F'(1, \mu_{2j+1}) = 0, \quad F(1, \mu_{2j+1}) = G'(1, \mu_{2j+1}) = 1.$$

One has

$$(9.4.19) \quad \Phi''(\lambda) = \frac{\partial^2 F}{\partial \lambda^2}(1, \lambda) + \frac{\partial^3 G}{\partial t \partial \lambda^2}(1, \lambda).$$

Differentiating (9.4.7) with respect to  $\lambda$

$$(9.4.20) \quad \frac{\partial G}{\partial \lambda} \frac{\partial F}{\partial t} + G \frac{\partial^2 F}{\partial t \partial \lambda} - \frac{\partial^2 G}{\partial t \partial \lambda} F - \frac{\partial G}{\partial t} \frac{\partial F}{\partial \lambda} = 0$$

and by virtue of (9.4.18)

$$(9.4.21) \quad \frac{\partial^2 G}{\partial t \partial \lambda} (1, \mu_{2j+1}) = -\frac{\partial F}{\partial \lambda} (1, \mu_{2j+1}).$$

Differentiating once more (9.4.20) with respect to  $\lambda$ , taking  $t = 1$  and  $\lambda = \mu_{2j+1}$ , and taking into account (9.4.18) and (9.4.21)

$$(9.4.22) \quad \Phi''(\mu_{2j+1}) = 2 \left( \left( \frac{\partial F}{\partial \lambda} (1, \mu_{2j+1}) \right)^2 + \frac{\partial G}{\partial \lambda} (1, \mu_{2j+1}) \frac{\partial^2 F}{\partial t \partial \lambda} (1, \mu_{2j+1}) \right).$$

But remembering (9.4.18), from (9.4.16)

$$(9.4.23) \quad \frac{\partial F}{\partial \lambda} (1, \mu_{2j+1}) = \int_0^1 G(s, \mu_{2j+1}) F(s, \mu_{2j+1}) g(s) ds$$

and in the same way one obtains

$$(9.4.24) \quad \frac{\partial G}{\partial \lambda} (1, \mu_{2j+1}) = \int_0^1 G^2(s, \mu_{2j+1}) g(s) ds$$

$$(9.4.25) \quad \frac{\partial^2 F}{\partial t \partial \lambda} (1, \mu_{2j+1}) = - \int_0^1 F^2(s, \mu_{2j+1}) g(s) ds.$$

Since  $F(s, \mu_{2j+1}) \sqrt{g(s)}$  and  $G(s, \mu_{2j+1}) \sqrt{g(s)}$  cannot be proportional, the Cauchy-Schwarz inequality (I, 4.5) shows that  $\Phi''(\mu_{2j+1}) < 0$ , as required.

The case where  $\Phi(\mu_{2j}) = -2$ ,  $\Phi'(\mu_{2j}) = 0$  is similarly treated, and this completes the proof of parts 1-3 of (9.4).

Finally the assertions of (4) will be proved. It follows from (9.2) that the functions  $v_j$  have an *even* number of zeros in  $[0, 1[$  and it follows similarly from (9.3) that the functions  $w_j$  have an *odd* number of zeros in  $[0, 1[$ . On the other hand from (8.5)  $G(t, \mu_j)$  has exactly  $j$  zeros in  $]0, 1[$ . Since  $\lambda_0 < \mu_0$ , it follows from the oscillation theorem (7.1) that  $v_0$  cannot have two zeros in  $[0, 1[$ ; since the number of these zeros is even,  $v_0$  has no zeros in  $[0, 1[$ . Since

$$\mu_{2j} < \lambda_{2j+1} \leq \mu_{2j+1} < \mu_{2j+2}$$

for  $j \geq 0$ , it again follows from (7.1) that  $v_{2j+1}$  and  $v_{2j+2}$  have  $N$  zeros in  $[0, 1[$ , where

$$2j + 1 \leq N \leq 2j + 3$$

and since  $N$  is even, it is equal to  $2j + 2$ .

Because  $\lambda'_1 \leq \lambda'_2 < \mu_1$ , the number of zeros of  $w_1$  and  $w_2$  is  $\leq 2$  by (7.1) and since this number is odd, it is equal to 1. Similarly, because

$$\mu_{2j-1} < \lambda'_{2j+1} \leq \lambda'_{2j+2} < \mu_{2j+1}$$

it again follows from (7.1) that the number  $N'$  of zeros of  $w_{2j+1}$  and  $w_{2j+2}$  in  $[0, 1[$  is such that

$$2j \leq N' \leq 2j + 2;$$

$N'$  being odd,  $N' = 2j + 1$ .

(9.5) The most important particular case for the application of (9.4) is that where  $g(t) = 1, f(t) = \cos 2\pi t$ ; the functions  $v_j$  and  $w_j$  are then called *Mathieu functions* and occur in numerous applications.

(9.6) For each integer  $k$ , there may also be required for suitable values of  $\lambda$ , periodic solutions of period  $kT$  of the equation (9.1). Naturally we shall again find the solutions of period  $T$  already obtained. However new solutions are also obtained by applying (9.4) to the interval  $[0, kT[$  instead of  $[0, T[$ : the number of zeros in  $[0, kT[$  of the periodic solutions of period  $T$  is a multiple of  $k$ , whereas (9.4) gives periodic solutions of period  $kT$  having in  $[0, kT[$  any even number of zeros.

## PROBLEMS

1. Let  $A$  be a diagonal matrix of order  $n$  and  $\lambda_1, \dots, \lambda_n$  the sequence of its eigenvalues (distinct or not). For every matrix  $B(t)$  continuous in  $[t_0, +\infty[$  and such that  $\int_{t_0}^{+\infty} \|B(t)\| dt$  is convergent, show that the equation

$$x' = (A + B(t))x$$

has a fundamental system of solutions  $u_1(t), \dots, u_n(t)$  defined in  $[t_0, +\infty[$  and such that  $u_k(t) \sim \exp(\lambda_k t) e_k$ , where  $(e_k)$  is the canonical base of  $\mathbf{C}^n$ . (Use problem 8 of Chap. XIII.)

2. Using Liouville's transformation show how the result of (4.2) can be generalized to the case of a development of the form (4.5.1) with  $k > 0$ . If  $k = 2h$  is even, obtain a solution having an asymptotic development of the form

$$(*) \quad u(t) = t^\alpha \exp(c_0 t^h + c_1 t^{h-1} + \dots + c_{h-1} t) \left( 1 + \frac{c'_1}{t} + \dots + \frac{c'_n}{t^n} + o\left(\frac{1}{t^n}\right) \right).$$

If  $k = 2h + 1$  is odd, obtain a development of the form

$$u(t) = v(t^{1/2})$$

where the development of  $v$  is of the form (\*). Study the analogous problem in the complex domain.

3. Show that if (4.2) is applied to the equation

$$x'' + \left(1 - \frac{a}{t^2}\right)x = 0,$$

the coefficients  $c_n$  which appear in the development (4.2.2) are such that the series  $\sum_{n=1}^{\infty} c_n z^n$  has zero radius of convergence, except if  $a$  has the form  $k(k+1)$  for some integer  $k > 0$ .

4. Consider the scalar equation

$$(1) \quad x'' - (1 + f(t))x = 0$$

where it is supposed that  $f$  is real and  $\lim_{t \rightarrow +\infty} f(t) = 0$ . Putting  $y = x'/x$ , the Riccati equation for  $y$  is

$$(2) \quad y' + y^2 = 1 + f(t).$$

Finally, if  $z = y - 1$

$$(3) \quad z' = -2z - z^2 + f(t).$$

(a) For each  $\varepsilon > 0$ , choose  $t_1$  so that  $|f(t)| \leq \varepsilon$  for  $t \geq t_1$ . Show that if one forms the successive approximations for (3) with

$$u_0(t) = \int_{t_1}^t e^{-2(t-s)} f(s) ds \quad \text{and} \quad u_n(t) = u_0(t) - \int_{t_1}^t e^{-2(t-s)} u_{n-1}^2(s) ds,$$

the  $u_n$  converge uniformly to a solution  $u(t)$  of (3) satisfying

$$|u(t)| \leq \varepsilon \quad \text{for } t \geq t_1.$$

Show further that for  $t \geq t_1$

$$|u(t)| \leq 2 \int_{t_1}^t e^{-2(t-s)} |f(s)| ds$$

and deduce that

$$\int_{t_1}^t |u(s)| ds \leq 4 \int_{t_1}^t |f(s)| ds.$$

Conclude that there is a solution  $v_1(t)$  of (1) satisfying the relations

$$(4) \quad \exp \left( t - c \int_{t_0}^t |f(s)| ds \right) \leq v_1(t) \leq \exp \left( t + c \int_{t_0}^t |f(s)| ds \right)$$

for a constant  $c$ , as soon as  $t$  is sufficiently large. Putting  $z = y + 1$  in (2), show similarly that there is a second solution  $v_2(t)$  of (1) satisfying

$$(5) \quad \exp \left( -t - c \int_{t_0}^t |f(s)| ds \right) \leq v_2(t) \leq \exp \left( -t + c \int_{t_0}^t |f(s)| ds \right).$$

(b) Suppose further that the integral  $\int_{t_0}^{+\infty} f^2(t) dt$  is convergent (although not necessarily the integral  $\int_{t_0}^{+\infty} |f(t)| dt$ ). In this case show that there is a solution  $v_1(t)$  of (1) such that

$$(6) \quad v_1(t) = \exp \left( t + \frac{1}{2} \int_{t_0}^t f(s) ds + o(1) \right);$$

also a solution  $v_2(t)$  such that

$$(7) \quad v_2(t) = \exp \left( -t - \frac{1}{2} \int_{t_0}^t f(s) ds + o(1) \right)$$

(majorize the squares of the solution  $u$  obtained in (a) with the help of the Cauchy-Schwarz inequality).

(c) If it is further supposed that the integral  $\int_{t_0}^{+\infty} |f(t)| dt$  is convergent, then (problem 1)

$$v_1(t) \sim e^t, \quad v_2(t) \sim e^{-t}.$$

(d) Examine in particular the case where  $f(t) = \sin t/t^\alpha$  with  $\frac{1}{2} < \alpha \leq 1$ .

5. Consider the equation

$$(1) \quad x'' + (1 + f(t))x = 0$$

where it is supposed that  $f$  is real,  $\int_{t_0}^{+\infty} |f'(t)| dt$  is convergent (which implies the existence of a finite limit for  $f(t)$  as  $t$  tends to  $+\infty$ ) and that

$$\lim_{t \rightarrow +\infty} f(t) = 0.$$

(a) Show that all the real solutions of (1) are defined for  $t \geq t_0$  and are bounded. (Multiplying the equation (1) by  $x'$  and integrating by parts, show that every solution  $u$  satisfies, for  $t$  sufficiently large, the inequality

$$u^2(t) \leq c + 2 \int_{t_0}^t |f'(s)| u^2(s) ds.$$

(b) Deduce a similar result for the solutions of

$$x'' + (1 + f(t) + g(t))x = 0$$

where  $\int_{t_0}^{+\infty} |g(t)| dt$  is convergent (cf. Chap. XIII, problem 8).

(c) Compare with the examples of problem 2, Chap. XIII, which show that the result of (a) does not extend to a system of two linear equations of the first order.

6. The function

$$u(t) = \left( \exp \left( \int_0^t g(s) ds \right) \cos t \right)$$

is a solution of the differential equation

$$x'' + (1 + f(t))x = 0$$

where

$$f(t) = 3g(t) \sin t - g'(t) \cos t - g^2(t) \cos^2 t.$$

If one takes  $g(t) = (1/t) \cos t$ ,  $u(t)$  is not bounded, but  $f(t)$  and  $f'(t)$  both tend to 0 with  $1/t$  (compare with problem 5).

Deduce that the equation

$$x'' + \left( 1 + \frac{\sin 2t}{t} \right) x = 0$$

possesses unbounded solutions (compare with the preceding equation using problem 8 of Chap. XIII).

7. The differential equation with real coefficients

$$(1) \quad x'' + (1 + f(t))x = 0$$

can be reduced to the system

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -(1 + f(t))x_1. \end{aligned}$$

Changing to polar coordinates

$$x_1 = r \cos(t + \theta), \quad x_2 = r \sin(t + \theta)$$

one obtains the non-linear system

$$(2) \quad \begin{cases} \theta' = \frac{1}{2} f(t) (1 + \cos 2(t + \theta)) \\ r'/r = \frac{1}{2} f(t) \sin 2(t + \theta). \end{cases}$$

Suppose that the improper integral  $\int_{t_0}^{+\infty} f(t) dt$  is convergent, as well as the two integrals

$$f_1(t) = \int_t^{+\infty} f(s) \cos 2s ds, \quad f_2(t) = \int_t^{+\infty} f(s) \sin 2s ds,$$

and lastly the two integrals  $\int_{t_0}^{+\infty} |f(t)f_j(t)| dt$  ( $j = 1, 2$ ). Show that under these conditions the equation (1) has a fundamental system of solutions  $u_1, u_2$  such that in the neighbourhood of  $+\infty$

$$\begin{aligned} u_1(t) &= \cos t + o(1), & u_1'(t) &= -\sin t + o(1) \\ u_2(t) &= \sin t + o(1), & u_2'(t) &= \cos t + o(1). \end{aligned}$$

(Introduce into the equations (2) the derivatives of

$$f_j(t) \cos 2\theta(t) \quad \text{and} \quad f_j(t) \sin 2\theta(t) \quad (j = 1, 2),$$

in order to show that for every solution, for which  $r \neq 0$ ,  $\theta(t)$  and  $r(t)$  tend to finite limits as  $t$  tends to  $+\infty$ . Show then that when  $\theta_1$  and  $\theta_2$  are two solutions of the first equation (2) not differing by a multiple of  $\pi$ , the limits of  $\theta_1$  and  $\theta_2$  cannot differ by a multiple of  $\pi$ . To do this, integrate the two members of this equation between  $t$  and  $+\infty$ , after having introduced the derivatives of  $f_1(t) \cos 2\theta(t)$  and  $f_2(t) \sin 2\theta(t)$  as indicated above).

Apply to the case where  $f(t) = (\sin at)/t^\beta$  with  $a \neq 2$  and  $\beta > \frac{1}{2}$ ; compare with problem 6.

8. In the differential equation with real coefficients

$$x'' + q(t)x = 0$$

suppose that  $q$  is differentiable,  $q(t) > 0$  and  $q'(t) > 0$  for  $t \geq t_0$ . Show that all the solutions of the equation are bounded in the neighbourhood of  $+\infty$  (same method as in problem 5(a)).

9. Show that in the differential equation

$$x'' + q(t)x = 0$$

if the integral  $\int_{t_0}^{+\infty} |q(t)| dt$  is convergent, the solutions cannot all be bounded.

10. Show that if, in the differential equation

$$x'' - q(t)x = 0$$

$q(t) > 0$  for every  $t \in \mathbf{R}$ , then no solution not identically zero can be bounded in the whole of  $\mathbf{R}$  (examine the signs and the variations of  $x$  and  $x'$ , looking at the different possible cases).

11. In the differential equation

$$x'' + q(t)x = 0$$

suppose that the integral  $\int_{t_0}^{+\infty} t|q(t)| dt$  is convergent. Show then that for each solution  $u$ ,  $\lim_{t \rightarrow +\infty} u'(t)$  exists and is finite, and that  $u(t) = at + b + o(1)$  in the neighbourhood of  $+\infty$ ,  $a$  and  $b$  being two constants not both zero. (Write

$$u(t) = at + b - \int_1^t (t-s)q(s)u(s) ds.$$

First deduce that  $u(t) = O(t)$ , and secondly observe that

$$u'(t) = a - \int_1^t q(s)u(s) ds.)$$

12. Let

$$w' + p(z)w' + q(z)w = 0$$

be a differential equation in which  $p$  and  $q$  are meromorphic and have only a finite number of poles  $a_1, \dots, a_r$  in  $\mathbf{C}$ . In order that the equation have only regular points (including the point at infinity), it is necessary and sufficient that  $p(z) = A(z)/\omega(z)$  and  $q(z) = B(z)/(\omega(z))^2$ , where  $\omega(z) = \prod_{k=1}^r (z - a_k)$ ,  $A(z)$  is a polynomial of degree  $\leq r - 1$  and  $B(z)$  a polynomial

of degree  $\leq 2(r - 1)$  (use Liouville's theorem).

Study the case where  $r = 3$ . It may be supposed that the regular points are 0, 1 and the point at infinity, by means of a suitable homographic transformation. By a change of unknown of the form  $w = w_1 z^\rho (1 - z)^\sigma$ , it may further be supposed that the characteristic equation at the point 0 and at the point 1 both have the root 0. The equation then has the form

$$z(1 - z)w'' + (\gamma - (\alpha + \beta + 1)z)w' - \alpha\beta w = 0$$

and is called the *hypergeometric equation*. The solution which takes the value 1 at  $z = 0$  is denoted by  $F(\alpha, \beta, \gamma; z)$ . Show that

$$\begin{aligned} F(\alpha, \beta, \gamma; z) = & 1 + \frac{\alpha\beta}{1\cdot\gamma}z + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1\cdot2\cdot\gamma(\gamma+1)}z^2 + \dots \\ & + \frac{\alpha(\alpha+1)\dots(\alpha+n-1)\beta(\beta+1)\dots(\beta+n-1)}{n!\gamma(\gamma+1)\dots(\gamma+n-1)}z^n + \dots \end{aligned}$$

the series having radius of convergence 1.

13. Study in detail the Liouville transformation in the complex domain, which reduces an equation of the form

$$w'' + \left(1 + \frac{q_1}{z} + \frac{q_2}{z^2} + \dots + o\left(\frac{1}{z^n}\right)\right)w = 0$$

to an equation of the same form but with  $q_1 = 0$ , and which therefore reduces to the case studied in (5.5).

14. Consider the differential equation with real coefficients

$$x'' + q(t)x = 0$$

where it is supposed that  $q(t) > 0$  for  $t \geq t_0$ . Show that if  $q'(t) = o(q(t)^{3/2})$ , then the number  $n(T)$  of zeros belonging to the interval  $[t_0, T]$  of a real, non-trivial solution of the equation is such that

$$n(T) \sim \frac{1}{\pi} \int_{t_0}^T \sqrt{q(t)} dt.$$

(Make the change of variable (3.2), then in the equation thus obtained change to polar coordinates in the equivalent system of two equations of the first order.)

15. Consider the differential equation with real coefficients

$$x'' + q(t)x = 0.$$

Show that if

$$q(t) \geq (1 + \alpha) \frac{1}{4t^2}, \quad \text{or} \quad q(t) \geq \frac{1}{4t^2} + (1 + \alpha) \frac{1}{4t^2(\log t)^2}$$

for  $t \geq t_0$  ( $\alpha$  constant  $> 0$ ), every solution  $u$  of the equation has infinitely many zeros in every interval  $[t_1, +\infty[$  for  $t_1 \geq t_0$ . (Apply successively the change of variable  $s = e^t$  starting from the equation  $x'' + m^2x = 0$  and apply (7.1).) Generalize.

16. Consider the differential equation with real coefficients

$$x'' + (ag(t) - f(t))x = 0$$

where  $g(t)$  and  $f(t)$  are periodic of period 1,  $g(t) > 0$  in  $\mathbf{R}$  and  $a > 0$ . Show that if  $\int_0^1 f(t) dt = 0$ , then every solution  $u(t)$  such that, in  $\mathbf{R}$ ,  $u(t+1) = \lambda u(t)$  for some real constant  $\lambda$ , vanishes at least once in the interval  $[0, 1]$ . (Use a *reductio ad absurdum* argument considering the Riccati equation satisfied by  $u'/u$ .)

17. Consider the differential equation with real coefficients

$$x'' + q(t)x = 0$$

where  $q(t)$  is a continuous function, periodic of period 1 and  $\geq 0$  in  $\mathbf{R}$ . Show that if  $\int_0^1 q(t) dt \leq 4$ , the equation has all its solutions bounded in  $\mathbf{R}$ . (Prove that there cannot be a solution  $u$  such that  $u(t+1) = \lambda u(t)$  for a real  $\lambda$ , using problem 16 above and problem 20 of Chap. I; then apply Floquet's theorem.)

18. Generalize the results of (9.4) when one has the boundary conditions

$$u(0) = au(1) + bu'(1)$$

$$u'(0) = cu(1) + du'(1)$$

with  $ad - bc = 1$ . (Compare with the boundary conditions

$$u(0) = 0, \quad au(1) + bu'(1) = 0.)$$

19. Suppose that in (9.4)  $\lambda_{2j+1} < \lambda_{2j+2}$ . Show that if  $\lambda = \lambda_{2j+1}$ , there is a second solution  $u_{2j+1}$  of (9.1) forming a fundamental system with  $v_{2j+1}$  such that

$$u_{2j+1}(t) = p_{2j+1}(t) + tv_{2j+1}(t)$$

where  $p_{2j+1}$  is periodic of period 1.

# Bessel functions

## 1. Integration of linear differential equations by integrals depending on a parameter

Consider a linear differential equation of order  $n$

$$(1.1) \quad a_0(z)w^{(n)} + a_1(z)w^{(n-1)} + \cdots + a_n(z)w = 0$$

where the variable  $z$  is complex, as is the unknown function  $w$ , and the coefficients  $a_j(z)$  are analytic in an open set  $D \subset \mathbf{C}$ . Occasionally one may obtain solutions of (1.1) of the form

$$(1.2) \quad u(z) = \int_L K(z, \zeta) d\zeta$$

where  $L$  is a path (or a path without endpoints) in  $\mathbf{C}$  and  $K(z, \zeta)$  is analytic. If one can obtain the successive derivatives of  $u$  by differentiating under the sign of integration,  $u$  will be a solution of (1.1) if

$$(1.3) \quad \int_L \left( a_0(z) \frac{\partial^n K}{\partial z^n} + a_1(z) \frac{\partial^{n-1} K}{\partial z^{n-1}} + \cdots + a_n(z) K(z, \zeta) \right) d\zeta = 0.$$

It may happen that this integral transforms by integration by parts to an integral which can be made zero by a suitable choice of  $L$ . It is then also necessary to verify that the integral (1.2) is not identically zero.

*Example* (1.4) Suppose that the  $a_j$  ( $0 \leq j \leq n$ ) are *polynomials of degree*  $\leq 1$ ,  $a_j(z) = b_j z + c_j$  ( $b_j, c_j$  complex constants), and try to make (1.3) vanish taking  $K(z, \zeta) = e^{z\zeta} v(\zeta)$ . The integral (1.3) can be written

$$\int_L \left( \sum_{j=0}^n (b_j z + c_j) \zeta^j \right) e^{z\zeta} v(\zeta) d\zeta.$$

Integrating by parts

$$\int_L z \zeta^j e^{z\zeta} v(\zeta) d\zeta = \zeta^j e^{z\zeta} v(\zeta) \Big|_L - \int_L e^{z\zeta} \frac{d}{d\zeta} (\zeta^j v(\zeta)) d\zeta$$

where the first term of the second member is an abbreviation for the difference between

the values of  $\zeta^j e^{z\zeta} v(\zeta)$  when  $\zeta$  is given the values equal to the terminal point and the initial point of  $L$ . Adding term by term, the problem reduces to satisfying the relation

$$e^{z\zeta} v(\zeta) Q(\zeta) \Big|_L - \int_L e^{z\zeta} (Q(\zeta) v'(\zeta) + P(\zeta) v(\zeta)) d\zeta = 0$$

where

$$P(\zeta) = \sum_{j=0}^n (jb_j \zeta^{j-1} - c_j \zeta^j), \quad Q(\zeta) = \sum_{j=1}^n b_j \zeta^j.$$

A solution of the problem can be sought by first taking for  $v(\zeta)$  a solution of the first order linear equation

$$Q(\zeta) v'(\zeta) + P(\zeta) v(\zeta) = 0$$

and then, with this function  $v(\zeta)$ , choosing  $L$  so that

$$e^{z\zeta} v(\zeta) Q(\zeta) \Big|_L = 0$$

and so that  $\int_L e^{z\zeta} v(\zeta) d\zeta$  is not identically zero ("Laplace's method").

As a first example of the application of this method, consider the equation

$$(1.5) \quad w'' - zw = 0.$$

One finds that  $v(\zeta) = e^{-\frac{1}{3}\zeta^3}$  and then takes for  $u(\zeta)$  the *Airy integral* (IX, 2.2), which, as has been seen, is not identically zero.

## 2. Hankel functions

The linear differential equation of the second order

$$(2.1) \quad w'' + \frac{1}{z} w' + \left(1 - \frac{\lambda^2}{z^2}\right) w = 0$$

is called the *Bessel equation* (of parameter  $\lambda \in \mathbf{C}$ ). We look for solutions of the form

$$u(z) = \int_L e^{-iz \sin \zeta} v(\zeta) d\zeta$$

for a suitable choice of  $L$  and  $v$ . Assuming that it is possible to differentiate under the integration sign, one must satisfy the equation

$$(2.2) \quad \int_L e^{-iz \sin \zeta} (z^2 \cos^2 \zeta - iz \sin \zeta - \lambda^2) v(\zeta) d\zeta = 0$$

which, putting  $F(z, \zeta) = e^{-iz \sin \zeta}$ , can also be written

$$(2.3) \quad \int_L \left( \frac{\partial^2 F}{\partial \zeta^2} + \lambda^2 F(z, \zeta) \right) v(\zeta) d\zeta = 0.$$

Now

$$\frac{\partial}{\partial \zeta} \left( Fv' - v \frac{\partial F}{\partial \zeta} \right) = Fv'' - v \frac{\partial^2 F}{\partial \zeta^2}$$

and therefore (2.3) can also be written

$$(2.4) \quad \int_L F(z, \zeta) (v''(\zeta) + \lambda^2 v(\zeta)) d\zeta + \left( \frac{\partial F}{\partial \zeta} v(\zeta) - F(z, \zeta) v'(\zeta) \right) \Big|_L = 0.$$

Proceeding as in (1.4), choose

$$v(\zeta) = e^{i\lambda\zeta}.$$

Then

$$(2.5) \quad -Fv' + v \frac{\partial F}{\partial \zeta} = e^{-iz \sin \zeta + i\lambda\zeta} (-iz \cos \zeta - i\lambda).$$

Take for  $L$  a path without endpoints such that when  $\zeta$  tends to one or the other of the “endpoints” of  $L$ , the limit of (2.5) is 0. This can be done only by *restricting* the domain of the variable  $z$  (the general theory (XIV, 5.7) shows that the solutions of (2.1) have in general a “branch point” at the point  $z = 0$ ).

One first assumes that  $\Re z > 0$  and takes for  $L$  either of the paths without endpoints  $L_1$  or  $L_2$  (Fig. 83); the functions

$$(2.6) \quad H_\lambda^1(z) = -\frac{1}{\pi} \int_{L_1} e^{-iz \sin \zeta + i\lambda\zeta} d\zeta$$

$$(2.7) \quad H_\lambda^2(z) = -\frac{1}{\pi} \int_{L_2} e^{-iz \sin \zeta + i\lambda\zeta} d\zeta$$

are called the *Hankel functions* of index  $\lambda$ . It will be seen that the integrals of the second members of (2.6) and (2.7) have a meaning and define non-trivial solutions of the Bessel equation (2.1) in the open half-plane  $\Re z > 0$ , forming a *fundamental system* of solutions (i.e. non-proportional).

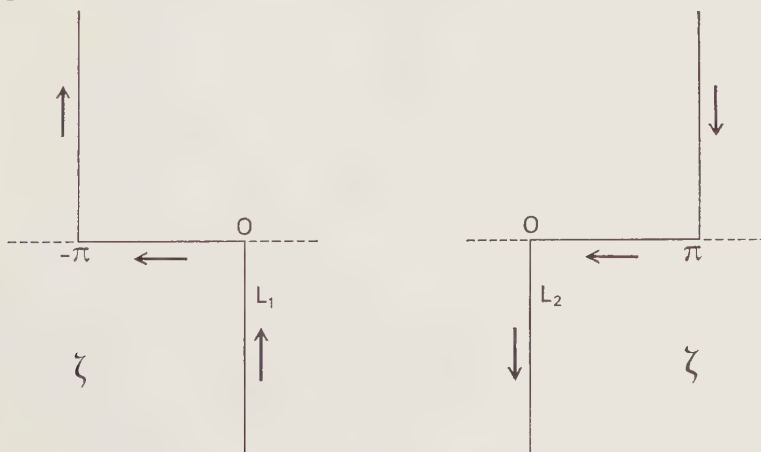


FIGURE 83

Let us put

$$\zeta = \xi + i\eta, \quad z = x + iy, \quad \lambda = a + ib, \quad \xi, \eta, x, y, a, b \text{ being real.}$$

For  $\xi = 0$ ,  $\sin \zeta = i \sinh \eta$ , hence

$$\mathcal{R}(-iz \sin \zeta + i\lambda \zeta) \sim -\frac{1}{2}x e^{-\eta}$$

as  $\eta$  tends to  $-\infty$ . Similarly for  $\xi = -\pi$ ,

$$\mathcal{R}(-iz \sin \zeta + i\lambda \zeta) \sim -\frac{1}{2}x e^{\eta}$$

as  $\eta$  tends to  $+\infty$ . This proves immediately the existence of the integral (2.6), and the validity of the differentiations under the integration sign. The reasoning is similar for (2.7). On the other hand, under the same conditions

$$|iz \cos \zeta + i\lambda| \sim |z| e^{|\eta|}$$

which is negligible compared with  $\exp(\frac{1}{2}x e^{\eta})$  for  $\xi = -\pi$  and  $\eta$  tending to  $+\infty$ , and compared with  $\exp(\frac{1}{2}x e^{-\eta})$  for  $\xi = 0$  and  $\eta$  tending to  $-\infty$ . The limit of (2.5) is thus 0 when  $\xi = 0$ ,  $\eta$  tends to  $-\infty$ , or when  $\xi = -\pi$ ,  $\eta$  tends to  $+\infty$ , and the function (2.6) is indeed a solution of (2.1). Similarly for (2.7). It will be seen in no. 3 that these solutions are not identically zero.

### 3. Analytic continuations and asymptotic developments of Hankel functions

The procedure used in no. 2 also gives solutions of (2.1) in other open sets of  $\mathbf{C}$ , provided we modify suitably the path of integration. Let  $\xi_0$  be a real number satisfying  $-\pi < \xi_0 < \pi$ , and let  $L(\xi_0)$  be the path analogous to  $L_1$ , but where the half-lines

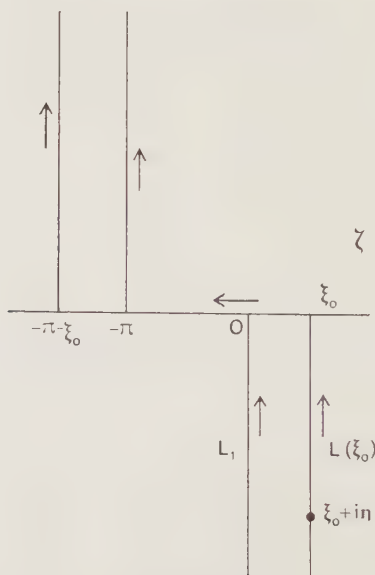


FIGURE 84

parallel to the imaginary axis pass through the points  $\xi_0$  and  $-\xi_0 - \pi$  (Fig. 84). For  $\xi = \xi_0$ ,

$$\Re(-iz \sin \zeta + i\lambda \zeta) = x \cos \xi_0 \sinh \eta + y \sin \xi_0 \cosh \eta - b\xi_0 - a\eta$$

and hence if

$$(3.1) \quad x \cos \xi_0 - y \sin \xi_0 > 0$$

then

$$\Re(-iz \sin \zeta + i\lambda \zeta) \sim -c e^{-\eta}$$

with  $c > 0$ , for  $\xi = \xi_0$  and  $\eta$  tending to  $-\infty$ . One may verify that the same condition (3.1) implies also

$$\Re(-iz \sin \zeta + i\lambda \zeta) \sim -c' e^{-\eta}$$

for  $\xi = -\pi - \xi_0$  and  $\eta$  tending to  $+\infty$ . The function

$$(3.2) \quad -\frac{1}{\pi} \int_{L(\xi_0)} e^{-iz \sin \zeta + i\lambda \zeta} d\zeta$$

is thus a solution of (2.1) defined in the half-plane (3.1), by the same reasoning as in no. 2. Now, for  $-\pi < \xi_0 < \pi$ , the half-plane (3.1) and the half-plane  $\Re z > 0$  have a non-empty intersection (Fig. 85).

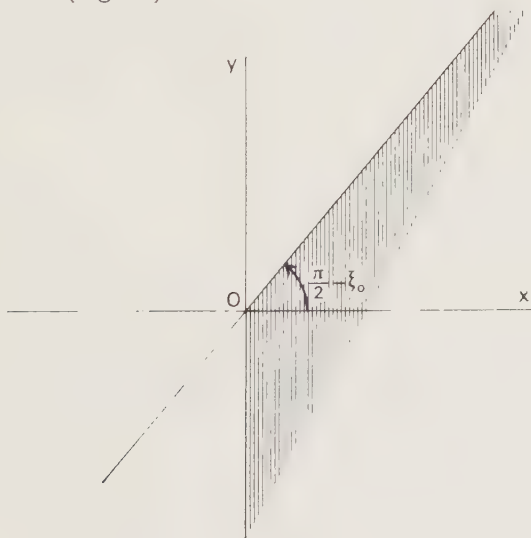


FIGURE 85

Let us show that the functions (2.6) and (3.2) coincide in this intersection. By Cauchy's theorem it will be sufficient to prove that when  $\gamma$  tends to  $+\infty$ , the integrals of the function  $\zeta \rightarrow e^{-iz \sin \zeta + i\lambda \zeta}$  taken along the paths  $t \rightarrow i\gamma + t$ ,  $-\pi - \xi_0 \leq t \leq -\pi$ , and  $t \rightarrow -i\gamma + t$ ,  $0 \leq t \leq \xi_0$  (assuming for example  $\xi_0 > 0$ ) tend to 0. Now, for  $\zeta = i\gamma + t$ ,

$$\Re(-iz \sin \zeta + i\lambda \zeta) = e^\gamma \left( \frac{x \cos t + y \sin t}{2} \right) + e^{-\gamma} \left( \frac{-x \cos t + y \sin t}{2} \right) - bt - ay$$

and when  $t$  varies from  $-\pi - \xi_0$  to  $-\pi$ , by hypothesis  $x \cos t + y \sin t$  does not vanish and remains  $< 0$ . The theorem of the mean at once shows that the integral

$$\int_{-\pi-\xi_0}^{-\pi} \exp(\Re(-iz \sin(i\gamma + t) + i\lambda(i\gamma + t))) dt$$

is majorized by a number of the form  $\exp(-c e^\gamma)$  as  $\gamma$  tends to  $+\infty$  ( $c$  constant  $> 0$ ), hence our assertion. One argues similarly for the path  $t \rightarrow -i\gamma + t$ ,  $0 \leq t \leq \xi_0$  when  $\gamma$  tends to  $+\infty$ . We have therefore obtained an analytic continuation of  $H_\lambda^1(z)$  into the union of the two half-planes (3.1) and  $\Re z > 0$ . In particular, if  $\xi_0 = \pi/2$  and  $\xi_0 = -\pi/2$ , one obtains an analytic continuation of  $H_\lambda^1(z)$  into the plane cut along the negative real axis. It should be noted that for  $\xi_0 = -\pi/2$ , we obtain the expression valid for  $\Im z > 0$

$$(3.3) \quad H_\lambda^1(z) = \frac{e^{-i(\lambda\pi/2)}}{\pi i} \int_{-\infty}^{+\infty} e^{iz \cosh x - \lambda x} dx.$$

Hence in particular, for  $z = it$ ,  $t > 0$

$$(3.4) \quad H_\lambda^1(it) = \frac{e^{-i(\lambda\pi/2)}}{\pi i} \int_{-\infty}^{+\infty} e^{-t \cosh x - \lambda x} dx$$

an integral to which Laplace's method can be applied (IV, 2) as  $t$  tends to  $+\infty$ . This gives the principal part

$$(3.5) \quad H_\lambda^1(it) \sim \frac{e^{-i(\lambda\pi/2)}}{\pi i} \sqrt{\frac{2\pi}{t}} e^{-t},$$

and shows that  $H_\lambda^1(z)$  is not identically zero.

Now apply the general theory (XIV, 5.3) observing that the linear change of unknown  $w = (1/\sqrt{z})v$  gives the equation

$$(3.6) \quad v'' + \left(1 + \frac{1 - 4\lambda^2}{z^2}\right)v = 0$$

in which (with the notations of (XIV, 5.3))  $q_0 = 1$ ,  $q_1 = 0$ .

Up to a constant factor, there is just one solution of (3.6) which tends to 0 when  $z = it$  and  $t$  tends to  $+\infty$ ; the general theory (XIV, 5.5) proves that there are asymptotic developments to an arbitrary precision

$$(3.7) \quad H_\lambda^1(z) = \sqrt{\frac{2}{\pi}} z^{-1/2} e^{i(z - \pi/4 - \lambda\pi/2)} \left(1 + \frac{a_1}{z} + \cdots + \frac{a_n}{z^n} + o\left(\frac{1}{z^n}\right)\right)$$

valid in the whole plane cut along the negative real axis as  $|z|$  tends to  $+\infty$  ( $z^{-1/2}$  being the principal determination of this function). An analogous calculation shows that for  $H_\lambda^2(z)$  one has asymptotic developments valid in the whole of the cut plane

$$(3.8) \quad H_\lambda^2(z) = \sqrt{\frac{2}{\pi}} z^{-1/2} e^{-i(z - \pi/4 - \lambda\pi/2)} \left(1 + \frac{b_1}{z} + \cdots + \frac{b_n}{z^n} + o\left(\frac{1}{z^n}\right)\right)$$

and this completes the proof that  $H_\lambda^1$  and  $H_\lambda^2$  form a fundamental system of solutions of (2.1) in the whole of the cut plane.

Note now that since equation (2.1) involves only the square of  $\lambda$ ,  $H_{-\lambda}^1(z)$  and  $H_{-\lambda}^2(z)$  are also solutions of this equation, which have respectively the same behaviour for  $z = it$ ,  $t$  tending to  $+\infty$  (resp.  $-\infty$ ) as  $H_{\lambda}^1$  (resp.  $H_{\lambda}^2$ ). Therefore comparing the principal parts of these functions

$$(3.9) \quad H_{-\lambda}^1(z) = e^{i\lambda\pi} H_{\lambda}^1(z), \quad H_{-\lambda}^2(z) = e^{-i\lambda\pi} H_{\lambda}^2(z),$$

relations which could in fact be deduced directly from the integral expressions given above. When  $\lambda$  and  $z$  are real

$$(3.10) \quad \overline{H_{\lambda}^1(z)} = -\frac{1}{\pi} \int_{L'_1} e^{iz \sin \zeta - i\lambda \zeta} d\zeta$$

where  $L'_1$  is the image of  $L_1$  under the mapping  $\zeta \rightarrow \bar{\zeta}$ . Observing that  $L_2$  is the image of  $L'_1$  under the mapping  $\zeta \rightarrow -\zeta$

$$\overline{H_{\lambda}^1(z)} = -\frac{1}{\pi} \int_{L_2} e^{-iz \sin \zeta + i\lambda \zeta} d\zeta.$$

In other words, for  $\lambda$  and  $z$  real

$$(3.11) \quad \overline{H_{\lambda}^1(z)} = H_{\lambda}^2(z), \quad \overline{H_{\lambda}^2(z)} = H_{\lambda}^1(z).$$

Finally, if  $\lambda = \frac{1}{2}$ , the equation (3.6) reduces to  $v'' + v = 0$ , and hence the only solution of (2.1) which tends to 0 when  $z = it$  and  $t$  tends to  $+\infty$ , is (up to a factor)  $(1/\sqrt{z}) e^{iz}$ . Comparison of the principal parts thus gives

$$(3.12) \quad H_{\frac{1}{2}}^1(z) = -i \sqrt{\frac{2}{\pi}} z^{-1/2} e^{iz}$$

and similarly

$$(3.13) \quad H_{\frac{1}{2}}^2(z) = i \sqrt{\frac{2}{\pi}} z^{-1/2} e^{-iz}.$$

#### 4. Bessel functions and Neumann functions

The *Bessel function of index*  $\lambda \in \mathbf{C}$  is defined as the function

$$(4.1) \quad J_{\lambda}(z) = \frac{1}{2}(H_{\lambda}^1(z) + H_{\lambda}^2(z))$$

and the *Neumann function of index*  $\lambda$  as the function

$$(4.2) \quad N_{\lambda}(z) = \frac{1}{2i}(H_{\lambda}^1(z) - H_{\lambda}^2(z)).$$

These functions are therefore defined and analytic in the plane cut along the negative real axis and form in this open set a fundamental system of solutions of the Bessel equation (2.1).

From the expressions (2.6) and (2.7)

$$(4.3) \quad J_{\lambda}(z) = -\frac{1}{2\pi} \int_L e^{-iz \sin \zeta + i\lambda \zeta} d\zeta$$

where  $L$  is the path of Fig. 86. If  $C$  is the image of  $L$  under the mapping  $\zeta \rightarrow e^{-i\zeta}$  (Fig. 87)

$$(4.4) \quad J_\lambda(z) = \frac{1}{2\pi i} \int_C \exp\left(\frac{1}{2}z\left(u - \frac{1}{u}\right)\right) u^{-\lambda-1} du;$$

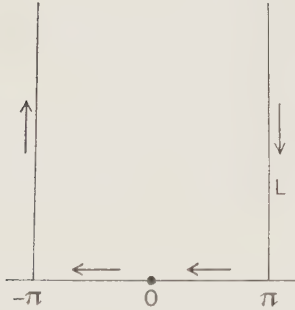


FIGURE 86

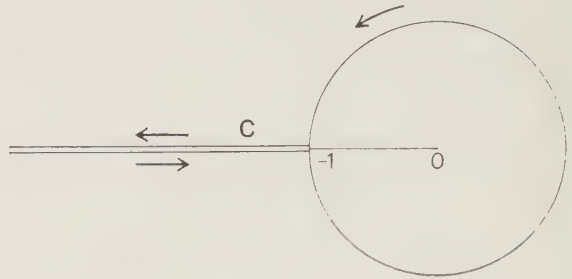


FIGURE 87

by Cauchy's theorem, the integral occurring in this formula does not change when  $C$  is replaced by its image under a homothety of ratio  $> 0$ . If  $z$  is *real and*  $> 0$ , by the change of variable  $zu = 2v$

$$(4.5) \quad J_\lambda(z) = \frac{1}{2\pi i} \left(\frac{z}{2}\right)^\lambda \int_C \exp\left(v - \frac{z^2}{4v}\right) v^{-\lambda-1} dv.$$

But the integral occurring in this formula is defined for *every*  $z \in \mathbf{C}$  (for  $\exp(-z^2/4v)$  is bounded on  $C$ ) and is an *entire function* of  $z$ . Since this entire function coincides with  $(2/z)^\lambda J_\lambda(z)$  for  $z$  real and  $> 0$ , it also coincides with  $(2/z)^\lambda J_\lambda(z)$  in the plane cut along the negative real axis by the principle of analytic continuation. In other words the formula (4.5) gives the value of  $J_\lambda(z)$  in the cut plane and proves that  $(2/z)^\lambda J_\lambda(z)$  can be continued into  $\mathbf{C}$  as an entire function. In fact for  $v \in \mathbf{C}$

$$(4.6) \quad \exp\left(-\frac{z^2}{4v}\right) = \sum_{n=0}^{\infty} \frac{(-z^2)^n}{2^{2n} n! v^n}.$$

Now, for every  $R > 1$

$$\int_{-\infty}^{-R} |t^{-n-\lambda-1}| e^t dt \leq c R^{\alpha-n-1} e^{-R}$$

for constants  $c$  and  $\alpha$  depending only on  $\lambda$ . The sum of the series

$$\sum_{n=0}^{\infty} \left| \frac{z^{2n}}{2^{2n} n!} \right| \int_{-\infty}^{-R} |t^{-n-\lambda-1}| e^t dt$$

is arbitrarily small with  $1/R$ . Since the series

$$\sum_{n=0}^{\infty} \frac{(-z^2)^n}{2^{2n} n!} v^{-n-\lambda-1} e^v$$

is normally convergent for  $1 \leq |v| \leq R$ , substituting this series for  $\exp(v - (z^2/4v))v^{-\lambda-1}$  in (4.5), it follows immediately that one can integrate term by term, and thus obtain the power series development of  $(2/z)^\lambda J_\lambda(z)$ :

$$(4.7) \quad J_\lambda(z) = \left(\frac{z}{2}\right)^\lambda \sum_{n=0}^{\infty} \frac{(-z^2)^n}{2^{2n} n! \Gamma(n + \lambda + 1)}$$

taking into account the expression for  $1/\Gamma(z)$  with the help of the Hankel integral (IX, 4.8.1).

Differentiating term by term the power series of the second member of (4.7), taking into account the functional equation of the gamma function, gives

$$(4.8) \quad \left(\frac{1}{z} \frac{d}{dz}\right)^k \left(\frac{J_\lambda(z)}{z^\lambda}\right) = (-1)^k \frac{J_{\lambda+k}(z)}{z^{\lambda+k}}$$

for every integer  $k \geq 1$ . In particular, for  $k = 1$ , this gives

$$(4.9) \quad J'_\lambda(z) = \frac{\lambda}{z} J_\lambda(z) - J_{\lambda+1}(z).$$

In a similar way the recurrence relation

$$(4.10) \quad J_{\lambda-1}(z) + J_{\lambda+1}(z) = \frac{2\lambda}{z} J_\lambda(z)$$

is verified using the relation

$$\frac{1}{(n+1)! \Gamma(n + \lambda + 1)} - \frac{1}{n! \Gamma(n + \lambda + 2)} = \frac{\lambda}{(n+1)! \Gamma(n + \lambda + 2)}.$$

By virtue of (3.9)

$$(4.11) \quad J_{-\lambda}(z) = \cos \lambda \pi J_\lambda(z) - \sin \lambda \pi N_\lambda(z)$$

hence, if  $\lambda$  is not an integer,  $J_\lambda$  and  $J_{-\lambda}$  form a fundamental system of solutions of (2.1). On the other hand, if  $\lambda = p$  is an integer

$$(4.12) \quad J_{-p}(z) = (-1)^p J_p(z)$$

and, conforming to the general theory (XIV, 5.7), one can verify in this case that in the plane cut along the negative axis the Neumann function  $N_p(z)$  has the form  $a J_p(z) \log z + z^{-p} F(z)$ , where  $a$  is a constant and  $F$  an entire function (problem 3). One should note that the roots of the characteristic equation (XIV, 5.7.2) of (2.1) relative to the point  $z = 0$  are  $\pm \lambda$ ; their difference can thus be an integer when  $\lambda = k/2$  ( $k$  an odd integer), without there being a logarithmic term in the Neumann function.

By virtue of (3.12) and (3.13)

$$(4.13) \quad J_{1/2}(z) = \sqrt{\frac{2}{\pi}} z^{-1/2} \sin z, \quad N_{1/2}(z) = -\sqrt{\frac{2}{\pi}} z^{-1/2} \cos z$$

and by virtue of (4.10) and (4.11), for  $k > 0$ ,  $J_{k+1/2}(z)$  and  $N_{k+1/2}(z)$  are linear combinations of the form

$$P\left(\frac{1}{z}\right)J_{1/2}(z) + Q\left(\frac{1}{z}\right)N_{1/2}(z)$$

where  $P$  and  $Q$  are polynomials of degree  $\leq k$ .

From (3.7) and (3.8) we deduce the asymptotic developments valid in the whole plane cut along the negative real axis

$$(4.14) \quad J_\lambda(z) = \sqrt{\frac{2}{\pi}} z^{-1/2} \cos\left(z - \frac{\pi}{4} - \frac{\lambda\pi}{2}\right) + O(|z|^{-3/2})$$

$$(4.15) \quad N_\lambda(z) = \sqrt{\frac{2}{\pi}} z^{-1/2} \sin\left(z - \frac{\pi}{4} - \frac{\lambda\pi}{2}\right) + O(|z|^{-3/2}).$$

By an immediate application of Rouché's theorem, for each  $\varepsilon > 0$ , and for sufficiently large  $n$ , there exists one and only one zero of  $J_\lambda(z)$  in the disc

$$\left| z - (2\lambda + 1)\frac{\pi}{4} - (2n + 1)\frac{\pi}{2} \right| \leq \varepsilon.$$

There are analogous results for the zeros of  $N_\lambda(z)$ .

## 5. Bessel functions of integer index

When  $\lambda = n$  is an integer  $\geq 0$ , in the formula (4.3) the integrals along the two half-lines of the path  $L$  parallel to the imaginary axis cancel each other, by virtue of the periodicity of  $\sin \zeta$  and  $e^{t\zeta}$ ; thus

$$(5.1) \quad J_n(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{tz \sin t} e^{-int} dt.$$

In other words  $J_n(z)$  is the  $n^{\text{th}}$  Fourier coefficient of the function  $t \mapsto e^{iz \sin t}$ ; since this function is entire and periodic, its Fourier series is convergent (X, 8.5). Therefore for any  $z \in \mathbf{C}$  and  $t \in \mathbf{R}$

$$(5.2) \quad e^{iz \sin t} = \sum_{n=-\infty}^{+\infty} J_n(z) e^{nit}$$

which is equivalent also to

$$(5.3) \quad \cos(z \sin t) = J_0(z) + 2 \sum_{n=1}^{\infty} J_{2n}(z) \cos 2nt$$

$$(5.4) \quad \sin(z \sin t) = 2 \sum_{n=0}^{\infty} J_{2n+1}(z) \sin(2n+1)t.$$

Since the function  $J_n$  is real in  $\mathbf{R}$ , we can also take the real part in (5.1) (for  $z$  real), which gives for all  $z \in \mathbf{C}$

$$(5.5) \quad J_n(z) = \frac{1}{\pi} \int_0^\pi \cos(nt - z \sin t) dt.$$

Note also that if we put  $-z = (\frac{2}{3}s)^{2/3}$  in equation (1.5), the equation becomes

$$w'' + \frac{1}{3s} w' + w = 0$$

and if we put  $w = s^{1/3}u$

$$u'' + \frac{1}{s} u' + \left(1 - \frac{1}{9s^2}\right)u = 0$$

which is the Bessel equation of parameter  $\frac{1}{3}$ . The Airy function  $\text{Ai}(z)$  can thus be expressed as a linear combination of  $J_{1/3}(z)$  and  $J_{-1/3}(z)$ ; the coefficients of this linear combination can be determined with the help of the asymptotic developments of  $\text{Ai}(z)$  and of the Bessel functions.

## PROBLEMS

1. Consider the linear equation of the fourth order

$$w^{(iv)} + \frac{2}{z} w''' + w = 0.$$

The only singular point of the coefficients is  $z = 0$  and the characteristic equation at this point is  $\rho(\rho - 1)^2(\rho - 2) = 0$ . Applying Laplace's method one obtains for a solution the integral

$$\int_L \frac{e^{z\xi} d\xi}{(1 + \xi^4)^{1,2}}$$

the path  $L$  being a loop such that  $j(\xi; L) = 1$  for two of the four zeros of  $\xi^4 + 1$ ,  $j(\xi; L) = 0$  for the other two. Show that one obtains in this way three solutions  $w_1 = u_1 + u_2$ ,  $w_2 = u_1 + v_1$ ,  $w_3 = u_2 - v_2$ , where

$$\begin{aligned} u_1(z) &= \int_0^1 \frac{1}{\sqrt{1-t^4}} e^{zt/\sqrt{2}} \cos \frac{zt}{\sqrt{2}} dt, \\ u_2(z) &= \int_0^1 \frac{1}{\sqrt{1-t^4}} e^{-zt/\sqrt{2}} \cos \frac{zt}{\sqrt{2}} dt, \\ v_1(z) &= \int_0^1 \frac{1}{\sqrt{1-t^4}} e^{zt/\sqrt{2}} \sin \frac{zt}{\sqrt{2}} dt, \\ v_2(z) &= \int_0^1 \frac{1}{\sqrt{1-t^4}} e^{-zt/\sqrt{2}} \sin \frac{zt}{\sqrt{2}} dt. \end{aligned}$$

A fourth solution is obtained by taking for  $L$  a path without endpoints enclosing one of

the zeros of  $\zeta^4 + 1$ , the integral being defined only if  $\Re z > 0$ . One obtains in this way the solution

$$w_4(z) = \int_0^{+\infty} \frac{e^{-zt} dt}{\sqrt{1+t^4}} - \frac{1}{\sqrt{2}} (u_2(z) + v_2(z)).$$

The functions  $w_1, w_2$  and  $w_3$  are entire functions. Show that when  $x$  is real and tends to 0 through values  $> 0$ , then  $w_4(x) \sim x \log 1/x$ . Show that for  $x$  real and tending to  $+\infty$

$$u_1 \pm iv_1 \sim \frac{e^{\mp i\pi/8}}{2} \sqrt{\frac{\pi}{x}} \exp(xe^{\pm i\pi/4})$$

$$u_2 \pm iv_2 \sim \frac{e^{\pm i\pi/4}}{x}.$$

Deduce that the four solutions  $w_1, w_2, w_3$  and  $w_4$  form a fundamental system in the half-plane  $\Re z > 0$ , and that  $w_3$  is the only solution which has, for  $x$  real, a finite derivative at the point 0 and tends to 0 as  $x$  tends to  $+\infty$ .

2. Show that for  $\Re \beta > 0$ ,  $\Re(\gamma - \beta) > 0$  and  $z$  distinct from a real number  $\geq 1$ , the hypergeometric function (Chap. XIV, problem 12) is given by

$$F(\alpha, \beta, \gamma; z) = \int_0^1 t^{\beta-1} (1-t)^{\gamma-\beta-1} (1-tz)^{-\alpha} dt.$$

3. Show that, for  $p$  an integer  $\geq 0$ ,

$$\begin{aligned} \pi N_p(z) &= 2 \left( \log \frac{z}{2} + \gamma - \frac{1}{2} \sum_{k=1}^n \frac{1}{k} \right) J_p(z) \\ &\quad - \sum_{n=0}^{\infty} (-1)^n \frac{(z/2)^{p+2n}}{n! (n+p)!} \sum_{k=1}^n \left( \frac{1}{k} + \frac{1}{k+p} \right) \\ &\quad - \sum_{k=0}^{p-1} \frac{(p-k-1)!}{k!} \left( \frac{z}{2} \right)^{-p+2k}. \end{aligned}$$

4. Evaluate explicitly and to an arbitrary precision the coefficients of the asymptotic development (4.14) of  $J_\lambda(z)$  (substitute in Bessel's equation).

5. Show that if  $a \neq b$  are two complex numbers and if  $\Re \lambda > -1$ ,

$$\begin{aligned} (a^2 - b^2) \int_0^x t J_\lambda(at) J_\lambda(bt) dt &= x \left( J_\lambda(ax) \frac{d}{dx} (J_\lambda(bx)) - J_\lambda(bx) \frac{d}{dx} (J_\lambda(ax)) \right) \\ 2a^2 \int_0^x t (J_\lambda(at))^2 dt &= (a^2 x^2 - \lambda^2) (J_\lambda(ax))^2 + \left( x \frac{d}{dx} (J_\lambda(ax)) \right)^2. \end{aligned}$$

(Use the relation  $\frac{d}{dt} (uv' - vu') = uw'' - wu''$ .)

Deduce that if  $J_\lambda(a) = J_\lambda(b) = 0$

$$\int_0^1 t J_\lambda(at) J_\lambda(bt) dt = 0, \quad \int_0^1 t (J_\lambda(at))^2 dt = \frac{1}{2} (J_{\lambda+1}(a))^2.$$

Conclude that if  $\lambda$  is real and  $> -1$ , the function  $J_\lambda(z)$  has no non-real zeros (if  $a$  were a non-real zero,  $b = \bar{a}$  would also be a zero.)

6. Show that

$$(J_0(z))^2 + 2(J_1(z))^2 + \cdots + 2(J_n(z))^2 + \cdots = 1$$

(use (5.2)). Deduce that for  $x$  real

$$|J_0(x)| \leq 1, \quad |J_n(x)| \leq 1/\sqrt{2} \quad \text{for } n \geq 1.$$

7. Show that for  $n$  an integer,  $u$  and  $v$  complex

$$J_n(u+v) = \sum_{p=-\infty}^{+\infty} J_p(u)J_{n-p}(v).$$



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- Uniformly bounded sequence of functions: V, 3.
- Uniformly continuous function: 0, 3 and 0, 5.
- Uniformly convergent (sequence, series): V, 2.
- Unit circle described  $n$  times: VII, 1.
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- Variation number of a sequence: II, problem 6.
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- Vector (complex): I, 1.
- Vector differential equation: XI, 4.
- Weierstrass approximation theorem: V, 5.
- Weierstrass convergence theorem: VII, 10.
- Weierstrass formula for the gamma function: IX, 4.
- Wronskian: XII, 2.
- Young's inequality: I, problem 4.
- Zero of a meromorphic function: VIII, 3.

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# Principal formulae

## Inequalities

$\left| \int_a^b f(t) dt \right| \leq \int_a^b |f(t)| dt$  ( $f$  complex piecewise-continuous function) (*inequality of the mean*)

$$\left| \sum_{j=1}^n a_j b_j \right|^2 \leq \left( \sum_{j=1}^n |a_j|^2 \right) \left( \sum_{j=1}^n |b_j|^2 \right)$$

$$\left| \int_a^b f(t) g(t) dt \right| \leq \left( \int_a^b |f(t)|^2 dt \right)^{1/2} \left( \int_a^b |g(t)|^2 dt \right)^{1/2}$$

(*Cauchy-Schwarz inequality*).

## Analytic functions

Definition of the integral  $\int_{\gamma} f(z) dz$ : if  $\gamma: [a, b] \rightarrow \mathbf{C}$ ,

$$\int_{\gamma} f(z) dz = \int_a^b f(\gamma(t)) \gamma'(t) dt.$$

*Index of  $a$  with respect to a loop  $\gamma$ :*

$$j(a; \gamma) = \frac{1}{2\pi i} \int_{\gamma} \frac{dz}{z - a}$$

*Cauchy's formula* ( $f$  analytic in a simply connected domain  $D$ ,  $\gamma$  a loop in  $D$ ,  $x \in D$  not on the image of the loop)

$$j(x; \gamma) f(x) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z) dz}{z - x}$$

$$j(x; \gamma) f^{(k)}(x) = \frac{k!}{2\pi i} \int_{\gamma} \frac{f(z) dz}{(z - x)^{k+1}} \quad \text{for } x \in D.$$

*Laurent series* for a function  $f$  analytic in  $\Delta - \{a\}$ :

$$f(z) = \sum_{n=0}^{\infty} c_n (z - a)^n + \sum_{n=1}^{\infty} \frac{d_n}{(z - a)^n}$$

with

$$c_n = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z) dz}{(z - a)^{n+1}}, \quad d_n = \frac{1}{2\pi i} \int_{\gamma} f(z) (z - a)^{n-1} dz$$

( $\gamma: t \rightarrow a + re^{it}$  contained in  $\Delta$ ).

*Theorem of residues* ( $f$  analytic in  $D' = D - \{a_1, \dots, a_n\}$ ,  $D$  simply connected,  $\gamma$  a loop in  $D'$ )

$$\int_{\gamma} f(z) dz = 2\pi i \sum_{k=1}^n j(a_k; \gamma) \operatorname{Res}_{a_k} f.$$

### Complex exponential and logarithmic functions

$$e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \dots + \frac{z^n}{n!} + \dots, \quad \text{convergent for } z \in \mathbf{C}$$

$$e^{z+z'} = e^z e^{z'}, \quad e^{-z} = 1/e^z \quad z, z' \in \mathbf{C}$$

$$|e^z| = e^{\Re z}, \quad e^{i(n/2)} = i$$

$$\cos z = \frac{1}{2}(e^{iz} + e^{-iz}), \quad \sin z = \frac{1}{2i}(e^{iz} - e^{-iz})$$

$$\log z = \log |z| + i \arg z, \quad \frac{d}{dz} (\log z) = \frac{1}{z}$$

for  $z$  in the plane cut along the negative real axis.

$$z^\lambda = e^{\lambda \log z}, \quad \frac{d}{dz} (z^\lambda) = \lambda z^{\lambda-1}$$

for any complex  $\lambda$ ,  $z$  in the plane cut along the negative real axis.

$$\log(1+z) = z - \frac{z^2}{2} + \dots + (-1)^{n-1} \frac{z^n}{n} + \dots$$

$$(1+z)^\lambda = 1 + \binom{\lambda}{1}z + \binom{\lambda}{2}z^2 + \dots + \binom{\lambda}{n}z^n + \dots$$

for  $|z| < 1$ .

$$\sin z = z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2\pi^2}\right) \quad \text{for } z \in \mathbf{C}$$

$$\cot z = \frac{1}{z} + 2z \sum_{n=1}^{\infty} \frac{1}{z^2 - n^2\pi^2} \quad \text{for } z/\pi \text{ not an integer.}$$

### The gamma function

$$\frac{1}{\Gamma(z)} = \lim_{n \rightarrow \infty} \frac{z(z+1)\dots(z+n)}{(n+1)^z n!} = z e^{\gamma z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-z/n}$$

for every  $z \in \mathbf{C}$ ; Euler's constant  $\gamma$  is given by

$$\gamma = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} - \log n\right) = 0.577\,215\,664\dots$$

$$\Gamma(z+1) = z\Gamma(z) \quad \text{for } -z \notin \mathbf{N}$$

$$\frac{\Gamma'(z)}{\Gamma(z)} = -\gamma - \frac{1}{z} + \sum_{n=1}^{\infty} \frac{z}{n(z+n)} \quad \text{for } -z \notin \mathbf{N}$$

$$\Gamma(n+1) = n! \quad \text{for } n \in \mathbf{N}, \quad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}, \quad \Gamma'(1) = -\gamma$$

$$\frac{1}{\Gamma(z)\Gamma(1-z)} = \frac{1}{\pi} \sin \pi z$$

$$\Gamma\left(\frac{z}{p}\right)\Gamma\left(\frac{z+1}{p}\right)\dots\Gamma\left(\frac{z+p-1}{p}\right) = (2\pi)^{(p-1)/2} p^{1-z} \Gamma(z) \quad (p \text{ integer} > 1)$$

(Legendre–Gauss formula).

$$\frac{1}{\Gamma(z)} = \frac{1}{2\pi i} \int_L u^{-z} e^u du \quad (z \in \mathbf{C}).$$

(Hankel's integral), where  $L: \mathbf{R} \rightarrow \mathbf{C}$  is a path without endpoints  $t \rightarrow r(t)e^{i\varphi(t)}$  contained in the plane cut along the negative real axis, with

$$\lim_{t \rightarrow \pm\infty} r(t) = +\infty,$$

$$\frac{\pi}{2} + \delta < \varphi(t) < \pi \quad \text{in the neighbourhood of } +\infty,$$

$$-\pi < \varphi(t) < -\frac{\pi}{2} - \delta \quad \text{in the neighbourhood of } -\infty \quad (\delta > 0).$$

$$\Gamma(z) = \int_0^{+\infty} t^{z-1} e^{-t} dt \quad \text{for } \Re z > 0$$

$$\mathbf{B}(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \quad \text{for } \Re x > 0, \Re y > 0$$

$$\Gamma(x) \sim \sqrt{2\pi} x^{x-1/2} e^{-x} \quad (x \text{ real tending to } +\infty).$$

(Stirling's formula).

$$z(z+1)\dots(z+n) \sim \frac{\sqrt{2\pi}}{\Gamma(z)} n^{n+z+1/2} e^{-z}, \quad \frac{\Gamma(n+z)}{\Gamma(n)} \sim n^z$$

( $n$  integer tending to  $+\infty$ ,  $z$  complex)

$$\binom{z}{n} \sim \frac{(-1)^n}{\Gamma(-z)} n^{-z-1} \quad (n \text{ integer tending to } +\infty, z \notin \mathbf{N})$$

$$\binom{2n}{n} \sim \frac{2^{2n}}{\sqrt{\pi n}} \quad (n \text{ integer tending to } +\infty)$$

$$\int_0^{+\infty} t^\alpha e^{-ct} dt = \frac{1}{\beta c^{(\alpha+1)/\beta}} \Gamma\left(\frac{\alpha+1}{\beta}\right) \quad \text{for } \alpha > -1, \beta > 0, c > 0$$

$$\int_0^{\pi/2} \sin^{2x-1} \theta \cos^{2y-1} \theta d\theta = \frac{1}{2} \mathbf{B}(x, y) \quad (x > 0, y > 0)$$

$$\int_0^{+\infty} t^{\lambda-1} e^{\pm it} dt = e^{\pm \frac{1}{2}\lambda\pi i} \Gamma(\lambda) \quad (\lambda \text{ real}, 0 < \lambda < 1).$$

### Bernoulli numbers and polynomials

$$\frac{1}{e^z - 1} = \frac{1}{z} - \frac{1}{2} + \sum_{n=1}^{\infty} (-1)^{n-1} \frac{B_n}{(2n)!} z^{2n-1} \quad (0 < |z| < 2\pi)$$

$$B_1 = \frac{1}{6}, \quad B_2 = \frac{1}{30}, \quad B_3 = \frac{1}{42}, \quad B_4 = \frac{1}{30}, \quad B_5 = \frac{5}{66}, \quad B_6 = \frac{691}{2730}$$

$$B_k = \frac{2(2k)!}{(2\pi)^{2k}} \sum_{n=1}^{\infty} \frac{1}{n^{2k}} \sim 4\sqrt{\pi} \frac{k^{2k+1/2}}{(2\pi)^{2k}}$$

$$\frac{e^{zx}}{e^z - 1} = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{\varphi_n(x)}{n!} z^{n-1} \quad (0 < |z| < 2\pi, x \in \mathbf{C})$$

$$\varphi_n(x) = x^n - \frac{n}{2} x^{n-1} + \binom{n}{2} B_1 x^{n-2} - \binom{n}{4} B_2 x^{n-4} + \binom{n}{6} B_3 x^{n-6} - \dots$$

$$\varphi_n(x+1) - \varphi_n(x) = nx^{n-1}$$

$$\varphi_n(1-x) = (-1)^n \varphi_n(x)$$

$$\varphi_{2k+1}(0) = 0, \quad \varphi_{2k}(0) = (-1)^{k+1} B_k$$

$$\varphi'_n(x) = n\varphi_{n-1}(x) \quad (n \geq 2)$$

$$\left. \begin{aligned} \varphi_{2k}(x) &= (-1)^{k+1} 2(2k)! \sum_{n=1}^{\infty} \frac{\cos 2n\pi x}{(2n\pi)^{2k}} \\ \varphi_{2k+1}(x) &= (-1)^{k+1} 2(2k+1)! \sum_{n=1}^{\infty} \frac{\sin 2n\pi x}{(2n\pi)^{2k+1}} \end{aligned} \right\} \quad (0 \leq x \leq 1)$$

for  $k \geq 1$ , the series being normally convergent.

$$\varphi_1(x) = x - \frac{1}{2} = - \sum_{n=1}^{\infty} \frac{\sin 2n\pi x}{n\pi}$$

for  $0 < x < 1$ , the series of the second member being uniformly convergent in  $[\alpha, 1 - \alpha]$  for  $0 < \alpha < \frac{1}{2}$ , and the partial sums being uniformly bounded in  $[0, 1]$ .

## Euler-Maclaurin formula

$$\begin{aligned} f(m) + f(m+1) + \dots + f(n) &= \int_m^n f(t) dt + \frac{1}{2}(f(m) + f(n)) \\ &+ \sum_{h=1}^r (-1)^{h-1} \frac{B_h}{(2h)!} (f^{(2h-1)}(n) - f^{(2h-1)}(m)) + R_r \end{aligned}$$

with

$$|R_r| \leq \frac{2}{(2\pi)^{2r}} \int_m^n |f^{(2r+1)}(t)| dt,$$

$m < n$  integers,  $f$  continuous in  $[m, n]$  together with its first  $2r$  derivatives, and having in  $[m, n]$  a piecewise-continuous  $(2r+1)^{\text{th}}$  derivative.

## Fourier series

$$c_m = \frac{1}{2\pi} \int_0^{2\pi} f(t) e^{-imt} dt \quad (m \in \mathbf{Z})$$

$$a_m = \frac{1}{\pi} \int_0^{2\pi} f(t) \cos mt dt, \quad b_m = \frac{1}{\pi} \int_0^{2\pi} f(t) \sin mt dt \quad (m \text{ integer } \geq 1)$$

$$a_0 = \frac{1}{2\pi} \int_0^{2\pi} f(t) dt$$

(Fourier coefficients of a function  $f$  piecewise-continuous in  $[0, 2\pi]$ ).

$$\sum_{m=-\infty}^{+\infty} |c_m|^2 = \frac{1}{2\pi} \int_0^{2\pi} |f(t)|^2 dt$$

(Parseval's relation).

$$a_0 + \sum_{m=1}^{\infty} (a_m \cos mx + b_m \sin mx) = \frac{1}{2}(f(x+) + f(x-))$$

if  $f$  is periodic of period  $2\pi$ , piecewise-continuous and piecewise-differentiable in  $\mathbf{R}$ , the convergence being uniform in every closed interval not containing a point of discontinuity of  $f$ , and the partial sums being uniformly bounded in  $\mathbf{R}$ .

### Matrix calculus

$$\frac{d}{dt}(A(t) \cdot \mathbf{f}(t)) = A'(t) \cdot \mathbf{f}(t) + A(t) \cdot \mathbf{f}'(t)$$

$$\frac{d}{dt}(A(t)B(t)) = A'(t)B(t) + A(t)B'(t)$$

( $A(t)$ ,  $B(t)$  matrices,  $\mathbf{f}(t)$  vector).

$$A \cdot \mathbf{f}(t) dt = A \cdot \int_a^b \mathbf{f}(t) dt$$

( $A$  constant matrix,  $\mathbf{f}(t)$  vector).

$$e^{Az} = \exp(Az) = \sum_{k=0}^{\infty} \frac{1}{k!} A^k z^k$$

( $A$  square matrix,  $z$  any complex number, the series being absolutely convergent).

$$e^0 = I, \quad e^{Iz} = I e^z, \quad e^{A(z+z')} = e^{Az} e^{Az'}, \quad e^{-Az} = (e^{Az})^{-1}$$

$$\frac{d^k}{dz^k} (e^{Az}) = A^k e^{Az} = e^{Az} A^k.$$

If

$$A = \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix}$$

( $B$  and  $C$  square matrices), then

$$e^{Az} = \begin{pmatrix} e^{Bz} & 0 \\ 0 & e^{Cz} \end{pmatrix}$$

### Integration of linear differential equations

A fundamental system of solutions of a homogeneous linear differential equation

$$\mathbf{x}' = A(t) \cdot \mathbf{x}$$

where  $A(t)$  is a square matrix of order  $n$ , is a system of  $n$  linearly independent solutions  $\mathbf{v}_1, \dots, \mathbf{v}_n$ ; the square matrix  $V(t)$  of order  $n$  whose columns are  $\mathbf{v}_1(t), \dots, \mathbf{v}_n(t)$  is then called the *fundamental matrix* of the equation. We have

$$\det(V(t)) = \det(V(s)) \exp \left( \int_s^t \text{Tr}(A(\xi)) d\xi \right).$$

The unique solution of the equation

$$\mathbf{x}' = A(t) \cdot \mathbf{x} + \mathbf{b}(t)$$

taking the value  $\mathbf{x}_0$  at a point  $s$ , is given by *Lagrange's formula*

$$\mathbf{v}(t) = V(t)V(s)^{-1} \cdot \mathbf{x}_0 + V(t) \cdot \int_s^t V(\xi)^{-1} \cdot \mathbf{b}(\xi) d\xi.$$

A *fundamental system* of solutions of a homogeneous linear differential equation of the second order

$$x'' + p(t)x' + q(t)x = 0$$

is a system of two *non-proportional* solutions  $u_1, u_2$ . The *wronskian*

$$W(t) = u_1(t)u_2'(t) - u_2(t)u_1'(t)$$

of this system is given by

$$W(t) = W(s) \exp \left( - \int_s^t p(\xi) d\xi \right)$$

and the solution of the equation

$$x'' + p(t)x' + q(t)x = f(t)$$

which vanishes at the point  $t_0$  is given by the formula

$$v(t) = \int_{t_0}^t \frac{u_1(s)u_2(t) - u_2(s)u_1(t)}{W(s)} f(s) ds.$$













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